Normal semi-transversal lightlike submanifolds of indefinite nearly Kaehler manifolds

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Abstract. The aim of present paper is to study normal semi-transversal lightlike submanifolds of indefinite nearly Kaehler manifolds. We find some necessary and sufficient conditions for an isometrically immersed semi-transversal lightlike submanifold of an indefinite nearly Kaehler manifold to be a normal semi-transversal lightlike submanifold.

M.S.C. 2010: 53C15, 53C40, 53C50.
Key words: Indefinite nearly Kaehler manifolds; semi-transversal lightlike submanifolds; normal semi-transversal lightlike submanifolds.

1 Introduction

The concept of CR-submanifolds of Kaehler manifolds was introduced by Bejancu [1], as a generalization of totally real and complex submanifolds and has been further developed by many others (for details, see [2, 3, 4]). The premise of CR-submanifolds has perceived several important contributions in complex and contact Riemannian (or pseudo-Riemannian) geometries and have been successfully applied in differential geometry and mathematical physics, particularly in, theory of general relativity. From last two decades, finding an interplay between Riemannian and semi-Riemannian geometries is a topic of chief interest. In the process of generalization of submanifold theory from Riemannian manifolds to semi-Riemannian manifolds, the lightlike submanifolds arise naturally in the semi-Riemannian category. In case of lightlike submanifolds, the normal bundle intersects with the tangent bundle and this characteristic feature makes the study of lightlike submanifolds more complicated and strikingly different from the study of non-degenerate submanifolds. As a result, one fails to use the results of non-degenerate submanifolds in case of lightlike submanifolds. Thus to generalize the concept of CR-submanifolds in lightlike geometry, Duggal and Bejancu [5] introduced the notion of CR-lightlike submanifolds of indefinite Kaehler manifolds and proved that this class of lightlike submanifolds has direct relation with physically important asymptotically flat space time, which further leads to Twistor theory of

Balkan Journal of Geometry and Its Applications, Vol.24, No.2, 2019, pp. 42-52.

Penrose and Heaven theory of Newman. But CR-lightlike submanifolds do not include complex and totally real lightlike submanifolds. Then Duggal and Sahin [6] introduced SCR-lightlike submanifolds of indefinite Kaehler manifolds, which contains complex and totally real subcases. But there was no inclusion relation between CR and SCR cases, therefore, Duggal and Sahin [7], introduced GCR-lightlike submanifolds of indefinite Kaehler manifolds, which behaves as an umbrella of complex, totally real, screen real and CR-lightlike submanifolds. Later on, Sahin [11] introduced the notion of semi-transversal lightlike submanifolds of indefinite Kaehler manifolds. Recently, Kumar [10] proved the existence of semi-transversal lightlike submanifolds in indefinite nearly Kaehler manifolds and proved various characterization results for semi-transversal lightlike submanifolds to be semi-transversal lightlike warped products.

In [2], Bejancu initiated the study of normal CR-submanifolds of Kaehler manifolds and proved several characterization theorems for a CR-submanifold of a Kaehler manifold to be a normal CR-submanifold. Haihua et. al. [9] investigated CR-submanifolds of nearly Kaehler manifolds and derived some results for a CR-submanifold to be normal. The available literature on lightlike submanifolds demonstrate that several classes of lightlike submanifolds have been introduced as a generalization of non-degenerate CR-submanifolds in lightlike geometry but no attempts have been made to generalize the idea of normal CR-submanifolds in lightlike geometry. Moreover, it is quite interesting to seek conditions under which a lightlike submanifold becomes a normal lightlike submanifold. Therefore, in this paper, we find some necessary and sufficient conditions for an isometrically immersed semi-transversal lightlike submanifold of an indefinite nearly Kaehler manifold to be a normal semi-transversal lightlike submanifold.

