Projectively flat Randers spaces with pseudo-Riemannian metric

Shyamal Kumar Hui, Akshoy Patra and Laurian-Ioan Pişcoran

Abstract. The projective flatness in the pseudo-Riemannian geometry and Finsler geometry is a topic that has attracted over time the interest of several geometers. A Finsler space (M, F) is composed by a differentiable manifold and a fundamental function $F(x, y) = \sqrt{a_{ij}(x)y^i y^j} + b_i y^i$, where $(x, y) \in TM - \{0\}$, where $a_{ij}(x)$ is a Riemannian metric tensor. In this paper we analyze the projectively flatness of the pseudo-Riemannian metric in Randers spaces. We also obtained the conditions of Randers space to be projectively flat.

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1 Introduction

In 1918, Finsler [6] studied a geometry of a space with a generalized metric, which is called a Finsler space. The geometry of a Finsler space is called Finsler geometry. Thereafter, Berwald [3], Synge [17] and Taylor [18] developed the theory of Finsler spaces as a generalization of Riemannian geometry. Finsler geometry is just Riemannian geometry without the quadratic restriction. Finsler geometry has applications in many area of mathematics as well as in biology, physics, geology etc. The Finsler geometry is also studied by many authors in different context.

Finsler metrics on an open subset in $\mathbb{R}$ with straight geodesics are said to be projective Finsler metrics. In 1961, Rapcsak [10] found the necessary and sufficient conditions that a Finsler space is projective to another Finsler space. This result is known as Rapcsak theorem, which plays an important role in the projective geometry of Finsler space. Thereafter projectively flat Finsler space have been studied by many Finslerists throughout the globe.

The Randers metric F is a special Finsler that arise in general relativity. This metric was introduced by Randers [9] and it has the form

$$F = F(x, y) = \alpha(x, y) + \beta(x, y), \tag{1.1}$$

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where $\alpha(x,y) = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric on the n-dimensional smooth manifold M and $\beta(x,y) = b_i dx^i$ is a 1-form on $(M, F = \alpha+\beta)$. The space $(M, F = \alpha+\beta)$ is called Randers space of dimensional n and (M, α) is called the associated Riemannian space.

The present paper deals with the study on projectively flatness of the pseudo-Riemannian metric in Randers spaces. After introduction, section 2 is concerned with some preliminaries, which will be required in the sequel. Section 3 consists the main results. The necessary and sufficient conditions of the 1-form β to be parallel with respect to the pseudo-Riemannian metric 'g' as well as with respect to α in Randers space $(M, F = \alpha + \beta)$ is obtained by replacing the components of α in a Finsler metric with components of the Hessian metric h. The Hessian structures plays an important role in differential geometry and its applications in economics, statistics etc. The Hessian type metrics have been intensively studied by several geometers such as [5], [12]-[16] and [18]-[23]. We also obtained the necessary and sufficient conditions of Randers space $(M, F = \alpha + \beta)$ to be projectively flat by replacing the components of α with the components of the Hessian metric. Finally, we have constructed two examples to illustrate the results.

2 Preliminaries

This section deals with some preliminaries, which will be required in the sequel.

Let (M, g) be an n-dimensional Riemannian manifold and $(U; x^1, x^2, \cdots, x^n)$ be a coordinate chart on M. The Christoffel symbols of the Levi-Civita connection is denoted by Γ^k_{ij}, is defined by [1]

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right).$$

Using the Christoffel symbols, the components of the Riemann curvature tensor R can be expressed in the following form [11]:

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ki}}{\partial x^j} - \frac{\partial \Gamma^l_{ji}}{\partial x^k} + \Gamma^r_{ki}\Gamma^l_{jr} - \Gamma^r_{ji}\Gamma^l_{kr},$$

while the Ricci tensor (Ric) is defined by $R_{ij} = R^l_{ilk}$.
For $f \in C^\infty(M)$, we consider [11]:

$$f_{,i} = \frac{\partial f}{\partial x^i}, \quad f_{,ij} = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^m_{ij} f_{,m}, \quad f_{,ijk} = \frac{\partial f_{,ij}}{\partial x^k} - \Gamma^l_{ki} f_{,lj} - \Gamma^l_{kj} f_{,li}. \tag{2.1}$$

Also we recall the following:

Definition 2.1. [1] The second covariant derivative of $f \in C^\infty(M)$,

$$\nabla^2_g f = \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij} f_{,k} \right) dx^i \otimes dx^j \tag{2.2}$$

is called the Hessian of f.

