Prescribing the mixed scalar curvature of a foliation

V. Rovenski

Abstract. One of the simplest curvature invariants of a foliated Riemannian manifold is the mixed scalar curvature $\mathrm{S}_{\mathrm{mix}}$ (i.e., an averaged sectional curvature over all planes that non-trivially intersect tangent and normal distributions). In this paper, we survey results on the nonlinear problem of prescribing $\mathrm{S}_{\mathrm{mix}}$ by a conformal change of the structure in tangent and normal to the leaves directions. Under certain geometrical assumptions and in two special cases: along a compact leaf and for a closed fibered manifold, the problem reduces to solution of a nonlinear leafwise elliptic equation for the conformal factor. Stable stationary solutions of the associated parabolic equation are expressed using spectral parameters of the Schrödinger operator. This is done using majorizing and minorizing nonlinear heat equations with constant coefficients and comparison theorems for solutions of Cauchy's problem for parabolic equations.

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1 Introduction

Foliations, which are defined as partitions of a manifold M into collections of submanifolds of the same dimension, called leaves, appeared in 1940s in the works of G. Reeb and Ch. Ehresmann. Since then, the subject has enjoyed a rapid development, see e.g. [5]. The leaves of a foliation $\mathcal{F}$ are tangent to an integrable distribution $\widetilde{\mathcal{D}} = T\mathcal{F}$ – subbundle of the tangent bundle TM. Foliations relate to such topics as vector fields, submersions, fiber bundles, pseudogroups, Lie groups actions; many models in physics are foliated. Although there are topological obstructions for existence of foliations on closed manifolds, we do not discuss them.

Extrinsic geometry of a foliated Riemannian manifold means the properties, which can be expressed in terms of the second fundamental form of the leaves. Several authors investigated the problem whether on a given Riemannian manifold there exists a totally geodesic foliation (i.e., of the simplest extrinsic geometry), as well as the inverse problem of determining whether one can find a Riemannian metric on a foliated

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manifold with respect to which the foliation becomes totally geodesic. The principal problem of extrinsic geometry of foliations is the following, see [6, 8, 14, 19]:

P1. *Given a foliation $\mathcal{F}$ of a manifold M and a geometric property (P) of a submanifold, find a Riemannian metric g on M such that $\mathcal{F}$ enjoys (P) for g.* To show how to reach the expected property using variations of metrics, e.g. conformal on $\widetilde{\mathcal{D}}$, we examine *extrinsic geometric flows* of metrics on foliations [16, 19, 20], and complete the problem **P1** by the following:

P2. *Given a foliation $\mathcal{F}$ of a manifold M and a geometric property (P) of a submanifold, find an extrinsic geometric flow such that the solution metrics g_t $(t \ge 0)$ converge, as $t \to \infty$, to a metric, for which $\mathcal{F}$ enjoys (P).*

Geometrical problems of prescribing curvature of a Riemannian manifold (M, g) using a conformal change of metric g have been popular for a long time, e.g., the problem of prescribing Gaussian curvature on a closed surface seems to be important, see [11]. The study of constancy of the scalar curvature was began by Yamabe in 1960 and completed by several mathematicians in 1986, see [2]. This geometrical problem is expressed in terms of the existence and multiplicity of solutions of a given elliptic PDE in (M, g). The Yamabe equation has a natural variational characterization, based on the normalized Einstein-Hilbert action (that yields the Einstein field equations through the principle of least action). Thus, the problem is equivalent to finding critical points of certain functional. R. Hamilton evolved the metric by the associated parabolic evolution equation $\partial_t g = s(g)g$ (conformal flow), where $-s(g)$ is the difference of the scalar curvature and its mean value. Fixed points of the normalized Yamabe flow are metrics of constant scalar curvature in the given conformal class.

One of the simplest curvature invariants of a foliated Riemannian manifold is the *mixed scalar curvature* $\mathrm{S}_{\rm mix}$ (i.e., an averaged sectional curvature over all planes that non-trivially intersect $\widetilde{\mathcal{D}}$ and the normal distribution $\mathcal{D}$), having strong relations with the extrinsic geometry, [14, 27]. The mixed sectional curvature is encoded in the Riccati equation (called the Raychaudhuri equation in relativity) and regulates the deviation of leaves along the leaf geodesics [14]; in the language of mechanics it measures the relative acceleration of particles moving forward on neighboring geodesics.

In this paper, we survey results [22]–[25] on the (Yamabe type) problem of prescribing the leaf-wise constant $\mathrm{S}_{\rm mix}$ of foliated Riemannian (and more general Riemann-Cartan) spaces. If either $\mathcal{D}$ or $\widetilde{\mathcal{D}}$ is spanned by a unit vector N, then $\mathrm{S}_{\rm mix}$ is simply the Ricci curvature $\mathrm{Ric}_{N,N}$. The notion of $\mathcal{D}$-*truncated* $(r, 2)$-tensor $\tilde{S}$ for $r = 0, 1$ will be helpful: $\tilde{S}(X, Y) := S(X^\perp, Y^\perp)$, where $^\top$ and $^\perp$ are projections on $\widetilde{\mathcal{D}}$ and $\mathcal{D}$. Thus, the $\mathcal{D}$-truncated metric is $g^\perp(X, Y) := \langle X^\perp, Y^\perp\rangle,\ X, Y \in \mathfrak{X}_M$.

The following question was posed in [10, Problem 16]:

Q1. *Given a foliated Riemannian manifold (M, g), does there exist smooth functions $u > 0$ and a leafwise constant Φ such that a $\mathcal{D}$-conformal metric $g' = g^\top \oplus u^2 g^\perp$ has the mixed scalar curvature equal to Φ ?*

We consider the following two approaches:

- evolving the metric by a $\mathcal{D}$-conformal flow g_t,
- exploring the factor u of a $\mathcal{D}$-conformal metric.

Both approaches reduce the problem to studying nonlinear leadwise elliptic or parabolic PDE's. In the case of a general foliation, the topology of the leaf through a point can change crucially with the point, this gives many difficulties in studying such

PDE's. Thus, we examine the following two formulations of the question:
(i) *to prescribe constant* $\mathrm{S}_{\rm mix}$ *on a given compact leaf* F of $\mathcal{F}$;
(ii) *to prescribe leafwise constant* $\mathrm{S}_{\rm mix}$ *on a closed* M, *fibered instead of being foliated*:

$$\mathcal{F} \text{ is defined by an orientable fiber bundle } \pi: M \to B. \tag{1.1}$$

Observe that if M is closed in (1.1) than all the leaves (fibers) are compact.

The results of [22], in shortened form, are as follows, see details in Section 4.

Theorem A. *Let* $\mathcal{F}$ *be a foliation of a closed Riemannian manifold* (M, g) *satisfying* (1.1) *and any of conditions*:

a) $\mathcal{F}$ *is harmonic and nowhere totally geodesic*,

b) $\mathcal{F}$ *is totally geodesic with integrable normal distribution* $\mathcal{D}$.

