Parallelizability of the Basic Four-Geometries

Simon Davis

Abstract. The conditions of parallelizability and a group structure related to the conformal embedding of four-dimensional hypersurfaces in ten dimensions are formulated for four-manifolds that describe quantum fluctuations of the metric. The set of basic four-geometries which satisfy the restrictions is given.

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1 Introduction

The class of manifolds that might arise in the path integral for quantum gravity may be refined for its evaluation and consistency with symmetries that arise in the theory. The absence of an algorithm for deciding the word problem for four-dimensional manifolds introduces an ambiguity in its definition. The presentation of homotopy groups generally does not establish the homeomorphism equivalence of different four-manifolds. Classes of four-manifolds with solvable word problems, however, would suffice to transform the path integral to a sum over homotopy classes. It has been proven that the set of parallelizable four-manifolds satisfies this condition [7]. Since orientable three-manifolds are parallelizable, this domain of the quantum path integral extends the class, which is unrestricted with the exception of orientation, in three dimensions. The Euler number must be set equal to zero, which restricts the range of topological characteristics. It follows that a standard summation of simply connected four manifolds over the Euler class and the Hirzebruch signature [21] is reduced to half of the invariants under the condition of parallelilzability.

The measure for the path integral for quantum gravity can be deduced from the prediction for the temperature of the cosmic microwave background radiation [8]. This integral includes asymptotically flat black-hole spaces with positive-definite signature. Conformally flat geometries occur in the vicinity of the horizons of extreme limits of these spaces. Therefore, Euclidean parallelizable and conformally flat manifolds could be included in the domain of the path integral. The word problem has not been solved yet for this class. Nevertheless, the path integral over these spaces may be decomposed

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into an integral over the conformal factor and an integral over parallelizable manifolds. The conformal factor Ω causes divergences since it can generate an arbitrarily large negative gravitational action, which can be removed through a choice of the contour of integration in the complex Ω plane [16]. The space-times with Lorentzian signature are known to be parallelizable. The models of matter then would be a solvable subsector of a gravitational theory. The space-times with Lorentzian signature are known to be parallelizable. The parallelizability of the Euclidean section then must be considered. An example is de Sitter space and the four-sphere. Since de Sitter space is noncompact, the analytic continuation to positive-definite signature yields the four-sphere metric and the manifold would be the sphere with antipodal points removed. This manifold is parallelizable in contrast with the four-sphere. More generally, the Euclidean section would be parallelizable because the a global frame of four smooth, linearly independent vector fields on spaces with Lorentzian signature can be continued to a set of smooth, nonvanishing vector fields and a complete metric [7] on the Riemannian geometry. The complete metric similarly can be defined only on the direct analytic continuation of the space and not necessarily on an entire compact manifold. The path integral over manifolds with a Lorentz signature is defined over a subset of the domain of the Euclidean path integral. It might be expanded to a set of compact manifolds that represent extreme limits of black hole space-times. There are extreme black hole space-times that have charges and masses of a special class of elementary particles [5]. The extension to the class of conformally flat manifolds then would provide a model of matter that is a solvable subsector of a Eucldiean path integral. Then, the properties of Euclidean sections of black-hole space-times must be determined to define the domain of the path integration.

The group symmetry for conformal embedding of hypersurfaces in higher dimensions will contain the G_2 invariance, required for the evolution of three-dimensional spaces in four-manifolds [39]. The inclusion in this groups will be required of the isometry groups of the basic four-geometries and may be achieved by considering the classification of four-dimensional Lie algebras together with the identification of the structure constants.

The condition of parallelizability is satisfied by a certain set of basic four-geometries, and it shall be determined if these manifolds are sufficient to form a basis for settting the boundary condition for the path integral. The minimum dimension for embedding of these parallelizable geometries also will be found. The parallelizable basic four-geometries will provide a method for summing over manifolds within the class characterized by vanishing Euler number. It will be found that those compact models of basic four-geometries that are not parallelizable admit conformally flat metrics.