2 Preliminaries

2.1 Lightlike submanifolds

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\bar{M}$ and g be the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M, then M is called a lightlike submanifold of $\bar{M}$, (see [5]). For a degenerate metric g on M, $TM^{\perp}$ is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces, but no longer complementary. In this case, there exists a subspace $Rad(T_xM) = T_xM \cap T_xM^{\perp}$, which is known as radical (null) subspace. If the mapping $Rad(TM) : x \in M \longrightarrow Rad(T_xM)$, defines a smooth distribution on M of rank $r > 0$, then the submanifold M of $\bar{M}$ is called an r-lightlike submanifold and $Rad(TM)$ is called the radical distribution on M.

Screen distribution $S(TM)$ is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, that is

$$TM = Rad(TM) \perp S(TM) \tag{2.1}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM

in $T\bar{M}\mid_M$ and to $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ respectively. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}). \tag{2.2}$$

$$T\bar{M}\mid_M = TM \oplus tr(TM) = (Rad(TM) \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{2.3}$$

For a quasi-orthonormal fields of frames on TM, we have

Theorem 2.1. *([5]). Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ of $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM)\mid_u)$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}\mid_u$, where u is a coordinate neighborhood of M such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \textit{for any} \quad i, j \in \{1, 2, .., r\}, \tag{2.4}$$

where $\{\xi_1, ..., \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$, then according to the decomposition (2.3), the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \tag{2.5}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^{\perp}_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called second fundamental form, A_U is a linear operator on M and is known as shape operator.

According to (2.2), considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$ respectively, then Gauss and Weingarten formulae become

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{2.6}$$

where we put $h^l(X, Y) = L(h(X, Y)), h^s(X, Y) = S(h(X, Y)), D^l_X U = L(\nabla^{\perp}_X U)$, $D^s_X U = S(\nabla^{\perp}_X U)$. As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued respectively, therefore they are called the lightlike second fundamental form and the screen second fundamental form on M. In particular,

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \quad \bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{2.7}$$

where $X \in \Gamma(TM), N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Using (2.6) and (2.7), we obtain

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{2.8}$$

$$\bar{g}(D^s(X, N), W) = \bar{g}(A_W X, N), \tag{2.9}$$

for any $X, Y \in \Gamma(TM)$, $W \in \Gamma(S(TM^{\perp}))$ and $N \in \Gamma(ltr(TM))$.

Let P be the projection morphism of TM on $S(TM)$, then using (2.1), we can induce some new geometric objects on the screen distribution $S(TM)$ on M as

$$\nabla_X PY = \nabla^*_X PY + h^*(X, Y), \quad \nabla_X \xi = -A^*_{\xi} X + \nabla^{*t}_X \xi, \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla^*_X PY, A^*_{\xi} X\}$ and $\{h^*(X, Y), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$, respectively. Using (2.6) and (2.10), we obtain

$$\bar{g}(h^l(X, PY), \xi) = g(A^*_{\xi} X, PY), \quad \bar{g}(h^*(X, PY), N) = g(A_N X, PY), \tag{2.11}$$

for any $X, Y \in \Gamma(TM), \xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$.

2.2 Indefinite nearly Kaehler manifolds

Let $\bar{M}$ be an indefinite almost Hermitian manifold with an almost complex structure $\bar{J}$ of type $(1,1)$ and Hermitian metric $\bar{g}$ such that for all $X, Y \in \Gamma(T\bar{M})$ (see [12]), we have

$$\bar{J}^2 = -I, \ \bar{g}(\bar{J}X, \bar{J}Y) = \bar{g}(X, Y).$$

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ with respect to $\bar{g}$, then the covariant derivative of $\bar{J}$ is defined by

$$(\bar{\nabla}_X \bar{J})Y = \bar{\nabla}_X \bar{J}Y - \bar{J}\bar{\nabla}_X Y, \tag{2.12}$$

for all $X, Y \in \Gamma(T\bar{M})$.
An indefinite almost Hermitian manifold $\bar{M}$ is called an indefinite nearly Kaehler manifold (see [8]), if

$$(\bar{\nabla}_X \bar{J})Y + (\bar{\nabla}_Y \bar{J})X = 0, \ \forall \ X, Y \in \Gamma(TM), \tag{2.13}$$

which is equivalent to

$$(\bar{\nabla}_X \bar{J})X = 0, \ \forall \ X \in \Gamma(TM). \tag{2.14}$$

It is well known that every Kaehler manifold is a nearly Kaehler manifold but converse is not true. S^6 with its canonical almost complex structure is a nearly Kaehler manifold but not a Kaehler manifold. Due to rich geometric and topological properties, the study of nearly Kaehler manifolds is as important as that of Kaehler manifolds.