Then there exists on M *a* $\mathcal{D}$*-conformal metric* g' *with leafwise constant* $\mathrm{S}'_{\rm mix}$.

Remark that there exist harmonic foliations of Lie groups with nowhere totally geodesic leaves with left-invariant metrics, and the metric can be chosen to be bundle-like, see [26]. Such foliations have leafwise constant mixed scalar curvature.

The Riemann-Cartan geometry uses a metric connection $\bar\nabla$, i.e., $\bar\nabla g = 0$, instead of the Levi-Civita connection ∇. Riemann-Cartan spaces $(M, g, \bar\nabla)$ appear in such topics as homogeneous and almost Hermitian spaces [9], and flows of metrics; in Einstein-Cartan theory of gravity the torsion of $\bar\nabla$ is represented by the spin tensor of matter, see e.g. [1, Chap. 17]. The difference $\mathfrak{T} := \bar\nabla - \nabla$ is the *contorsion tensor*.

One may extend the question **Q1** to Riemann-Cartan manifolds:

Q2. *Given a foliated Riemann-Cartan space, does there exist smooth functions* $u > 0$ *and a leafwise constant* Φ *such that* $(\widetilde{\mathcal{D}}, \mathcal{D})$*-conformal Riemann-Cartan structure,*

$$g' = g^\top \oplus u^2 g^\perp, \quad \mathfrak{T}' = u^2\, \mathfrak{T}^\top \oplus \mathfrak{T}^\perp, \tag{1.2}$$

has the mixed scalar curvature equal to Φ? Here we denote $\mathfrak{T}^\top(X, Y) = (\mathfrak{T}(X, Y))^\top$ and $\mathfrak{T}^\perp(X, Y) = (\mathfrak{T}(X, Y))^\perp$ for all $X, Y \in \mathfrak{X}_M$.

The main result of [25], in shortened form, is as follows, see Sections 3 and 5–7.

Theorem B. *Let* $(M, g, \nabla + \mathfrak{T})$ *be a foliated closed Riemann-Cartan space with the space-like leaves.* (*i*) *If the following conditions hold:*

$$\tilde H = 0, \quad H = 0 \quad \text{and} \quad (\mathrm{Tr}^\top \mathfrak{T})^\top = 0\,, \tag{1.3}$$

then there exist smooth solutions of **Q2**: with $u > 0$ and constant Φ on a leaf F.

(*ii*) *If the following conditions hold:* $\mathrm{S}_{\mathfrak{T}^\top} \le 0$ (*see Definition 2.3 below*), (1.1),

$$\tilde H = 0, \quad H = 0, \quad \mathrm{Tr}^\top \mathfrak{T} = 0 \quad \text{and} \quad (\mathrm{Tr}^\perp \mathfrak{T})^\perp = 0\,, \tag{1.4}$$

then there exist smooth solutions of **Q2**: *with* $u > 0$ *and some leafwise constant* Φ. *Under some conditions, for any such* Φ *the solution* u *is unique in certain domain.*

Under assumptions (1.3), the factor u in (1.2) obeys a leafwise elliptic PDE

$$\mathcal{H}(u) = \Psi_1\, u^{-1} - \Psi_2\, u^{-3} + \Psi_3\, u^3 \tag{1.5}$$

with known functions Ψ_i, see (3.4), and the Schrödinger operator

$$\mathcal{H} = -\Delta^\top - (\beta^\top + \Phi)\,\mathrm{id}^\top\,. \tag{1.6}$$

Observe that for $\bar\nabla = \nabla$ (Riemannian case) we have $\Psi_3 = 0$. The key role in our study play spectral parameters of the Schrödinger operator (1.6). The spectrum $\lambda_0 \le \lambda_1 \le \dots$ of $\mathcal{H}$ on compact leaves is discrete, the least eigenvalue λ_0 of $\mathcal{H}$ is simple, its eigenfunction e_0 (called the *ground state*) can be chosen positive and

$$-\max{}_F(\beta^\top + \Phi) \le \lambda_0 \le -\min{}_F(\beta^\top + \Phi)\,. \tag{1.7}$$

One may add to Φ a real constant to provide $\beta^\top + \Phi < 0$ without change of e_0; then $\mathcal{H}$ becomes invertible in $L^2(F)$ and $\mathcal{H}^{-1}$ is bounded. In case of (1.1), the leafwise constant λ_0 and the function e_0 are smooth on M.

We are looking for such solutions of (1.5) that are stable stationary solutions (attractors) of the associated Cauchy's problem for parabolic PDE:

$$\partial_t u + \mathcal{H}(u) = \Psi_1\, u^{-1} - \Psi_2\, u^{-3} + \Psi_3\, u^3, \qquad u|_{t=0} = u_0 > 0\,. \tag{1.8}$$

The paper is organized as follows. Section 2 surveys properties of $\mathrm{S}_{\mathrm{mix}}$ and illustrates our method in the case of generalized products. In Section 3 we derive the transformation of $\mathrm{S}_{\mathrm{mix}}$ under $\mathcal{D}$-conformal change of Riemann-Cartan structure. This yields, under assumptions (1.3), the elliptic PDE (1.5) on a leaf. Section 4 presents results of [23, 24] on prescribing $\mathrm{S}_{\mathrm{mix}}$ by $\mathcal{D}$-conformal flow of metrics. Section 5 uses $\mathcal{D}$-conformal change of metric and is supported by results on stable stationary solutions to the non-linear problem (1.8) associated with the elliptic PDE on a closed Riemannian manifold, and compactness in $C(F)$ of the set of all such solutions. The key role play spectral parameters of the Schrödinger operator, see Section 6. Among several tools of analysis on closed manifolds, used in prescribing $\mathrm{S}_{\mathrm{mix}}$, are "majorizing" and "minorizing" nonlinear heat equations with constant coefficients (Section 7) and comparison theorems for solutions of parabolic equations.

2 The mixed scalar curvature

Let M^{n+p} be a connected closed (i.e., compact without boundary) manifold, equipped with a pseudo-Riemannian metric $g = \langle\cdot,\cdot\rangle$ and a p-dimensional foliation $\mathcal{F}$. A pseudo-Riemannian metric of index q is an element $g \in \mathrm{Sym}^2(M)$ (of the space of symmetric $(0,2)$-tensors) such that each g_x $(x \in M)$ is a non-degenerate bilinear form of index q on the tangent space T_xM. When $q = 0$, g is a Riemannian metric, and a Lorentz metric when $q = 1$. Assume that g is non-degenerate on complementary orthogonal distributions $\widetilde{\mathcal{D}}$ (tangent to $\mathcal{F}$) and $\mathcal{D}$ (orthogonal to $\mathcal{F}$). Let $\mathfrak{X}_M$, $\mathfrak{X}^\top$ and $\widetilde{\mathfrak{X}}$ be the modules over $C^\infty(M)$ of vector fields on M, $\widetilde{\mathcal{D}}$ and $\mathcal{D}$, respectively. The integrability tensor of $\mathcal{D}$ is given by

$$\tilde T_{X,Y} = (1/2)\,[X,\,Y]^\top, \quad X,Y \in \mathfrak{X}^\perp.$$

The extrinsic geometry is determined by the second fundamental forms

$$h_{X,Y} = (\nabla_X Y)^\perp\ (X,Y \in \mathfrak{X}^\top), \quad \tilde h_{X,Y} = (1/2)(\nabla_X Y + \nabla_Y X)^\top, \quad X,Y \in \mathfrak{X}^\perp.$$