2 Isometry groups of the Basic Four-Geometries

There are nineteen basic four-geometries [27]:

$$
\begin{array}{l}
S^4 \quad \mathbb{CP}^2 \quad S^2 \times S^2 \\
S^3 \times \mathbb{E}^1 \quad S^2 \times \mathbb{E}^2 \quad S^2 \times \mathbb{H}^2 \\
\mathbb{E}^4 \quad Nil^3 \times \mathbb{E}^1 \quad Nil^4 \quad Sol^4_{m,n} \quad Sol^4_0 \quad Sol^4_1 \\
\mathbb{H}^3 \times \mathbb{E}^1 \quad \mathbb{H}^2 \times \mathbb{E}^2 \quad \tilde{SL} \times \mathbb{E}^1 \quad \mathbb{H}^2 \times \mathbb{H}^2 \quad \mathbb{H}^4 \quad \mathbb{H}^2(\mathbb{C}) \quad \mathbb{F}^4.
\end{array} \tag{2.1}
$$

The basic four-geometries may be classified according to compact models, covering spaces, the existence of solvable Lie groups for the fundamental group and aspherical manifolds.

The generalization to four-dimensional hypersurfaces embedded in a manifold of higher dimensions of the group structure would be determined by the symmetries of the product of the tangent bundle and a normal vector. Since this space is nine-dimensional, it would be the fundamental representation of $SO(9)$. Furthermore, this symmetry can be increased to an F_4 group.

Nilmanifolds and solvmanifolds in four dimensions have been demonstrated to be parallelizable. The product of the subbundle of the tangent bundle and the time coordinate must admit a F_4 structure. Since $SO(4) \subset F_4$, the structure groups of the tangent bundles of the four-dimensional infrasolvmanifolds might be examined.

The manifold $Sol^4_{m,n} = \mathbb{R}^3 \rtimes_{\theta_{m,n}} \mathbb{R}$, where m and n are integers such that the polynomial $f_{m,n} = X^3 - mX^2 + nX - 1$ has distinct roots e^a, e^b and e^c, $a < b < c$, has the metric $ds^2 = e^{-2at}dx^2 + e^{-2bt}dy^2 + e^{-2ct}dz^2 + dt^2$ and $\theta_{m,n} = diag(e^{at}, e^{bt}, e^{ct})$ [23]. An isometry between the metrics exists if $(a, b, c) = \lambda(a', b', c')$.

Theorem 2.1. *The spaces $Sol_{m,n}$ and $Sol_{m',n'}$ are diffeomorphic when the roots of the algebraic equations are related by a rational power $\lambda = \frac{r}{s}$, where $r = 2^t$ and $t \leq log_2 m$.*

Proof. Suppose that $(a, b, c) = \lambda(a', b', c')$. Then

$$\begin{aligned} (2.2) \qquad & e^{-2at}dx^2 + e^{-2bt}dy^2 + e^{-2ct}dz^2 + dt^2 \\ &= e^{-2\lambda a' t}dx^2 + e^{-2\lambda b' t}dy^2 + e^{-2\lambda c' t}dz^2 + dt^2 \\ &= \frac{1}{\lambda^2}\left[e^{-2a't'}(dx')^2 + e^{-2b't'}(dy')^2 + e^{-2c't'}(dz')^2 + dt'^2\right], \end{aligned}$$

where $x' = \lambda x$, $y' = \lambda y$, $z' = \lambda z$ and $t' = \lambda t$. Since m, n, m' and n' are integers, e^a, e^b, e^c, $e^{a'}$, $e^{b'}$ and $e^{c'}$ cannot be linear independent over the algebraic numbers, and an algebraic relation between (a, b, c) and (a', b', c') must exist.

It may be recalled that four exponentials conjecture states that one of the numbers $e^{x_1 y_1}$, $e^{x_1 y_2}$, $e^{x_2 y_1}$ and $e^{x_2 y_2}$ must be transcendental if x_1, x_2 and y_1, y_2 are linearly independent over $\mathbb{Q}$. Setting

$$\begin{aligned} (2.3) \qquad x_1 y_1 &= a \\ x_2 y_1 &= a' \\ x_1 y_2 &= b \\ x_2 y_2 &= b', \end{aligned}$$

it follows that, since e^a, e^b, $e^{a'}$ and $e^{b'}$ are algebraic numbers, $\frac{x_2}{x_1} \in \mathbb{Q}$ or $\frac{y_2}{y_1} \in \mathbb{Q}$ or both ratios are rational. If $\frac{x_1}{x_2} \in \mathbb{Q}$ and $\frac{y_1}{y_2} \notin \mathbb{Q}$, it would follows that e^b is not a rational power of e^a. Consequently, e^b would equal α^β where α is algebraic and β is either an irrational algebraic or transcendental number. This number therefore will be transcendental by a theorem on exponent of the denominator q for the upper bound for the difference between α^β and a fraction with this denominator. The roots of the algebraic equation defining $Sol^4_{m,n}$ cannot be transcendental, and $\frac{y_1}{y_2}$ must be a rational number. Similarly, a contradiction occurs if $\frac{x_1}{x_2} \notin \mathbb{Q}$ and $\frac{y_1}{y_2} \in \mathbb{Q}$. It follows that $\frac{x_1}{x_2}$, $\frac{y_1}{y_2} \in \mathbb{Q}$. Setting $\frac{x_1}{x_2} = \lambda$ and $\frac{y_1}{y_2} = \lambda'$, the factor of λ defines the proportionality constant in $a = \lambda a'$ and $b = \lambda b'$. A proof of the four exponentials conjecture is given in a recent manuscript [6].