3 Semi-transversal lightlike submanifolds

Definition 3.1. ([10]). Let M be a lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$, then M is called a semi-transversal lightlike submanifold of $\bar{M}$, if the following conditions are satisfied:

(A) $Rad(TM)$ is transversal with respect to $\bar{J}$, that is, $\bar{J}Rad(TM) = ltr(TM)$.

(B) There exists a real non-null distribution $D \subset S(TM)$ such that

$$S(TM) = D \oplus D^{\perp}, \quad \bar{J}D^{\perp} \subset S(TM^{\perp}), \quad \bar{J}(D) = D,$$

where $D^{\perp}$ is orthogonal complementary to D in $S(TM)$.

Thus we obtain that the tangent bundle TM of a semi-transversal lightlike submanifold is decomposed as $TM = D \perp D'$, where $D' = D^{\perp} \perp Rad(TM)$.
Let M be a semi-transversal lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$. Let Q, P_1, P_2 and P be the projections on D, $Rad(TM)$, $D^{\perp}$ and D', respectively. Then for any $X \in \Gamma(TM)$, we have

$$X = QX + P_1X + P_2X. \tag{3.1}$$

Applying $\bar{J}$ to (3.1), we obtain

$$\bar{J}X = fX + \omega_1 X + \omega_2 X, \tag{3.2}$$

and we can rewrite (3.2) as

$$\bar{J}X = fX + \omega X, \tag{3.3}$$

where fX and ωX are the tangential and transversal components of $\bar{J}X$, respectively. Similarly,

$$\bar{J}V = BV + CV, \tag{3.4}$$

for any $V \in \Gamma(tr(TM))$, where BV and CV are the sections of TM and $tr(TM)$, respectively.
According to definition of semi-transversal lightlike submanifold, considering the decomposition $T\bar{M} = D \oplus D' \oplus \bar{J}D' \oplus \mu$, using (3.3) and (3.4), we have $fX \in \Gamma(D)$, $\omega X \in \Gamma(\bar{J}D')$, $BV \in \Gamma(D')$ and $CV \in \Gamma(\mu)$, for any $X \in \Gamma(TM)$ and $V \in \Gamma(\bar{J}D' \oplus \mu)$.
Moreover, the covariant derivatives of f and ω are respectively, given by

$$(\nabla_X f)Y = \nabla_X fY - f\nabla_X Y, \quad (\nabla^t_X \omega)Y = \nabla^t_X \omega Y - \omega \nabla_X Y, \tag{3.5}$$

for any $X, Y \in \Gamma(TM)$.

Lemma 3.1. *([12]). If $\bar{M}$ is a nearly Kaehler manifold, then*

$$(\bar{\nabla}_X \bar{J})Y + (\bar{\nabla}_{\bar{J}X} \bar{J})\bar{J}Y = 0, \quad (\bar{\nabla}_X \bar{J})Y = \frac{1}{4}\bar{J}[\bar{J}, \bar{J}](X, Y), \tag{3.6}$$

for any $X, Y \in \Gamma(T\bar{M})$, where $[\bar{J}, \bar{J}](X, Y)$ is the torsion tensor or the Nijenhuis tensor of $\bar{J}$ given by

$$[\bar{J}, \bar{J}](X, Y) = [\bar{J}X, \bar{J}Y] - \bar{J}[X, \bar{J}Y] - \bar{J}[\bar{J}X, Y] - [X, Y]. \tag{3.7}$$

Firstly, we will prove a basic lemma for later use.