The traces $H = \mathrm{Tr}_g\, h$ and $\tilde H = \mathrm{Tr}_g\, \tilde h$ are the mean curvature vectors of $\widetilde{\mathcal{D}}$ and $\mathcal{D}$. A pseudo-Riemannian manifold may admit many kinds of geometrically interesting foliations, among them are *totally umbilical*, *harmonic*, or *totally geodesic*, if $h = (1/p)Hg^\top$, $H = 0$, or $h = 0$, resp., see e.g. [14]. A foliation is *conformal*, *transversely harmonic*, or *Riemannian*, if $\tilde h = (1/n)\tilde H g^\perp$, $\tilde H = 0$ or $\tilde h = 0$, resp. Parallel circles or winding lines on a flat torus and a Hopf field of great circles on a sphere S^3 are examples of geodesic foliations, but there is no metric making harmonic a Reeb foliation of S^3. Totally geodesic foliations of codimension-one on closed non-negatively curved space forms are completely understood: they are given by parallel hyperplanes in the case of a flat torus T^n and they do not exist for spheres S^n. If the codimension is > 1, examples of geometrically distinct totally geodesic foliations are abundant.

As usual, $R_{X,Y} = \nabla_Y\nabla_X - \nabla_X\nabla_Y + \nabla_{[X,Y]}$ is the curvature tensor. The sectional curvature of a plane $\sigma = X\wedge Y$ is $K_\sigma = \langle R_{X,Y}X, Y\rangle/(\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2)$.

Definition 2.1. The *mixed scalar curvature* of the curvature tensor $\bar R$ of a connection $\bar\nabla$ on a foliated pseudo-Riemannian manifold (M,g) is the function

$$\bar{\mathrm{S}}_{\mathrm{mix}} = \frac{1}{2}\sum\nolimits_{a,i}\epsilon_a\epsilon_i\big(\langle\bar R_{E_a,\mathcal{E}_i}E_a,\mathcal{E}_i\rangle + \langle\bar R_{\mathcal{E}_i,E_a}\mathcal{E}_i,E_a\rangle\big), \tag{2.1}$$

where $\{E_i,\,\mathcal{E}_a\}_{i\le n,\,a\le p}$ is a local orthonormal adapted frame, i.e., $\{E_a\}\subset\widetilde{\mathcal{D}}$ and $\{\mathcal{E}_i\}\subset\mathcal{D}$ and $\epsilon_a = \langle E_a,E_a\rangle$, $\epsilon_i = \langle\mathcal{E}_i,\mathcal{E}_i\rangle$. The definition (2.1) does not depend on the order of distributions and on the choice of a local frame. In particular, the *mixed scalar curvature of the curvature tensor for the Levi-Civita connection* is the following function on M: $\mathrm{S}_{\mathrm{mix}} = \sum_{a,i}\epsilon_a\epsilon_i\,\langle R_{E_a,\mathcal{E}_i}E_a,\mathcal{E}_i\rangle$.

Notice that $\mathrm{S}_{\mathrm{mix}} = \mathrm{Tr}_g\,\mathrm{Ric}^\perp$, where

$$\mathrm{Ric}^\perp_{X,Y} = \sum\nolimits_a\epsilon_a\,\langle R_{X^\perp,\,E_a}Y^\perp,E_a\rangle \tag{2.2}$$

is the partial Ricci tensor, see [13] and Remark 4.3 below. The formula, see [27],

$$\mathrm{S}_{\mathrm{mix}} = \mathrm{div}(\tilde H + H) + \|\tilde H\|^2 + \|H\|^2 + \|\tilde T\|^2 - \|\tilde h\|^2 - \|h\|^2, \tag{2.3}$$

where $\|h\|^2 = \sum_{a,b}\|h_{E_a,E_b}\|^2$, $\|\tilde T\|^2 = \sum_{i,j}\|\tilde T_{\mathcal{E}_i,\mathcal{E}_j}\|^2$, etc. shows that $\mathrm{S}_{\mathrm{mix}}$ belongs to the extrinsic geometry. Applying the Divergence Theorem for a closed manifold, yields the integral formula, which provides decomposition criteria for foliations with an integrable distribution $\mathcal{D}$ under constraints on the sign of $\mathrm{S}_{\mathrm{mix}}$, see [27] and [14].

For example: (a) If $\mathcal{F}$ is a compact harmonic foliation of a Riemannian manifold (M,g) with an integrable normal distribution $\mathcal{D}$ and $\mathrm{S}_{\mathrm{mix}}\ge 0$, then M splits on $\mathcal{F}$. (b) If $\mathcal{F}$ and $\mathcal{F}^\perp$ are complementary orthogonal totally umbilical foliations of a closed oriented (M,g) with $\mathrm{Sc}_{\mathrm{mix}}\le 0$, then M splits along the foliations. In [15], such integral formulas and splitting results are generalized to metric-affine manifolds.

Remark 2.2. The following new action on foliations was introduced in [3] as analog of the Einstein-Hilbert action with the scalar curvature replaced by $\mathrm{S}_{\mathrm{mix}}$:

$$J_{\mathrm{mix}}:\ g\mapsto\int_\Omega\Big\{\frac{1}{2\mathfrak{a}}\,(\mathrm{S}_{\mathrm{mix}}(g) - 2\,\Lambda) + \mathcal{L}(g)\Big\}\,\mathrm{d}\,\mathrm{vol}_g\,. \tag{2.4}$$

Here Λ is the cosmological constant, $\mathcal{L}$ – Lagrangian describing the matter contents, and $\mathfrak{a}$ – the coupling constant. The integral is taken over M if it converges; otherwise, one integrates over arbitrarily large, relatively compact domain Ω containing supports of variations g_t with $g_0 = g$. The physical meaning of (2.4) has been discussed in [3] for the case of a globally hyperbolic spacetime (M^4, g) when $\widetilde{\mathcal{D}} = \mathrm{Span}(N)$ and hence $\mathrm{S}_{\,\mathrm{mix}} = \langle N, N\rangle \,\mathrm{Ric}_{N,N}$. A *spacetime* is described as a time-orientable manifold, equipped with a Lorentzian metric, there also exists a timelike unit vector field N, whose orthogonal distribution is not necessarily integrable. The "mixed gravitational field equations" have been recently derived [17] for a spacetime (e.g. stably causal and globally hyperbolic spacetimes are naturally foliated, [4, 7]), in fact, for a pseudo-Riemannian manifold endowed with a non-degenerate distribution. The Euler-Lagrange equations for (2.4) have the form of Einstein field equation,