One of the numbers $e^{x_1 y_1}$, $e^{x_1 y_2}$, $e^{x_1 y_3}$, $e^{x_2 y_1}$, $e^{x_2 y_2}$ and $e^{x_3 y_3}$ must be transcendental if (x_1, x_2) and (y_1, y_2, y_3) are two sets of numbers linearly independent over $\mathbb{Q}$ by the six exponentials theorem [31][37]. Suppose that

$$\begin{aligned} &x_1 y_1 = a \qquad x_1 y_2 = b \qquad x_1 y_3 = c \\ &x_2 y_1 = a' \qquad x_2 y_2 = b' \qquad x_2 y_3 = c'. \end{aligned} \tag{2.4}$$

Then e^a, e^b, e^c, $e^{a'}$, $e^{b'}$ and $e^{c'}$ will be algebraic numbers if and only if $\frac{x_1}{x_2} \in \mathbb{Q}$ or $\frac{y_1}{y_2}$, $\frac{y_1}{y_3} \in \mathbb{Q}$ or each of these three ratios is a rational number. When $\frac{y_1}{y_2} \notin \mathbb{Q}$, e^b is an irrational power of e^a. Again, a contradiction with the algebraicity of e^a and e^b. Similarly, if $\frac{y_1}{y_3} \notin \mathbb{Q}$, one of the numbers e^a or e^c would have to be transcendental. All of the ratios $\frac{x_1}{x_2}$, $\frac{y_1}{y_2}$ and $\frac{y_1}{y_3}$ must be rational. It follows that $a = \lambda a'$, $b = \lambda b'$ and $c = \lambda c'$, where $\frac{x_1}{x_2} = \lambda$.

The polynomial $X^3 - mX^2 + nX - 1$ can be factored as

$$(X-e^a)(X-e^b)(X-e^c) = X^3-(e^a+e^b+e^c)X^2+(e^{a+b}+e^{a+c}+e^{b+c})X-e^{a+b+c} \tag{2.5}$$

if

$$\begin{aligned} a + b + c &= 0 \\ e^a + e^b + e^c &= m \\ e^{a+b} + e^{a+c} + e^{b+c} &= n \end{aligned} \tag{2.6}$$

Similarly, $X^3 - m'X^2 + n'X - 1 = (X - e^{a'})(X - e^{b'})(X - e^{c'})$ when

$$\begin{aligned} a' + b' + c' &= 0 \\ e^{a'} + e^{b'} + e^{c'} &= m' \\ e^{a'+b'} + e^{a'+c'} + e^{b'+c'} &= n'. \end{aligned} \tag{2.7}$$

The first conditions in Eqs.(2.6) and (2.7) are equivalent since $a+b+c = \lambda(a'+b'+c')$. The final conditions are

$$\begin{aligned} e^{-a} + e^{-b} + e^{-c} &= n \\ e^{-a'} + e^{-b'} + e^{-c'} &= n'. \end{aligned} \tag{2.8}$$

Setting $\eta_1 = e^a$, $\eta_2 = e^b$ and $\eta_3 = e^c$,

$$\eta_1 \eta_2 = \frac{1}{\eta_3} \qquad \eta_1 \eta_3 = \frac{1}{\eta_2} \qquad \eta_2 \eta_3 = \frac{1}{\eta_1} \tag{2.9}$$