Lemma 3.2. *Let M be a semi-transversal lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$. Then we have*

$$(\nabla_X f)Y = A_{\omega Y}X + Bh(X, Y) + \frac{1}{4}(\bar{J}[\bar{J}, \bar{J}](X, Y))^T, \tag{3.8}$$

$$(\nabla^t_X \omega)Y = Ch^s(X, Y) - h(X, fY) + \frac{1}{4}(\bar{J}[\bar{J}, \bar{J}](X, Y))^\perp, \tag{3.9}$$

for any $X, Y \in \Gamma(TM)$.

Proof. For any $X, Y \in \Gamma(TM)$, using (2.5), (3.3) and (3.4) in (2.12), we obtain

$$\begin{aligned}(\bar{\nabla}_X \bar{J})Y =& \nabla_X(fY) + h(X, fY) - A_{\omega Y}X + \nabla^t_X(\omega Y) \\ & - f(\nabla_X Y) - \omega(\nabla_X Y) - Bh(X, Y) - Ch^s(X, Y).\end{aligned} \tag{3.10}$$

Then from (3.5) and (3.10), we get

$$(\bar{\nabla}_X \bar{J})Y = (\nabla_X f)Y + (\nabla^t_X \omega)Y + h(X, fY) - A_{\omega Y}X - Bh(X, Y) - Ch^s(X, Y). \tag{3.11}$$

Further using (3.6) in (3.11), we have

$$(\nabla_X f)Y + (\nabla^t_X \omega)Y + h(X, fY) - A_{\omega Y}X - Bh(X, Y) - Ch^s(X, Y) = \frac{1}{4}\bar{J}[\bar{J}, \bar{J}](X, Y).$$

Thus on comparing the tangential and transversal components, the assertion follows. □

4 Normal semi-transversal lightlike submanifolds

Define a tensor field S as

$$S(X,Y) = [f,f](X,Y) - 2Bd\omega(X,Y),\ \forall\ X,Y \in \Gamma(TM), \tag{4.1}$$

where

$$[f,f](X,Y) = [fX,fY] + f^2[X,Y] - f([fX,Y] + [X,fY]) \tag{4.2}$$

and

$$d\omega(X,Y) = \frac{1}{2}\{\nabla^t_X(\omega Y) - \nabla^t_Y(\omega X) - \omega[X,Y]\}. \tag{4.3}$$

Since ∇ and ∇^t are torsion free, therefore from (4.2) and (4.3), we obtain

$$[f,f](X,Y) = (\nabla_{fX}f)Y - (\nabla_{fY}f)X - f((\nabla_X f)Y - (\nabla_Y f)X) \tag{4.4}$$

and

$$d\omega(X,Y) = \frac{1}{2}\{(\nabla^t_X\omega)Y - (\nabla^t_Y\omega)X\}. \tag{4.5}$$

Then using (4.4) and (4.5) in (4.1), we derive

$$\begin{aligned} S(X,Y) =&(\nabla_{fX}f)Y - (\nabla_{fY}f)X - f((\nabla_X f)Y - (\nabla_Y f)X)\\ &- B\{(\nabla^t_X\omega)Y - (\nabla^t_Y\omega)X\}. \end{aligned} \tag{4.6}$$

Further using (3.8) and (3.9) in (4.6), we have

$$\begin{aligned} S(X,Y) =&A_{\omega Y}fX - fA_{\omega Y}X - A_{\omega X}fY + fA_{\omega X}Y + \frac{1}{4}(\bar{J}[\bar{J},\bar{J}](fX,Y))^T\\ &- \frac{1}{4}(\bar{J}[\bar{J},\bar{J}](fY,X))^T - \frac{1}{2}f(\bar{J}[\bar{J},\bar{J}](X,Y))^T - \frac{1}{2}B(\bar{J}[\bar{J},\bar{J}](X,Y))^{\perp}. \end{aligned} \tag{4.7}$$

Now, we define a normal semi-transversal lightlike submanifold of an indefinite nearly Kaehler manifold as follows:

Definition 4.1. A semi-transversal lightlike submanifold M of an indefinite nearly Kaehler manifold $\bar{M}$ is said to be normal, if the tensor field S vanishes identically on M, that is, if

$$S(X,Y) = 0,\ \forall\ X,Y \in \Gamma(TM).$$

Theorem 4.1. *Let M be a semi-transversal lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$ with the totally real distribution D' being integrable. Then M is normal, if and only if*

$$\begin{aligned} 0 =&A_{\omega Y}fX - fA_{\omega Y}X - A_{\omega X}fY + fA_{\omega X}Y + \frac{1}{4}(\bar{J}[\bar{J},\bar{J}](fX,Y))^T\\ &- \frac{1}{4}(\bar{J}[\bar{J},\bar{J}](fY,X))^T - \frac{1}{2}f(\bar{J}[\bar{J},\bar{J}](X,Y))^T - \frac{1}{2}B(\bar{J}[\bar{J},\bar{J}](X,Y))^{\perp}, \end{aligned}$$

for any $X,Y \in \Gamma(TM)$.

Proof. The proof of assertion follows directly from Definition 4.1 and (4.7). □

Theorem 4.2. *Let M be a semi-transversal lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$ and*

$$[\bar{J}, \bar{J}](X, Y) \in \Gamma(\mu), \tag{4.8}$$

for any $X, Y \in \Gamma(TM)$. Then M is normal, if and only if

$$A_{\omega Y} fX = fA_{\omega Y} X, \tag{4.9}$$

for any $X \in \Gamma(D)$ and $Y \in \Gamma(D')$.

Proof. Using (4.8) in (4.7), we derive

$$S(X, Y) = A_{\omega Y} fX - fA_{\omega Y} X - A_{\omega X} fY + fA_{\omega X} Y, \tag{4.10}$$

for any $X, Y \in \Gamma(TM)$.
Assume that M be a normal semi-transversal lightlike submanifold of an indefinite nearly Kaehler manifold, then for any $X \in \Gamma(D)$ and $Y \in \Gamma(D')$ from (4.10), we obtain (4.9).
Conversely, let M be a semi-transversal lightlike submanifold of an indefinite nearly Kaehler manifold satisfying (4.9). For $X, Y \in \Gamma(D)$, using (4.10), we get $S(X, Y) = 0$. Now for $X \in \Gamma(D)$ and $Y \in \Gamma(D')$, from (4.10), we obtain $S(X, Y) = A_{\omega Y} fX - fA_{\omega Y} X$, which on using (4.9) reduces to $S(X, Y) = 0$. Similarly, for $X \in \Gamma(D')$ and $Y \in \Gamma(D)$, from (4.10), we have $S(X, Y) = 0$.
Finally for $X, Y \in \Gamma(D')$, using (4.10), we get

$$S(X, Y) = f(A_{\omega X} Y - A_{\omega Y} X). \tag{4.11}$$

Next for any $X, Y \in \Gamma(D')$ from (2.5) and (2.12), we obtain $(\bar{\nabla}_X \bar{J})Y = -A_{\omega Y} X + \nabla^t_X(\omega Y) - \bar{J}(\bar{\nabla}_X Y)$, then considering inner product with $Z \in \Gamma(D)$, we obtain

$$g(A_{\omega Y} X, Z) = -\bar{g}((\bar{\nabla}_X \bar{J})Y, Z) - \bar{g}(\bar{J}(\bar{\nabla}_X Y), Z). \tag{4.12}$$

On interchanging the role of X and Y in (4.12), we get

$$g(A_{\omega X} Y, Z) = -\bar{g}((\bar{\nabla}_Y \bar{J})X, Z) - \bar{g}(\bar{J}(\bar{\nabla}_Y X), Z). \tag{4.13}$$

Now subtracting (4.12) from (4.13), we derive

$$g(A_{\omega X} Y - A_{\omega Y} X, Z) = 2\bar{g}((\bar{\nabla}_X \bar{J})Y, Z) + \bar{g}(\bar{J}[X, Y], Z). \tag{4.14}$$

Then for any $X, Y \in \Gamma(D')$, (4.8) gives that D' is integrable. Further using (3.6) and (4.8) in (4.14), we obtain $g(A_{\omega X} Y - A_{\omega Y} X, Z) = 0$, then non-degeneracy of D yields that $A_{\omega X} Y = A_{\omega Y} X$, thus from (4.11), we have $S(X, Y) = 0$, which completes the proof. □