$$\mathrm{Ric}_{\,\widetilde{\mathcal{D}}} - (1/2)\,\mathrm{S}_{\,\widetilde{\mathcal{D}}}\cdot g + \Lambda\, g = \mathfrak{a}\,\Theta,$$

where Θ is the stress-energy tensor, while Ric and scalar curvature are replaced by a new Ricci type tensor $\mathrm{Ric}_{\,\widetilde{\mathcal{D}}}$ and its trace $\mathrm{S}_{\widetilde{\mathcal{D}}} = \mathrm{S}_{\,\mathrm{mix}} + \frac{p-n}{n+p-2}\,\mathrm{div}(\tilde H - H)$. For $\widetilde{\mathcal{D}}$ spanned by a unit vector field N, this $\mathrm{Ric}_{\widetilde{\mathcal{D}}}$ is given by, see [17, 21],

$$\begin{aligned}
\mathrm{Ric}_{\,\widetilde{\mathcal{D}}\,|\,\mathcal{D}\times\mathcal{D}} &= \nabla_N\,\tilde h_{sc} - \tau_1 \tilde h_{sc} - \epsilon_N (2\,(\tilde T^\sharp_N)^2 + [\tilde T^\sharp_N, \tilde A_N])^\flat,\\
\mathrm{Ric}_{\,\widetilde{\mathcal{D}}}(\,\cdot\,, N)_{\,|\,\mathcal{D}} &= \mathrm{div}^\perp(\tilde T^\sharp_N) + 2\,(\tilde T^\sharp_N H^\perp)^\flat,\\
\mathrm{Ric}_{\,\widetilde{\mathcal{D}}}(N, N) &= \epsilon_N(N(\tau_1) - \tau_2) - \|\tilde T\|^2,
\end{aligned}$$

and its trace is $\mathrm{S}_{\,\widetilde{\mathcal{D}}} = \epsilon_N\,\mathrm{Ric}_{N,N} + \mathrm{div}(\epsilon_N \tau_1 N - H^\perp)$. Here, $(1,1)$-tensors $\tilde A_N$ (the shape operator of $\mathcal{F}$) and $\tilde T^\sharp_N$ are adjoint to $\tilde h = \tilde h_{sc}\,N$ and $\tilde T$, and $\tau_i = \mathrm{Tr}\,\tilde A^i_N$. There also exists an equivalent form of $\mathrm{Ric}_{\,\widetilde{\mathcal{D}}}$ involving the Jacobi operator R_N.

The $\mathcal{K}$-sectional curvature of a $(1,2)$-tensor $\mathcal{K}$ was defined in [12] for statistical manifolds. The algebraic analogue of $\mathrm{S}_{\mathrm{mix}}$ for $\mathcal{K}$ on a foliation is the following.

Definition 2.3 (see [25])**.** The *mixed scalar curvature of a* $(1,2)$-*tensor* $\mathcal{K}$ on a foliated manifold is an averaged $\mathcal{K}$-sectional curvature over all mixed planes:

$$\mathrm{S}_{\,\mathcal{K}} := \frac{1}{2}\sum\nolimits_{a,i} \epsilon_a\epsilon_i\big(\langle[\mathcal{K}_i,\, \mathcal{K}_a]\,E_a, \mathcal{E}_i\rangle + \langle[\mathcal{K}_a,\, \mathcal{K}_i]\,\mathcal{E}_i, E_a\rangle\big). \tag{2.5}$$

If $\mathcal{K}_X$ is (anti-) symmetric then (2.5) reads $\mathrm{S}_{\,\mathcal{K}} = \sum_{a,i}\epsilon_a\epsilon_i\,\langle[\mathcal{K}_i,\, \mathcal{K}_a]\,E_a, \mathcal{E}_i\rangle$.

For *Riemann-Cartan spaces*, $\mathfrak{T}_X$ $(X \in TM)$ is anti-symmetric:

$$(\bar\nabla_X\, g)(Y, Z) = \langle \mathfrak{T}_X Y, Z\rangle + \langle \mathfrak{T}_X Z, Y\rangle = 0 \quad (X, Y, Z \in TM). \tag{2.6}$$

By (2.5), the *mixed scalar curvature of contorsion tensor* $\mathfrak{T}$ in the Riemann-Cartan case is $\mathrm{S}_{\,\mathfrak{T}} := \sum_{a,i}\epsilon_a\epsilon_i\langle[\mathfrak{T}_a, \mathfrak{T}_i]\,\mathcal{E}_i, E_a\rangle$. We will use notations $\mathrm{Tr}^\perp\,\mathfrak{S} := \sum_i \epsilon_i \mathfrak{S}_i\,\mathcal{E}_i$ and $\mathrm{Tr}^\top\,\mathfrak{S} := \sum_a \epsilon_a \mathfrak{S}_a E_a$ for traces of a $(1,2)$-tensor $\mathfrak{S}$. Since $\mathfrak{T}^\top$ satisfies the equality $\langle[\mathfrak{T}^\top_i, \mathfrak{T}^\top_a]\,E_a, \mathcal{E}_i\rangle = 0$, by (2.5) with $\mathcal{K} = \mathfrak{T}^\top$ and any $\bar\nabla$ we get

$$\mathrm{S}_{\,\mathfrak{T}^\top} := \sum\nolimits_{a,i}\epsilon_a\epsilon_i\,\langle[\mathfrak{T}^\top_a, \mathfrak{T}^\top_i]\,\mathcal{E}_i, E_a\rangle. \tag{2.7}$$

Comparing the curvature tensor $\bar R_{X,Y} = [\bar\nabla_Y, \bar\nabla_X] + \bar\nabla_{[X,Y]}$ of $\bar\nabla = \nabla + \mathfrak{T}$, with similar formula for $R_{X,Y}$, we find the equality

$$\bar R_{X,Y} = R_{X,Y} + (\nabla_Y \mathfrak{T})_X - (\nabla_X \mathfrak{T})_Y + [\mathfrak{T}_Y, \mathfrak{T}_X]. \tag{2.8}$$

The tensor $\bar R$ has symmetries, e.g. $\langle \bar R_{X,Y}Z,U\rangle = -\langle \bar R_{X,Y}U,Z\rangle$ and $\langle \bar R_{X,Y}Z,U\rangle = -\langle \bar R_{Y,X}Z,U\rangle$. Using (2.8) and Definition 2.1, we obtain

$$\bar{\mathrm S}_{\rm mix} = \mathrm S_{\rm mix} + \mathrm S_{\mathfrak T} + Q, \tag{2.9}$$

where

$$Q = \frac12 \sum\nolimits_{a,i} \epsilon_a\epsilon_i[\langle(\nabla_i\mathfrak T)_a E_a,\mathcal E_i\rangle - \langle(\nabla_a\mathfrak T)_i E_a,\mathcal E_i\rangle + \langle(\nabla_a\mathfrak T)_i\mathcal E_i, E_a\rangle - \langle(\nabla_i\mathfrak T)_a\mathcal E_i, E_a\rangle].$$

In [15] we proved integral formulas with $\bar{\mathrm S}_{\rm mix}$ and splitting results for foliated Riemann-Cartan spaces. The leaves of a foliated $(M,g,\bar\nabla)$ are submanifolds with induced metric $g^\top$ and metric connection $\bar\nabla^\top_X Y := (\bar\nabla_X Y)^\top$ for $X,Y\in\mathfrak X^\perp$. Since, see (2.6),

$$g^\top(\mathfrak T^\top_X Y,Z) + g^\top(\mathfrak T^\top_X Z,Y) = \langle\mathfrak T_X Y,Z\rangle + \langle\mathfrak T_X Z,Y\rangle = 0 \quad (X,Y,Z\in\mathfrak X^\top),$$

the leaves (with metric $g^\top$ and connection $\bar\nabla^\top$) are themselves Riemann-Cartan spaces. We will demonstrate our approach to question **Q2** (see Section 5) in the case of generalized products, which appear in examples of geometry and relativity.