If η_1, η_2 and η_3 are integers, $\eta_1\eta_2$, $\eta_1\eta_3$ and $\eta_2\eta_3$ are integers, while $\frac{1}{\eta_1}$, $\frac{1}{\eta_2}$ and $\frac{1}{\eta_3}$ are fractions with magnitude less than one unless $|\eta_1| = |\eta_2| = |\eta_3| = 1$. Selecting the integers m and n to be positive integers, the following values are derived:

$$\begin{aligned}
(2.10)\qquad &\eta_i,\ \eta_j = 1,\ \eta_k = -1 && i,\ j,\ k \textit{ not equal} && m = 1,\ n = -1 \\
&\eta_i,\ \eta_j = -1,\ \eta_k = 1 && i,\ j,\ k \textit{ not equal} && m = -1,\ n = -1 \\
&\eta_1 = \eta_2 = \eta_3 = 1 && && m = 3,\ n = 3 \\
&\eta_1 = \eta_2 = \eta_3 = -1 && && m = -3,\ n = 3.
\end{aligned}$$

The space $Sol^4_{1,-1}$ would not be present given the condition $m \leq n$. Therefore, with the exception of $Sol^4_{-1,-1}$, $Sol^4_{3,3}$ and $Sol^4_{-3,3}$, the values of η_i, $i = 1, 2, 3$ must be chosen to be not integral. Since

$$(2.11)\qquad \begin{aligned}
\eta_1 + \eta_2 + \eta_3 &= m \\
\eta_1^\lambda + \eta_2^\lambda + \eta_3^\lambda &= m' \\
\eta_1\eta_2 + \eta_1\eta_3 + \eta_2\eta_3 &= n \\
(\eta_1\eta_2)^\lambda + (\eta_1\eta_3)^\lambda + (\eta_2\eta_3)^\lambda &= n',
\end{aligned}$$

$$(2.12)\qquad n = \frac{m^2 - (\eta_1^2 + \eta_2^2 + \eta_3^2)}{2}$$

which is integer if $\eta_1^2 + \eta_2^2 + \eta_3^2 \in \frac{(1-(-1)^m)}{2} + 2\mathbb{Z}$. Then m' also would be integer if $\lambda = 2$, and

$$(2.13)\qquad n' = \frac{(\eta_1^2 + \eta_2^2 + \eta_3^2)^2 - (\eta_1^4 + \eta_2^4 + \eta_3^4)}{2} = \frac{(m')^2 - (\eta_1^4 + \eta_2^4 + \eta_3^4)}{2}.$$

The value of n' is integer if $\eta_1^4 + \eta_2^4 + \eta_3^4 \in \frac{1-(-1)^{m'}}{2} + 2\mathbb{Z}$. A sequence of powers $\lambda_t = 2^t$ then yield integers which correspond to the coefficients in the algebraic equations representing infrasolvmanifolds diffeomorphic to $Sol_{m,n}$. Given a value of m, the minimum index m_{min} satisfies $m_{min}^{2^t} \approx m$ or $m_{min} \approx m^{\frac{1}{2^t}}$ such that m_{min} is integer. For a fixed value of n, the minimum index $n_{min} \approx n^{\frac{1}{2^t}}$, when it is integer.

An s^{th} root of η_1, η_2 and η_3 must be equal to a single radical extension of $\mathbb{Q}$ to achieve the cancelation of the noninteger parts. It follows that $\eta_i = \left(\frac{p_i}{q_i}\right)^s \rho$, $i = 1,\ 2,\ 3$ and such that $\eta_i \in \mathbb{Q}(\rho^{\frac{1}{s}})$. Then $\lambda_{s,t}$ may have the form $\frac{2^t}{s}$. Since the equations are also conditions on η_i are satisfied for $s = 1$, the integer s would not affect the minimum values m_{min} and n_{min} such that $Sol_{m_{min},n_{min}} \simeq Sol_{m,n}$. □

The set of basic four-geometries which are compatible with the condition on the tangent bundle compatible with a conformal evolution in a higher embedding space may be established from the restriction of the isometry groups in $SO(9)$.

Theorem 2.2. *The basic Four-Geometries that are consistent with the conformal evolution of a four-manifold embedded in higher dimensions with an isometry group that is a subgroup of $SO(9)$ have the compact models S^4, $\mathbb{CP}^2$, $S^2 \times S^2$, $S^3 \times S^1$ and $S^1 \times S^1 \times S^1 \times S^1$.*

Proof. The isometry group of S^4 is $SO(5)$, which is a proper subgroup of $SO(9)$ and F_4. Since $\mathbb{CP}^2 \simeq S^5/S^1$, and the complex projective space is represented also by $SU(3)/(SU(2)\times U(1))$ and $SU(3)/\mathbb{Z}_3 \subset SO(6) \subset SO(9)$ is a subgroup of F_4. It would be included amongst the four-manifolds that could arise in the evolution in higher dimensions. The isometry group of $S^2\times S^2$, $SO(3)\times SO(3)$, is included in $SO(9)$.