Suppose that the Hessian $h = \nabla_g^2 f$ with components $h_{pq} = f_{,pq}$ is non-degenerate and of constant signature. Then h is a pseudo-Riemannian metric which produces the Levi-Civita connection ∇_h and the Christoffel symbols $\overline{\Gamma}_{ij}^k$ [2]. It is known that ([1], [2]) the components of the Levi-Civita connection ∇_h, are given by the following formula:

$$\overline{\Gamma}_{ij}^p = \Gamma_{ij}^p + \frac{1}{2} f^{,kp} \left[f_{,ijk} + (R_{ikj}^m + R_{jki}^m) f_{,m} \right], \tag{2.3}$$

where $f_{,}^{pk}$ are the contravariant components of the pseudo-Riemannian metric $h_{,pk} = f_{,pk}$. In Randers space $(M, F = \alpha + \beta)$ [11]
(i) the 1-form β is said to be parallel with respect to α if

$$b_{i|j} = \frac{\partial b_i}{\partial x^j} - b_k \Gamma_{ij}^k = 0. \tag{2.4}$$

(ii) the 1-form β is said to be closed if

$$b_{i|j} - b_{j|i} = 0,$$

which means:

$$\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i} = 0.$$

Remark 2.2. [11] Let $\overline{G}^i$ and G^i be the geodesic coefficients of $F = \alpha + \beta$ and α respectively. If β is parallel with respect to α, i.e. $b_{i|j} = 0$, then

$$\overline{G}^i = G^i = \frac{1}{2} \Gamma_{jk}^i y^j y^k. \tag{2.5}$$

Definition 2.3. [11] A Finsler space T^n is projective to another Finsler space $\overline{T}^n$ if and only if there exists a positively homogeneous scalar field $P(x, y)$ of degree 1 in x, y such that

$$\overline{G}^i(x, y) = G^i(x, y) + P(x, y) y^i.$$

The scalar field $P = P(x, y)$ is called the projective factor of the projective change.

Definition 2.4. [10] If there exists a projective change $F \to \overline{F}$ of a Finsler space $T^n = (M, F)$ such that the Finsler space $\overline{T}^n = (M, \overline{F})$ is a locally Minkowski space, then T^n is called locally projectively flat.

In 1961, Rapcsak [10] proved the following relation between $\overline{G}^i$ and G^i as

$$\overline{G}^i = G^i + \frac{\overline{F}_{|k} y^k}{2\overline{F}} y^i + \frac{\overline{F}}{2} \overline{g}^{il} \left\{ \frac{\partial \overline{F}_{|k}}{\partial y^l} y^k - \overline{F}_{|l} \right\}, \tag{2.6}$$

where $\overline{F}_{|k} = \frac{\partial \overline{F}}{\partial x^k} - \frac{\partial G^r}{\partial y^k} \frac{\partial \overline{F}}{\partial y^r}$, denotes the covariant derivative of $\overline{F}$ on $T^n = (M, F)$. Also we recall the following:

Definition 2.5. [1] The function ϕ is called Hessian-harmonic if it satisfies $\Delta_h \phi = 0$.

In local coordinates, the above relation can be written as

$$f_{,}^{ij}\left(\frac{\partial^2\phi}{\partial x^i\partial x^j}-\overline{\Gamma}_{ij}^{k}\frac{\partial\phi}{\partial x^k}\right)=0. \tag{2.7}$$

Some special projectively flat metrics have been studied as the class of (α,β)-metrics by Cheng and Shen [4], Matsumoto [7], Park and Lee [8], Shen and Zhao [12] and many others.