Example 2.4 (Generalized products). The *doubly-twisted product* of Riemann-Cartan manifolds $(B,g_B,\mathfrak T_B)$ and $(F,g_F,\mathfrak T_F)$ with positive warping functions $u,v\in C^\infty(B\times F)$ is a manifold $M=B\times F$ with the metric $g = v^2 g^\top\oplus u^2 g^\perp$ and the contorsion tensor $\mathfrak T = u^2\mathfrak T^\top\oplus v^2\mathfrak T^\perp$, where

$$\begin{aligned} g^\top(X,Y) &= g_B(X^\top,Y^\top), \quad g^\perp(X,Y) = g_F(X^\perp,Y^\perp),\\ \mathfrak T^\top_X Y &= (\mathfrak T_B)_{X^\top}Y^\top, \quad {\mathfrak T^\perp}_X Y = (\mathfrak T_F)_{X^\perp}Y^\perp. \end{aligned}$$

For $v=1$ we have the *Riemann-Cartan twisted product* (a *Riemann-Cartan warped product* when also $u\in C^\infty(B)$). One may show that $\bar\nabla g=0$, see (2.6), for the linear connection $\bar\nabla = \nabla+\mathfrak T$. Hence, $(M,g,\bar\nabla)$ is a Riemann-Cartan space, denoted by $B\times_{(v,u)}F$. The *leaves* $B\times\{y\}$ and the *fibers* $\{x\}\times F$ of a Riemann-Cartan doubly-twisted product $B\times_{(v,u)}F$ are totally umbilical with respect to $\bar\nabla$ and ∇. Indeed, $\tilde h = -(\nabla\log u)^\top g^\perp$ and $h = -(\nabla\log v)^\perp g^\top$; hence, $\tilde H = -n\,\nabla^\top(\log u)$ and $H = -p\,\nabla^\perp(\log v)$, where we denote $\nabla^\top(\log u) = (\nabla\log u)^\top$, etc. Since

$$\begin{aligned} \operatorname{div}\tilde H &= -n\,(\Delta^\top u)/u - (n^2-n)\,|\nabla^\top u|^2/u^2,\\ |\tilde H|^2 - \|\tilde h\|^2 &= (n^2-n)\,|\nabla^\top u|^2/u^2,\\ \operatorname{div} H &= -p\,(\Delta^\perp v)/v - (p^2-p)\,|\nabla^\perp v|^2/v^2,\\ |H|^2 - \|h\|^2 &= (p^2-p)\,|\nabla^\perp v|^2/v^2, \end{aligned}$$

then (2.3) reduces to $\mathrm S_{\rm mix} = -n\,(\Delta^\top u)/u - p\,(\Delta^\perp v)/v$. Also $Q = n\,u\,\langle\operatorname{Tr}\mathfrak T^\top,\nabla u\rangle + p\,v\,\langle\operatorname{Tr}(\mathfrak T^\perp),\nabla v\rangle$ and $\mathrm S_{\mathfrak T}=0$. Put $\beta^\top = \frac{p}{n}\,(v^{-1}\Delta^\perp v - v\langle\operatorname{Tr}(\mathfrak T^\perp),\nabla v\rangle$. By the above and (2.9) we get the linear elliptic PDE along a leaf for function u,

$$-\Delta^\top u - (\beta^\top + \bar{\mathrm S}_{\rm mix}/n)\,u + u^2\langle\operatorname{Tr}\mathfrak T^\top,\nabla u\rangle = 0. \tag{2.10}$$

Let B be a closed manifold with $g_B>0$ and $\operatorname{Tr}\mathfrak T_B = 0$. So, $\operatorname{Tr}\mathfrak T^\top = 0$, and (2.10) becomes the eigenvalue problem for operator $\mathcal H$ in (1.6) with $\beta^\top = (p/n)\,v^{-1}\Delta^\perp v$. Thus, $B\times_{(v,e_0)}F$ has leafwise constant $\bar{\mathrm S}_{\rm mix} = n\lambda_0$, where λ_0 is the least eigenvalue of $\mathcal H$ and $e_0>0$ is its simple eigenvector.

3 $\mathcal{D}$-conformal change of a metric

Here, we show how the extrinsic geometry is transformed under $\mathcal{D}$-*conformal* change of a metric. We find the transformation of $\mathrm{S}_{\rm mix}$ under $\mathcal{D}$-conformal change of Riemann-Cartan structure. This yields, under assumptions (1.3), the elliptic PDE (1.5) on any leaf. Recall [22] that the shape operator $\tilde A_U$ and the skew-symmetric operator $\tilde T^\sharp_U$ of $\mathcal{D}$ are given by $\langle\tilde A_U(X),Y\rangle=\langle\tilde h(X,Y),U\rangle$ and $\langle\tilde T^\sharp_U(X),Y\rangle=\langle\tilde T(X,Y),U\rangle$.

Lemma 3.1 (see [20, 22]). *Given a foliation $\mathcal{F}$ on (M,g) and $\phi\in C^1(M)$, put $g'=g^\top+e^{2\phi}g^\perp$. Then*

$$h'=e^{-2\phi}h,\quad H'=e^{-2\phi}H,\quad \tilde h'=e^{2\phi}\big(\tilde h-(\nabla^\top\phi)\,g^\perp\big),\quad \tilde H'=\tilde H-n\nabla^\top\phi,$$
$$\tilde A'_U=\tilde A_U-U(\phi)\,\mathrm{id}^\perp,\quad \tilde T'^\sharp_U=e^{-2\phi}\,\tilde T^\sharp_U\quad (U\in\mathfrak{X}^\top).$$

Hence, $\mathcal{D}$-conformal variations preserve total umbilicity, harmonicity, and total geodesy of $\mathcal{F}$, and preserve total umbilicity of the normal distribution $\mathcal{D}$.