The isometry group of $S^3\times\mathbb{E}^1$, $SO(4)\times\mathbb{R}$ is noncompact, while that of the compact model $S^3\times S^1$, $SO(4)\times SO(2)$, is a subgroup of $SO(9)$. The compact model of $S^2\times\mathbb{E}^2$, $S^2\times S^1\times S^1$, has a symmetry group $SO(3)\times SO(2)\times SO(2)$ and rank 3, and it may be included in $SO(9)$. The space $S^2\times\mathbb{H}^2$ has the isometry group $SO(3)\times SO(2,1)$, which is not a subgroup of $SO(9)$.

The compact model of $\mathbb{E}^4$, $S^1\times S^1\times S^1\times S^1$ has an isometry group $U(1)\times U(1)\times U(1)\times U(1)$ of rank 4, which may be included in F_4. The space $\mathbb{E}^2\times\mathbb{H}^2$ has a noncompact isometry group $\mathbb{R}^2\times SO(2)\times SO(2,1)$ which is not a subgroup of $SO(9)$. The isometries of $\mathbb{H}^3\times\mathbb{E}^1$, $SO(3,1)\times\mathbb{R}$ and the symmetries of the metric on $\tilde{SL}\times\mathbb{E}^1$ do not form subgroups of $SO(9)$.

The isometry groups of $\mathbb{H}^2\times\mathbb{H}^2$ and $\mathbb{H}^4$, $SO(2,1)\times SO(2,1)$ and $SO(4,1)$, are not proper subgroups of $SO(9)$. Similarly, the isometry group of $\mathbb{H}^2(\mathbb{C})$, $SU(2,1)/\mathbb{Z}_3$ [2], is not a proper subgroup of $SO(9)$.

The isometry groups of the Nil^3 and Sol^3 manifolds in three dimensions are not proper subgroups of G_2, because the generators of the Heisenberg group are nilpotent and the commutation relations of the Sol group are not isomorphic to a subalgebra of $\mathcal{L}G_2$. The reduction of the $Nil^4_{m,n}$ and $Sol^4_{m,n}$ groups to three dimensions, including $Sol^4_{m,m}\simeq Sol^3\times\mathbb{E}^1$ [23], proves that the isometry groups of the four-dimensional Nil and Sol geometries are not subgroups of $G_2\subset SO(9)$.

The isometry group of F^4, $\mathbb{R}^2\rtimes PSL(2,\mathbb{R})$ [23], is not a subgroup of $SO(9)$. Therefore, this basic four-geometry cannot be included amongst those manifolds that are included in the conformally evolve in a higher-dimensional embedding space. □

The reduction of the group of hyperbolic motions to the Nil and Sol groups for a given signature is required for the embedding of the geometries in the hyperbolic subspace of $\mathbb{RP}^n$. The representation of this hyperbolic space in $\mathbb{R}^{n+1}$ then provides an embedding of Nil^4 and Sol^4 in $\mathbb{R}^5$.

3 Parallelizability

Both S^4 and $\mathbb{CP}^2$ are not parallelizable. The Euler characteristic of $S^2\times S^2$ is the square of $\chi(S^2)=2$, and $S^2\times S^2$ is not parallelizable. The Euler characteristic of the nontrivial S^2 bundle over S^2, $S^2\tilde{\times}S^2$, also equals 4, and the space is not parallelizable.

The conditions for parallelizability of four-manifolds are $\chi(M)=0$ and $\pi_3(M)=\mathbb{Z}$ [23]. The manifold $S^3\times\mathbb{E}^1$ is parallelizable with $\chi(S^3\times\mathbb{E}^1)=0$ and $\pi_3(S^3\times\mathbb{E}^1)=\mathbb{Z}$. An $S^2\times\mathbb{E}^2$ manifold M fibred over S^1 has $\chi(M)=0$, $\pi_1(M)$ virtually isomorphic with $\mathbb{Z}^2$, $\chi(M)=0$ with infinite $\pi/[\pi,\pi]$. The third homotopy group of the covering space $\pi_3(S^2\times\mathbb{E}^2)\simeq\pi_3(S^2)\simeq\mathbb{Z}$. Consequently, $S^2\times\mathbb{E}^2$ manifolds are parallelizable. An S^2 or $\mathbb{RP}^2$ bundle over a surface of genus $g\geq 2$ has the covering space $S^2\times\mathbb{H}^2$. Given that $\pi_1(\Sigma_g)$ acts trivially on $H_*(S^2)$ and the multiplicative property of the Euler characteristic of fibre bundles [29], $\chi(M)=\chi(S^2)\chi(\Sigma_g)=4(1-g)$ and the compact fibration is not parallelizable for $g \geq 2$.