Corollary 4.3. *A semi-transversal lightlike submanifold M of an indefinite nearly Kaehler manifold $\bar{M}$ satisfying (4.8) is normal, if and only if*

(i) $\bar{g}(h^s(X, fY), \omega Z) + \bar{g}(h^s(fX, Y), \omega Z) = 0,$

(ii) $\bar{g}(h^s(fX, W), \omega Z) = 0$,

for any $X, Y \in \Gamma(D)$ *and* $Z, W \in \Gamma(D^{\perp})$.

Suppose $\{E_1, E_2, E_3, ..., E_q\}$ is a local field of orthogonal frames for $D^{\perp}$. Denote A_i, the fundamental tensor of Weingarten with respect to $V_i = \bar{J}E_i$, then from above theorem, we have

Corollary 4.4. *Let* M *be a semi-transversal lightlike submanifold of an indefinite nearly Kaehler manifold* $\bar{M}$ *and*

$$[\bar{J}, \bar{J}](X, Y) \in \Gamma(\mu), \tag{4.15}$$

for any $X, Y \in \Gamma(TM)$. *Then* M *is normal if and only if the fundamental tensors of Weingarten* A_i *commute with* f *on invariant distribution, that is, if and only if*

$$A_i \circ f = f \circ A_i. \tag{4.16}$$

Next using (2.5), we derive

$$\nabla_X E_i = fA_{\bar{J}E_i}X - B\nabla^t_X \bar{J}E_i - \frac{1}{4}(\bar{J}[\bar{J}, \bar{J}](X, E_i))^T \tag{4.17}$$

and

$$\nabla^t_X \bar{J}E_i = w\nabla_X E_i + Ch^s(X, E_i) + \frac{1}{4}(\bar{J}[\bar{J}, \bar{J}](X, E_i))^{\perp}. \tag{4.18}$$

Definition 4.2. A vector field X is said to be a D-Killing vector field, if

$$g(\nabla_Z X, Y) + g(Z, \nabla_Y X) = 0,$$

for any $Y, Z \in \Gamma(D)$.

Now we are ready to give a necessary and sufficient condition for a semi-transversal lightlike submanifold to be normal. Thus we have

Theorem 4.5. *A necessary and sufficient condition for a semi-transversal lightlike submanifold of an indefinite nearly Kaehler manifold and* $[\bar{J}, \bar{J}](X, Y) \in \Gamma(\mu)$ *for* $X, Y \in \Gamma(TM)$ *to be normal is that* $E_i, (i = 1, 2, 3, ..., q)$ *be* D*-Killing vector fields.*

Proof. For $Y, Z \in \Gamma(D)$, using (4.17), we obtain

$$g(\nabla_Z E_i, Y) + g(Z, \nabla_Y E_i) = g(fA_{\bar{J}E_i}Z, Y) + g(Z, fA_{\bar{J}E_i}Y), \tag{4.19}$$

Now using (2.8), we have

$$\begin{aligned} g(Z, fA_{\bar{J}E_i}Y) &= -g(fZ, A_{\bar{J}E_i}Y) = -\bar{g}(h^s(Y, fZ), \bar{J}E_i) \\ &= -\bar{g}(\bar{\nabla}_{fZ}Y, \bar{J}E_i) = \bar{g}(Y, \bar{\nabla}_{fZ}\bar{J}E_i) \\ &= -g(Y, A_{\bar{J}E_i}fZ). \end{aligned} \tag{4.20}$$

Thus from (4.19) and (4.20), we derive

$$g(\nabla_Z E_i, Y) + g(Z, \nabla_Y E_i) = g(fA_{\bar{J}E_i}Z - A_{\bar{J}E_i}fZ, Y). \tag{4.21}$$

Hence, the result follows from Theorem 4.2 and (4.21). □

The Lie derivative of f with respect to $Y \in \Gamma(TM)$ is given by

$$(L_Y f)X = [Y, fX] - f[Y, X], \tag{4.22}$$

for any $X \in \Gamma(TM)$.
The normal semi-transversal lightlike submanifold can be characterized by another tensor field S^* defined by

$$S^*(Y, X) = (L_Y f)X, \tag{4.23}$$

for any $X, Y \in \Gamma(TM)$.