Lemma 3.2 ([22]). *Let $\mathcal{F}$ be a foliation of a pseudo-Riemannian manifold (M,g). Then, after transformation $g'=g^\top\oplus u^2\,g^\perp$ the mixed scalar curvature on any harmonic leaf F becomes*

$$\mathrm{S}'_{\rm mix}=\mathrm{S}_{\rm mix}-nu^{-1}\Delta^\top u+2u^{-1}\langle\tilde H,\nabla u\rangle+(u^{-4}-1)\|\tilde T\|^2_g-(u^{-2}-1)\|h\|^2_g. \tag{3.1}$$

The proof of Lemma 3.2 is based on Lemma 3.1 and the following calculations:

$$\|h'\|^2_{g'}=e^{-2\phi}\|h\|^2_g,\quad \|\tilde T'\|^2_{g'}=e^{-4\phi}\|\tilde T\|^2_g, \|\tilde h'\|^2_{g'}=\|\tilde h\|^2_g+n\|\nabla^\top\phi\|^2_g-2\,\tilde H(\phi),$$
$$\|\tilde H'\|^2_{g'}=\|\tilde H\|^2_g+n^2\|\nabla^\top\phi\|^2_g-2n\tilde H(\phi),\quad \mathrm{div}'^\top\,\tilde H'=\mathrm{div}^\top\,\tilde H-n\Delta^\top\phi\,.$$

Now, let $\mathcal{F}$ be a foliation of a Riemann-Cartan space $(M,g,\bar\nabla=\nabla+\mathfrak{T})$ with the space-like leaves. $(\widetilde{\mathcal{D}},\mathcal{D})$-conformal structures (1.2) preserve the splitting $TM=\widetilde{\mathcal{D}}\oplus\mathcal{D}$. By (2.6), $g'(\mathfrak{T}'_XY,\,Z)+g'(\mathfrak{T}'_XZ,\,Y)=u^2\,[\langle\mathfrak{T}_XY,\,Z\rangle+\langle\mathfrak{T}_XZ,\,Y\rangle]=0$. Hence, g' is parallel w.r.t. $\nabla'+\mathfrak{T}'$, where ∇' is the Levi-Civita connection of g'. Put

$$b_{\mathfrak{T}}=-\sum\nolimits_{i,a}\epsilon_i\,\epsilon_a\,\langle\tilde T(\mathfrak{T}_iE_a+\mathfrak{T}_a\mathcal{E}_i,\,\mathcal{E}_i),\,E_a\rangle\,.$$

If either $\mathcal{D}$ is integrable or $\bar\nabla$ and ∇ are projectively equivalent (i.e., the systems of geodesics for both connections coincide) then $b_{\mathfrak{T}}=0$. By Lemmas 3.1 and 3.2, we get

Proposition 3.3 ([25]). *After transformation* (1.2), *the mixed scalar curvature of the Riemann-Cartan manifold $(M,g',\bar\nabla'=\nabla'+\mathfrak{T}')$ on any ∇-harmonic leaf F becomes*

$$\begin{aligned}\bar{\mathrm{S}}'_{\rm mix}&=\bar{\mathrm{S}}_{\rm mix}+n\langle(\mathrm{Tr}^\top\mathfrak{T})^\perp,\nabla u\rangle\,u^{-1}+\langle(\mathrm{Tr}^\perp\mathfrak{T})^\perp,\nabla u\rangle\,u^{-1}\\&+n\,u\langle(\mathrm{Tr}^\top\mathfrak{T})^\top,\nabla u\rangle-(u^2-1)\langle\mathrm{Tr}^\top\mathfrak{T},\tilde H\rangle-nu^{-1}\Delta^\top u+2u^{-1}\langle\tilde H,\nabla u\rangle\\&+(u^{-4}-1)\|\tilde T\|^2_g-(u^{-2}-1)(\|h\|^2_g-b_{\mathfrak{T}})-(u^2-1)\,\mathrm{S}_{\mathfrak{T}^\top}\,.\end{aligned}\tag{3.2}$$

Observe that (3.2) is the second order PDE for the function $u>0$,

$$\begin{aligned}&-\Delta^\top u+(2/n)\langle\tilde H,\nabla u\rangle-(\beta^\top+\Phi)u=\Psi_1u^{-1}-\Psi_2u^{-3}+\Psi_3u^3-\langle(\mathrm{Tr}^\top\mathfrak{T})^\perp,\nabla u\rangle\\&-(1/n)\langle(\mathrm{Tr}^\perp\mathfrak{T})^\perp,\nabla u\rangle-u^2\langle(\mathrm{Tr}^\top\mathfrak{T})^\top,\nabla u\rangle+n^{-1}(u^3-u)\langle\mathrm{Tr}^\top\mathfrak{T},\tilde H\rangle,\end{aligned}\tag{3.3}$$

where $n\Phi = \bar{\rm S}\,'_{\rm mix}$ is the mixed scalar curvature after transformation (1.2) and Ψ_i and $\beta^\top$ are

$$\begin{aligned} \beta^\top &= \Psi_2 - \Psi_1 - \bar{\rm S}_{\rm mix}/n - {\rm S}_{\mathfrak{T}^\top}/n, \\ \Psi_1 &= (\|h\|^2_g - b_{\mathfrak{T}})/n, \quad \Psi_2 = \|\tilde T\|^2_g/n \ge 0, \quad \Psi_3 = {\rm S}_{\mathfrak{T}^\top}/n\,. \end{aligned} \tag{3.4}$$

Example 3.1. Let $\widetilde{\mathcal{D}}$ be spanned by a vector field $N \ne 0$. A flow of N is called *geodesic* if the orbits are geodesics ($H=0$), and is *Riemannian* if $\tilde h = 0$. Let $\langle N, N\rangle = \epsilon_N \in \{-1,1\}$, then ${\rm S}_{\rm mix} = \epsilon_N\,{\rm Ric}_{N,N}$. For the Riemann-Cartan case with $\bar\nabla = \nabla + \mathfrak{T}$, we get $\bar{\rm S}_{\rm mix} = \epsilon_N\,\overline{\rm Ric}_{N,N}$, where

$$\begin{aligned} \overline{\rm Ric}_{N,N} &= {\rm Ric}_{N,N} + \sum\nolimits_i \epsilon_i\big[\langle(\nabla_N\mathfrak{T})_i\mathcal{E}_i, N\rangle + \langle(\nabla_{\mathcal{E}_i}\mathfrak{T})_N N, \mathcal{E}_i\rangle \\ &+ \langle\mathfrak{T}_i N, \mathfrak{T}_N\mathcal{E}_i\rangle - \langle\mathfrak{T}_N N, \mathfrak{T}_i\mathcal{E}_i\rangle\big]. \end{aligned}$$

Define the functions $\tilde\tau_i = {\rm Tr}(\tilde A^{\,i})$ $(i \ge 0)$, where $\tilde A : \mathcal{D} \to \mathcal{D}$ is the shape operator. Assume $H = 0$ on a compact leaf (a closed geodesic) $F \in \mathcal{F}$. On F, the Ricci curvature of ∇ in the N-direction is transformed by (1.2) as, see (3.1),

$${\rm Ric}'_{N,N} = {\rm Ric}_{N,N} - n\,u^{-1}N(N(u)) + 2\,u^{-1}\tilde\tau_1 N(u) + (u^{-4}-1)\|\tilde T\|^2_g\,.$$