The manifolds $Nil^3 \times \mathbb{E}^1$, Nil^4, $Sol^4_{m,n}$, Sol^4_0 and Sol^4_1 have finite coverings which are parallelizable, being solvable Lie geometries. It has been proven for $\beta_1(\pi) \geq 2$, when these geometries are mapping tori of Nil^3, and closed infrasolvmanifolds with $\beta_1 = 1$ have been found to have fundamental groups that are torsion free, poly-Z groups of Hirsch length 4 [24]. It is known that affine manifolds with solvable fundamental groups have finite coverings that are parallelizable [17]. The conditions for parallelizability and the existence of spin structures will be distinguished since Riemann surfaces of arbitrary genus have spin structures and do not have a global frame of two smooth nonvanishing vector fields for $g = 0$ or $g \geq 2$. A smooth frame of four nonvanishing vector fields exists on $\mathbb{E}^4$ and $\widetilde{SL} \times \mathbb{E}^1$. Amongst the compact parallelizable four-manifolds are the double principal circle bundles over the torus [13], which are compact models of $\mathbb{E}^4$.

The quotients of the aspherical manifolds have the following characteristics [23]:

(3.1)

$$M \sim \mathbb{H}^2 \times \mathbb{E}^2 \; orbifold \quad \sqrt{\pi} \simeq \mathbb{Z}^2 \;\; [\pi; \sqrt{\pi}] = \infty \; [\pi : C_\pi(\sqrt{\pi})] < \infty \; e^{\mathbb{Q}}(\pi) = 0 \quad \chi(M) = 0$$

$$M \simeq \widetilde{SL} \times \mathbb{E}^1 \; manifold \; \sqrt{\pi} \simeq \mathbb{Z}^2 \;\; [\pi : \sqrt{\pi}] = \infty \; [\pi : C_\pi(\sqrt{\pi})] < \infty \; e^{\mathbb{Q}}(\pi) \neq 0 \quad \chi(M) = 0$$

$$M \simeq \mathbb{H}^3 \times \mathbb{E}^1 \; manifold \quad \chi(M) = 0 \quad \pi \; has \; a \; normal \; subgroup \; \rho \times \mathbb{Z} \; of \; finite \; index,$$

$$\rho \; has \; an \; infinite \; index \; normal \; subgroup \; and \; no \; noncyclic \; abelian \; subgroup$$

$$M \simeq reducible \; \mathbb{H}^2 \times \mathbb{H}^2 \; manifold \qquad \pi_2(M) = 0 \qquad \chi(M) \neq 0$$

$$M \simeq closed \; orientable \; \mathbb{H}^4 \; manifold \qquad \sigma(M) = 0 \qquad \chi \in 2\mathbb{Z}^+$$

$$M \simeq closed \; orientable \; \mathbb{H}^2(\mathbb{C}) \; manifold \qquad \chi(M) = 3\sigma(M) > 0$$

The third homotopy groups of the quotients of $\mathbb{H}^2$ and $\mathbb{H}^3$ by discontinuous groups would not be isomorphic to $\mathbb{Z}$. These manifolds are not parallelizable even though the Euler characteristic vanishes.

The dimensions of the union of the vector space algebra and the derived algebra defined by the commutators form the vector $(2, 3, 4)$ for Engel distributions [12]. Parallelizable four-manifolds admit a countable number of stable Engel distributions and represent tangent planes to a countable number of surfaces [27].

Stable prime decomposition of four-manifolds requires $S^2 \times S^2$ [30], which is not restricted to the class of parallelizable manifolds. A path integral over the class of parallelizable four-manifolds would not consist of basic geometries such that there exists a unique prime decomposition of a manifold with the addition of sums of copies of $S^2 \times S^2$. It has been found also that topological 4-manifolds can be smoothed through the connected sum with copies of $S^2 \times S^2$ and E_8 homology manifolds [15]. Therefore, this smoothing procedure does not exist for the class of parallelizable basic four-geometries. Since the intersection form of a connected sum of four-manifolds is the direct sum of intersection forms, the replacement of a topological sum of E_8 homology manifold M_{E_8} and basic four-geometries by another sum could only preserve the homeomorphism rather than the diffeormorpism type [15]. These manifolds are not diffeomorphic, and the smoothing method is not valid with $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$ which also do not belong to the class of parallelizable basic four-geometries.