3 Main Results

In this section, we consider a pseudo-Riemannian manifold (M,g) with Hessian $h=\nabla_g^2 f$, where $f:M\to\mathbb{R}$. The pseudo-Riemannian g has the components of its curvature tensor field R_{ijk}^m. We replace now the components of α in a Finsler metric with the components of the Hessian metric h. Then we get the following theorem:

Theorem 3.1. *Suppose the* 1*-form* β *is parallel with respect to* g*. Then* β *is parallel with respect to* h *if and only if*

$$f_{,ijk}+(R_{ijk}^m+R_{jki}^m)f_{,m}=0.$$

Proof. Using (2.4), we know that 1-form β is parallel with respect to α if

$$b_{i|j}=\frac{\partial b_i}{\partial x^j}-b_k\overline{\Gamma}_{ij}^{k}=0.$$

Using (2.3), after we replace $\overline{\Gamma}_{ij}^{k}$ in the above relation, we obtains:

$$b_{i|j}=\frac{\partial b_i}{\partial x^j}-b_k\left\{\Gamma_{ij}^p+\frac{1}{2}f^{,kp}\left[f_{,ijk}+(R_{ijk}^m+R_{jki}^m)f_{,m}\right]\right\}. \tag{3.1}$$

But we know that $\frac{\partial b_i}{\partial x^j}-b_k\Gamma_{ij}^p=0$.

So, (3.1), reduces to

$$\frac{1}{2}f^{,kp}\left[f_{,ijk}+(R_{ijk}^m+R_{jki}^m)f_{,m}\right]=0$$

and from the above equation we get the desired result. □

Remark 3.1. For the conformal non-homothetic deformation of the pseudo-Riemannian Hessian metric, Bercu et al. [1] have deduced the necessary conditions for the function f belongs to the set of functions which satisfy the condition: $\nabla_g^2 f=e^{2u}g$. They obtained the following condition:

$$\overline{\Gamma}_{ij}^{p}=\Gamma_{ij}^p+\delta_i^p u_{,j}+\delta_j^p u_{,i}-g_{ij}g^{pk}u_{,k}. \tag{3.2}$$

From (3.2), we can state the following:

Corollary 3.2. *Let* α *given by the above non-homothetic deformation of a pseudo-Riemannian metric. The* 1*-form* β *is parallel with* α*, if the following equality holds:*

$$\delta_i^p u_{,j}+\delta_j^p u_{,i}-g_{ij}g^{pk}u_{,k}=0.$$

Remark 3.2. For the following theorems and results, we use the fact that the Hessian h is non-degenerate, i.e. $f_{,}^{ij} \neq 0$.

Theorem 3.3. *Let the Hessian $h = \nabla_g^2 f$ of a smooth function f on an n-dimensional pseudo-Riemannian manifold (M, g) admits a Hessian harmonic function ϕ. If the components of α in a Finsler metric is replaced with the components of the Hessian metric h then the 1-form β is parallel with α if and only if $\frac{\partial^2\phi}{\partial x^i \partial x^j} = \Gamma_{ij}^k \frac{\partial \phi}{\partial x^k}$.*

Proof. The relation (2.7) can be rewritten as

$$f_{,}^{ij}\left\{\frac{\partial^2\phi}{\partial x^i\partial x^j} - \left[\Gamma_{ij}^k + \frac{1}{2}f_{,}^{kp}\left(f_{,ijp} + (R_{ipj}^m + R_{jpi}^m)f_{,m}\right)\right]\frac{\partial\phi}{\partial x^k}\right\} = 0. \tag{3.3}$$

By virtue of Theorem 3.1, we have the relation $f_{,}^{kp}\left[f_{,ijp} + (R_{ipj}^m + R_{jpi}^m)f_{,m}\right] = 0$ and hence (3.3) yields

$$f_{,}^{ij}\left(\frac{\partial^2\phi}{\partial x^i\partial x^j} - \Gamma_{ij}^k\frac{\partial\phi}{\partial x^k}\right) = 0,$$

which concludes the necessary condition. The converse is obvious. Hence the theorem is proved. □

We now assume that the Finsler space $\overline{T}^n = (M, \overline{F})$ is locally Minkowski space. So there exist a projective change $F \to \overline{F}$ of a Finsler space $\overline{T}^n = (M, F)$ such that T^n is locally projectively flat.