Theorem 4.6. *Let M be a semi-transversal lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$ satisfying*

(i) $P(\nabla_X Y) = 0,$ *for any* $X \in \Gamma(D),\ Y \in \Gamma(D'),$

(ii) $[\bar{J}, \bar{J}](X, Y) \in \Gamma(\mu),$ *for any* $X, Y \in \Gamma(TM).$

Then M is normal if and only if we have

$$S^*(Y, X) = 0, \tag{4.24}$$

for all $X \in \Gamma(D)$ and $Y \in \Gamma(D')$.

Proof. From Theorem 4.2, it follows that M is a normal semi-transversal lightlike submanifold, if and only if, $S(X, Y) = 0$, for all $X \in \Gamma(D)$ and $Y \in \Gamma(D')$. For any $X \in \Gamma(D)$ and $Y \in \Gamma(D')$, using (4.1) and (4.2), we derive

$$S(X, Y) = f([Y, fX] - f[Y, X]) - 2Bd\omega(X, Y). \tag{4.25}$$

Taking into account ∇^t is torsion free and using (3.9), (4.5) becomes

$$\begin{aligned} d\omega(X, Y) &= \frac{1}{2}\{(\nabla^t_X \omega)Y - (\nabla^t_Y \omega)X\} \\ &= \frac{1}{2}h(Y, fX) + \frac{1}{4}(\bar{J}[\bar{J}, \bar{J}](X, Y))^{\perp}, \end{aligned}$$

which further gives

$$2Bd\omega(X, Y) = Bh(Y, fX). \tag{4.26}$$

Now using (4.22), (4.23) and (4.26) in (4.25), we have

$$S(X, Y) = f(S^*(Y, X)) - Bh(fX, Y). \tag{4.27}$$

Now for any $X \in \Gamma(D)$ and $Y \in \Gamma(D')$, using (3.9), we obtain

$$h(Y, fX) = \omega\nabla_Y X + Ch^s(X, Y) - \frac{1}{4}(\bar{J}[\bar{J}, \bar{J}](X, Y))^{\perp}.$$

Applying $\bar{J}$ on both sides, we get

$$Bh(Y, fX) + Ch^s(Y, fX) = -P\nabla_Y X + \bar{J}Ch^s(X, Y) + \frac{1}{4}([\bar{J}, \bar{J}](X, Y))^{\perp},$$

then comparing the tangential components, we obtain

$$Bh(Y, fX) = -P(\nabla_Y X).$$

Thus, (4.27) reduces to

$$S(X, Y) = f(S^*(Y, X)) + P(\nabla_Y X). \tag{4.28}$$

Since M is normal, therefore we must have

$$f(S^*(Y, X)) = 0, \quad P(\nabla_Y X) = 0,$$

which implies that

$$QS^*(X, Y) = 0, \quad P(\nabla_Y X) = 0. \tag{4.29}$$

Again from (4.22) and (4.23), we get

$$P(S^*(Y, X)) = P(\nabla_Y fX - \nabla_{fX} Y), \tag{4.30}$$

which on using hypothesis alongwith second part of (4.29), yields that $P(S^*(Y, X)) = 0$ and hence $S^*(Y, X) = 0$.
Conversely, suppose that M is a semi-transversal lightlike submanifold of indefinite nearly Kaehler manifold $\bar{M}$ satisfying (4.24). Then using hypothesis and (4.24), from (4.30), we get

$$P(\nabla_Y fX) = 0. \tag{4.31}$$

Thus using (4.24) and (4.31) in (4.28), we obtain $S(X, Y) = 0$, which completes the proof. □

Acknowledgment

Sangeet Kumar is grateful to SERB-DST, New Delhi for the financial funding under grant no. ECR/2017/000786.

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