The dissolving of many simply connected symplectic spin four-manifolds into connected sums of $S^2 \times S^2$ [32] would not occur within a domain of integration of the path integral over the class of parallelizable Four-Geometries. A set of manifolds of this kind $H(k,n)$ constructed from Horikawa surfaces generates a lattice in the (χ, c_1^2) plane with $\chi(H(k,n)) = 8k + 2n - 1$ and $c_1^2(H(k,n)) = 16k - 8$ for $k \geq 1$, $n \geq 1$ [19]. The Euler characteristic cannot be reduced to zero, and fixing k also determines c_1^2. A more general spectrum is provided by the equalities for χ and c_1^2 allowing arbitrary values of the signature satisfying $\sigma \equiv 0$ $(mod\ 16)$ for smooth manifolds [38] Generalization to nonspin manifolds have been given [28].

4 Embedding into higher dimensions

The conditions for embedding of a spin four-manifold in S^5 or equivalently $\mathbb{R}^5$ have been related to the homotopy group, the second Stiefel-Whitney class and the signature. There exist four-manifolds with homotopy groups given by products of finite cyclic groups of odd prime order that cannot be embedded even homotopically in $\mathbb{R}^5$ and may be embedded smoothly in $\mathbb{R}^6$ [3]. It is evident that products of geometric simple manifolds of lower dimension can be embedded in $\mathbb{R}^5$. Furthermore, the conditions of $w_2 = 0$ and $\sigma = 0$, which suffice for the embedding in six dimensions, may be transferred to four dimensions with a set of restrictions on the fundamental group.

Theorem 4.1 *All of the stable parallelizable basic Four-Geometries with a spin structure can be embedded in* $\mathbb{R}^5$*.*

Proof. First consider the geometrically simple manifolds. The round sphere metric on S^4 is induced from the embedding metric of $\mathbb{R}^5$. The manifolds $\mathbb{CP}^2$ and $S^2 \times S^2$ similarly can be embedded in five dimensions.

Since S^2 and S^3 may be embedded in $\mathbb{R}^3$ and $\mathbb{R}^4$ respectively, $S^3 \times \mathbb{E}^1$ and $S^2 \times \mathbb{E}^2$ may be embedded in five dimensional Euclidean space. It has been proven that $\mathbb{H}^2$ cannot be smoothly immersed isometrically in $\mathbb{R}^3$ [22]. There does exist an isometric immersion [9]. It is known that the Cartesian product of n closed orientable two-dimensional manifolds may be embedded in $\mathbb{R}^{2n+1}$ [1] This proof can be extended to noncompact orientable two-dimensional manifolds because the obstruction of the Stiefel-Whitney classes of the normal bundle vanish [20]. Then $S^2 \times \mathbb{H}^2$, and similarly $\mathbb{H}^2 \times \mathbb{E}^2$ may be embedded in $\mathbb{R}^5$.

The trivial embedding of $\mathbb{E}^4$ and the not globally smooth embedding of $\mathbb{H}^4$ in five dimensions induce metrics on both of these maximally symmetric spaces. It follows from the embedding of $\mathbb{H}^3$ in $\mathbb{R}^4$ that there is a local diffeomorphism and a globally continuous homeomorphism from $\mathbb{H}^3 \times \mathbb{E}^1$ to $\mathbb{R}^5$.

There exists classes of Nil^n and Sol^n geometries that are limits of hyperbolic cone structures [36][25]. More generally, these metrics arise as limits of metrics on hyperbolic geometries that can be embedded in $\mathbb{RP}^n$ [4]. It is known that $\mathbb{RP}^n$ cannot be embedded smoothly into $\mathbb{R}^{n+1}$, and there is no similar embedding of $\mathbb{RP}^{2^k}$ into $\mathbb{R}^{2^{k+1}-1}$ [26], such that eight dimensions is required for $\mathbb{RP}^4$. Nevertheless, the hyperbolic limit yields an embedding in $\mathbb{R}^{n+1}$. Then $Nil^3 \times \mathbb{E}^1$, Nil^4 and $Sol^4_{m,n}$, Sol^4_0 and Sol^4_1 may be embedded in $\mathbb{R}^5$. The group $\widetilde{SL}$ does not arise in a limit of hyperbolic geometry, and yet $\widetilde{SL} \times \mathbb{E}^1$ is a parallelizable geometry. The stable parallelizable manifolds are characterized by $w_2(M) = 0$ and $\sigma(M) = 0$, and since $w_2(E \oplus F) = \sum_i w_i(E) \cup w_{2-i}(F)$, $w_1(\widetilde{SL}) = w_2(\widetilde{SL}) = 0$, $w_2(\widetilde{SL} \times \mathbb{E}^1) = 0$. The signature of the product $\widetilde{SL} \times \mathbb{E}^1$ equals zero [11]. Then, $\widetilde{SL} \times \mathbb{E}^1$ can be embedded in $\mathbb{R}^5$.