Replacing the geodesic coefficients G^i of the metric g with the geodesic coefficients of $h = \nabla_g^2 f$, we obtain

$$\overline{G}^i = G^i + \frac{1}{2}f_{,}^{kp}\left[f_{,ijk} + (R_{ikj}^m + R_{jki}^m)f_{,m}\right] + \frac{\overline{F}_{|k}y^k}{2F}y^i + \frac{\overline{F}}{2}\overline{g}^{il}\left\{\frac{\partial\overline{F}_{|k}}{\partial y^l}y^k - \overline{F}_{|l}\right\}. \tag{3.4}$$

We now prove the following:

Theorem 3.4. *Let $T^n = (M, F)$ be a Randers space with $F = \alpha + \beta$. If we replace the components of α in this Randers metric with the components of the Hessian metric $h = \nabla_g^2 f$, then the Randers space (M, F) is locally projectively flat if and only if*
(i) $f_{,ijk} + (R_{ikj}^m + R_{jki}^m)f_{,m} = 0$;
(ii) *β is parallel with respect to α.*

Proof. It is known that for a locally projectively flat Randers space, the relation (2.6) holds. Replacing the geodesic coefficients G^i of the metric g with the geodesic coefficients of $h = \nabla_g^2 f$, we obtain the relation (3.4). From (2.6) and (3.4) and imposing the third condition i.e, β is parallel with respect to α, we get the theorem. The converse is obvious. □

Example 3.3. The geodesic coefficients of F and α, for Matsumoto metric $F = \frac{\alpha^2}{\beta}$, when β is closed are related by the following relation [7]:

$$\overline{G}^i = G^i - \frac{b_{j|k}y^jy^k}{2\beta}y^i - \frac{F}{2}g^{ij}b_{r|k}y^ky^r\frac{\partial}{\partial y^j}\left(\frac{\alpha}{\beta}\right)^2. \tag{3.5}$$

Now, replacing the geodesic coefficients G^i of α with those of the metric $h = \nabla_g^2 f$ and using Theorem 3.4, we obtain

$$\overline{G}^i = G^i + \frac{1}{2} f_{,}^{kp} \left[f_{,ijk} + (R^m_{ikj} + R^m_{jki}) f_{,m} \right] - \frac{F}{2} g^{ij} b_{r|k} y^k y^r \frac{\partial}{\partial y^j} \left(\frac{h}{\beta} \right)^2 . \tag{3.6}$$

If (M, F) is locally projectively flat then from (3.6), we get

$$\frac{\partial}{\partial y^j} \left(\frac{h}{\beta} \right)^2 = 0 \;\Rightarrow\; \left(\frac{h}{\beta} \right)^2 = c,$$

where c is a constant. From the above equation, one obtains:

$$h^2 = \beta^2 c \;\Rightarrow\; \frac{\partial^2 f_i}{\partial y^i \partial y^j} = \Gamma^k_{ij} y^i y^k + \beta^2 c$$

and from this equation, we can deduce the function f.

Example 3.4. Now, taking the second Matsumoto metric of (α, β) type, $F = \frac{\alpha^2}{\alpha - \beta}$, the coefficients of $\overline{G}^i$ can be expressed in the following way [7]:

$$\overline{G}^i = G^i + \frac{1}{2} f_{,}^{kp} \left[f_{,ijk} + (R^m_{ikj} + R^m_{jki}) f_{,m} \right] - \frac{F}{2} g^{ij} b_{r|k} y^k y^r \frac{\partial}{\partial y^j} \left(\frac{\alpha}{\alpha - \beta} \right)^2 .$$

Now, if we proceed in the same way as we did in previous example, we obtain $h^2 = (h - \beta)^2 c$, where c is a constant. In the same way, if we replace $h = \frac{\partial^2 f_i}{\partial y^i \partial y^j} - \Gamma^k_{ij} y^i y^k$ we can obtain the function f.

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Authors' addresses:

Shyamal Kumar Hui
Department of Mathematics, The University of Burdwan
Golapbag, Burdwan–713104, West Bengal, India.
E-mail: skhui@math.buruniv.ac.in

Akshoy Patra
Department of Mathematics, Govt. College of Engineering and Textile Technology
Berhampore, Murshidabad–742101, West Bengal, India.
E-mail:akshoyp@gmail.com

Laurian-Ioan Pişcoran
North University Center of Baia Mare, Technical University of Cluj Napoca, Department of Mathematics and Computer Science, Victoriei 76, 430122 Baia Mare, Romania.
E-mail: plaurian@yahoo.com