Note that the vector field $\mathfrak{T}_X N$ belongs to $\mathfrak{X}^\perp$ for any $X \in \mathfrak{X}_M$, hence ${\rm S}_{\mathfrak{T}^\top} \equiv 0$ and $b_{\mathfrak{T}} = -\epsilon_N \sum_i \epsilon_i\langle\tilde T(\,\mathfrak{T}_i N + \mathfrak{T}_N\mathcal{E}_i,\,\mathcal{E}_i),\,N\rangle$, where $\{\mathcal{E}_i\}_{i\le n}$ is a local orthonormal frame on $\mathcal{D}$. Extending u from a compact leaf F onto M with the property $(\nabla u)^\perp = 0$ along F, we reduce (3.3) to $-N(N(u)) + \frac{2}{n}\,\tilde\tau_1 N(u) - (\beta^\top + \Phi)\,u = \Psi_1\,u^{-1} - \Psi_2\,u^{-3}$, where $\beta^\top = \Psi_2 - \Psi_1 - \frac1n\,\overline{\rm Ric}_{N,N}$, $\Psi_1 = -\frac1n\,b_{\mathfrak{T}}$, $\Psi_2 = \frac1n\,\|\tilde T\|^2_g \ge 0$ and $\Phi = \frac1n\,\overline{\rm Ric}\,'_{N,N}$.

4 $\mathcal{D}$-conformal flows of metrics

This section is aimed to prescribing ${\rm S}_{\rm mix}$ by a $\mathcal{D}$-conformal flow of metrics. Evolution equations provide an important tool to study physical phenomena. A *geometric flow* of metrics on M is a solution g_t of an evolution equation $\partial_t g = S(g)$, where a symmetric $(0,2)$-tensor $S(g)$ is usually related to some kind of curvature. This corresponds to a dynamical system in the infinite-dimensional space of all appropriate geometric structures on M. A flow of conformal metrics is determined by $S(g) = s(g)\,g$, where s is a smooth function on the space of metrics.

A $\mathcal{D}$-*conformal* flow $\partial_t g = s(g)\,g^\perp$ on a foliation depends on a function $s(g)$ on the space of metrics. The flow preserves total umbilicity, total geodesy and harmonicity of the leaves. Based on the inequality $n\,\|\tilde h\|^2 \ge \|\tilde H\|^2$ (with the equality when $\mathcal{D}$ is totally umbilical) we introduce the following *measure of non-umbilicity* of $\mathcal{D}$:

$$\beta^\top := n^{-2}\big(n\,\|\tilde h\|^2 - \|\tilde H\|^2\big) \ge 0.$$

For $p = 1$, we have $\beta^\top = n^{-2}\sum_{i<j}(k_i - k_j)^2$, where k_i are the principal curvatures of $\mathcal{D}$. The following normalized flow of metrics on a harmonic foliation (with leafwise constant $\Phi : M \to \mathbb{R}$) was studied in [22]–[24]:

$$\partial_t g = -2\big({\rm S}_{\rm mix}(g) - \Phi\big)\,g^\perp. \tag{4.1}$$

Note that (4.1) with $\Phi = 0$ reduces to (4.5), that looks like the normalized Ricci flow on surfaces, but uses the truncated metric $g^\perp$ instead of g. The $\mathcal{D}$-conformal flow (4.1), is 'Yamabe-type' analogue to (4.8). This yields the leafwise *forced Burgers equation* for the mean curvature vector $\tilde H$,

$$\partial_t \tilde H + \nabla^\top \|\tilde H\|^2 = p\nabla^\top(\operatorname{div}^\top \tilde H) + X \tag{4.2}$$

with $X = p\nabla^\top(\|\tilde T\|^2 - \|h\|^2 - n\beta^\top)$. If the vector $\tilde H_0$ is leafwise conservative,

$$\tilde H_0 = -n(\nabla \log u_0)^\top, \tag{4.3}$$

for a potential function $u_0 > 0$, compare Example 2.4, then (4.2) yields the non-linear heat equation $(1.8)_1$ with $\Psi_3 = 0$. Under certain assumptions about spectral parameters of $\mathcal{H}$, (4.1) has a unique global solution, whose $\mathrm{S}_{\rm mix}$ converges exponentially fast to a leafwise constant. Based on variational formulas (for $\mathcal{D}$-conformal metrics $g_t = s_t\, g_t^\perp$), $\partial_t h = -s\, h$, $\partial_t H = -s\, H$ and $\partial_t \tilde H = -(n/2)\,\nabla^\top s$, we get the following.

Proposition 4.1 (**Conservation laws**, [24]). *Let g_t $(t \ge 0)$ be $\mathcal{D}$-conformal metrics on a foliated manifold such that $\tilde H$ is leafwise conservative for u_0. Then the functions $\beta^\top$, $\|h\|^2/\|\tilde T\|$ and the vector field $\tilde H - \frac{n}{2}(\nabla \log \|\tilde T\|)^\top$ are t-independent.*

For u_0 and the ground state e_0 of $\mathcal{H}$ in (1.6), define

$$d_{u_0,e_0} := \min_F (u_0/e_0)/\max_F (u_0/e_0) \in (0,1].$$

The next theorems are central in [24].

Theorem 4.2. *Let $\mathcal{F}$ be a harmonic foliation on a closed Riemannian manifold (M, g_0) with conditions (1.1) and $\tilde H_0$ obeys (4.3). If Φ obeys*

$$\Phi \ge n\lambda_0 + d^{-4}_{u_0,e_0} \max_M \|\tilde T\|^2_{g_0}, \tag{4.4}$$

then (4.1) has a unique smooth solution g_t $(t \ge 0)$, and for any α in the interval $(0, \min\{\lambda_1 - \lambda_0,\, 2\,(\frac{\Phi}{n} - \lambda_0)\})$ we get the leaf-wise convergence in C^∞, as $t \to \infty$, with the exponential rate $n\alpha$:

$$\mathrm{S}_{\rm mix}(g_t) \to n\lambda_0 - \Phi \le 0, \qquad \tilde H_t \to -n\nabla^\top \log e_0, \qquad h(g_t) \to 0.$$

For $\tilde T = 0$, condition (4.4) becomes $\Phi \ge n\lambda_0$, and we have the following.

Corollary 4.3. *Let $\mathcal{F}$ be a harmonic foliation with integrable normal distribution on a closed Riemannian manifold (M, g_0) with assumptions (1.1) and $\tilde H_0$ obeys (4.3). If $\Phi \ge n\lambda_0$, then the statement of Theorem 4.2 holds.*

Theorem 4.4. *Let $\mathcal{F}$ be a harmonic foliation on a closed Riemannian manifold (M, g_0) with assumptions (1.1), and $\tilde H_0$ obeys (4.3). Suppose that*

$$d^2_{u_0/e_0} > \sqrt{2}\, \max_M \|\tilde T\|_{g_0} / \min_M \|h\|_{g_0}.$$

Then the interval

$$I_0 = (\max\big\{0,\, 3d^{-4}_{u_0,e_0} \max_M \|\tilde T\|^2_{g_0} - \min_M \|h\|^2_{g_0}\big\},\ \frac14 d^4_{u_0,e_0} \min_M \|h\|^4_{g_0} / \max_M \|\tilde T\|^2_{g_0})$$

is nonempty, and for any Φ obeying $n\lambda_0 - \Phi \in I_0$, (4.1) has a unique smooth solution $g_t\,(t \ge 0)$, and it converges in C^∞ exponentially fast to a limit metric $\bar g = \lim_{t\to\infty} g_t$; and we have the exponential convergence $\mathrm{S}_{\rm mix}(g_t) \to \Phi$, as $t \to \infty$, in C^∞ on the leaves.