Since the $\chi(M) = 3\sigma(M) > 0$ for a closed, orientable $H^2(\mathbb{C})$-manifold [40], it cannot be embedded smoothly with a spin structure in five dimensions. The covering space $H^2(\mathbb{C})$, however, can be described as a unit ball in two complex dimensions [23], which may be embedded in $\mathbb{C}^2$ and therefore $\mathbb{R}^5$ through the isotopy between $\mathbb{C}^2$ and $\mathbb{R}^4$.

Even though $\mathbb{F}^4$ cannot be represented by a closed four-dimensional geometry, one noncompact model is the tangent bundle of the hyperbolic plane, and the Euler characteristic vanishes [23]. The embedding of $\mathbb{H}^2 \times \mathbb{E}^2$ therefore suffices for an embedding of $\mathbb{F}^4$ in $\mathbb{R}^5$. □

Therefore, the basic four-geometries may be combined and embedded in a a five-dimensional space that satisfies the topological rigidity theorem. The bounded homotopy equivalence of five-dimensional universally contractible coarse manifolds with bounded geometry and finite decomposition complexity will be equivalent to a bounded homeomorphism [18]. The geometries generated in four dimensions through quantum fluctuations of the metric can be embedded in a fixed space in higher dimensions.

5 Conclusions

The class of parallelizable four-manifolds satisfies the conditions of vanishing Euler character and $\pi_3(M) = \mathbb{Z}$. Homotopy equivalence of simply connected four-manifolds is determined by the intersection form [33] and a topological classification, with the Kirby-Siebenmann invariant, has been given when these manifolds are closed [14] Since indefinite forms are classified by the rank b_2, signature σ and parity, which would be fixed by homotopy equivalence, and $b_2 = 2 + \chi$ since the first and third homology groups vanish, there can be many homotopy classes for each pair of values of χ and σ [34]. It is known that simply connected smooth four-manifolds are determined up to homeomorphism equivalence by χ, σ and the parity of the intersection form [10]. A path integral over these geometries can be reduced to a summation over these two topological characteristics after restriction of the orientation of the representatives of the second homology class. The vanishing of the Euler number of the manifold, however, would the path integral only to a summation over the Hirzebruch signature. Furthermore, it must satisfy congruence conditions such as $\sigma \equiv 0 \ (mod\ 16)$ for spin manifolds.

The summation over σ can be refined by delineating those basic four-geometries that belong to the set of manifolds with vanishing χ. The list of geometries has been described in §3 after establishing the conditions on the indices that characterize diffeomorphism classes of infrasolvmanifolds. Therefore, the enumeration of the different types of basic geometries provides a further summation over the numbers of basic four-geometries. The summations over each type could be evaluated if these geometries arise as gravitational instantons in the limit of large separation. A simultaneous sum over the numbers of geometrical components also can include connected sums representing the four-manifold. The Euler characteristic of the connected sum $M_1 \# M_2$ would not remain zero if M_1 and M_2 are parallelizable. Parallellizability of the entire four-manifold is preserved only if the topological sums are defined over a larger class of components including $S^2 \times S^2$. Therefore, the extent of the quantum gravitational fluctuations will determine the form of the path integral. An example of a larger class of four-manifolds would include those geometries characterized by conformally parallel structures. Furthermore, the basic four-geometries are parallelizable or conformally flat. While this category would include four-dimensional conformally flat spaces, a solution to the word problem, valid for parallelizable manifolds and necessary for distinguishing homotopy classes, remains to be given. Since the path integral may be evaluated over this class of four-manifolds, it will be necessary to consider the extension to the Euclidean sections of black hole space-times that yield a dominant contribution from critical points of the gravitational action.

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Author's address:

Simon Davis
Research Foundation of Southern California
8861 Villa La Jolla Drive #13595
La Jolla, CA 92039, USA.
E-mail: sbdavis@resfdnsca.org