If $\tilde{T}=0$ and $h\neq 0$ then I_0 looks simpler, and we get the following.

Corollary 4.5. *Let $\mathcal{F}$ be a harmonic foliation on a closed Riemannian manifold (M,g_0) with* (1.1)*. Let the normal distribution be integrable, $h\neq 0$, and $\tilde{H}_0$ obeys* (4.3)*. If $\Phi\le n\lambda_0$, then the claim of Theorem* 4.4 *holds.*

Example 4.1. (a) Let (M,g) be a surface of Gaussian curvature K, endowed with a unit geodesic vector field N. Then (4.1) looks like the normalized Ricci flow,

$$\partial_t g=-2\,(K(g)-\Phi)\,g^{\perp}, \tag{4.5}$$

but uses the truncated metric $g^{\perp}$ instead of g. The geodesic curvature k of curves orthogonal to N obeys $\partial_t k=K_{,x}$ along a trajectory $\gamma(x)$ of N. The above yields the *Burgers equation* $\partial_t k+(k^2)_{,x}=k_{,xx}$, which serves as the model equation for solitary waves, and is used for describing advection-diffusion processes in gas and fluid dynamics. The non-linear Burgers equation reduces to the heat equation $\varphi_{,t}=\varphi_{,xx}$ using the Cole-Hopf transformation $k=-(\log\varphi)_{,x}$. When k and K are known, the metrics may be recovered as $g_t^{\perp}=g_0^{\perp}\exp\big(-2\int_0^t(K(s,t)-\Phi)\,\mathrm{d}s\big)$.

(b) For the Hopf fibration $\pi: S^{2m+1}\to\mathbb{C}P^m$ of a unit sphere we get $\mathrm{S}_{\rm mix}=2\,m$. Thus, the canonical metric on S^{2m+1} is a fixed point of the flow (4.1) with $\Phi=2\,m$.

Now, we shall examine, when for the warped product initial metric on $B\times_\varphi\bar{M}$ the solution of (4.1) converges to the metric with leafwise constant $\mathrm{S}_{\rm mix}$. Look at what happens when B has a boundary, e.g., B is a ball in $\mathbb{R}^p$. Let $\mu(t,x):=\varphi(t,x)_{|\,\partial B}$ be twice continuously differentiable in t, and there exist limits

$$\lim_{t\to\infty}\mu(t,x)=0,\quad \lim_{t\to\infty}\partial_t\mu(t,x)=0,\quad \lim_{t\to\infty}\partial_t^2\mu(t,\,\cdot)=0 \tag{4.6}$$

uniformly for $x\in\partial B$. Define $\nu(t):=\max\{\|\mu(t,\,\cdot)\|_{C^0(\partial B)},\ \|\partial_t\mu(t,\,\cdot)\|_{C^0(\partial B)}\}$.

Theorem 4.6. *Let the warped product metrics g_t on $B\times_\varphi\bar{M}$ with $\dim B=p$ and $\dim\bar{M}=n$ solve* (4.1) *and any of conditions (i) – (iii) are satisfied:*

(i) $\Phi<0$ and $(4.6)_{1,2}$, *(ii) $0\le\Phi<\lambda_1$, $p<4$ and* (4.6)*,*

(iii) $\Phi=\lambda_1$, $p<4$, (4.6) *and $\int_0^\infty\nu(\tau)\,\mathrm{d}\tau<\infty$.*

Then g_t exist for all $t\ge 0$, and converge, as $t\to\infty$, uniformly on $B\times\bar{M}$ in C^0-norm to the metric $g_\infty=\mathrm{d}x^2+\varphi_\infty^2(x)\,\bar{g}$ with $\mathrm{S}_{\rm mix}(g_\infty)=n\,\Phi$. In cases (i)–(ii), (4.1) *has a single point global attractor, while for case (iii), g_∞ depends on initial and boundary conditions.*

Example 4.2 (Rotation surfaces). For a surface (M^2,g), foliated by curves, we have $\mathrm{S}_{\rm mix}=K$ – the Gaussian curvature. The metric on a rotation surface in $\mathbb{R}^3$ belongs to the class of warped products. Let $M_t^2\subset\mathbb{R}^3$:

$$[\varphi(t,x)\cos\theta,\ \varphi(t,x)\sin\theta,\ \psi(t,x)],\qquad 0\le x\le l,\ |\theta|\le\pi,\ \varphi\ge 0,$$

be a family of rotation surfaces such that $(\partial_x\varphi)^2+(\partial_x\psi)^2=1$. The profile curves $\theta=\mathrm{const}$ are unit speed geodesics tangent to the vector field N. The θ-curves are circles in $\mathbb{R}^3$ of geodesic curvature $k=-(\log\varphi)_{,\,x}$. The metric $g_t=\mathrm{d}x^2+\varphi^2(t,x)\mathrm{d}\theta^2$ is rotational symmetric of curvature $K=-\varphi_{,\,xx}/\varphi$. Let g_t obeys (4.5). Then φ solves

$$\partial_t\varphi=\varphi_{,\,xx}+\Phi\,\varphi,\quad \varphi(0,x)=\varphi_0(x),\quad \varphi(t,\,0)=\mu_0(t)\ge 0,\quad \varphi(t,\,l)=\mu_1(t)\ge 0,$$

where $\varphi(x) > 0$ for $x \in (0,l)$, $\mu_0, \mu_1 \in C^1[0,\infty)$ and there exist $\lim_{t\to\infty} \mu_j(t) = \tilde\mu_j \in [0,\infty)$. The solution of stationary problem with $B = [0,l]$ and $\lambda_1 = (\pi/l)^2$ is

$$\tilde\varphi(x) = \begin{cases} \frac{\tilde\mu_1 \sin(\sqrt{\Phi}\,x) + \tilde\mu_0 \sin(\sqrt{\Phi}\,(l-x))}{\sin(\sqrt{\Phi}\,l)} & \text{if } \ 0 < \Phi < \lambda_1,\\ \tilde\mu_0 + (\tilde\mu_1 - \tilde\mu_0)(x/l) & \text{if } \Phi = 0,\\ \frac{\tilde\mu_1 \sinh(\sqrt{-\Phi}\,x) + \tilde\mu_0 \sinh(\sqrt{-\Phi}\,(l-x))}{\sinh(\sqrt{-\Phi}\,l)} & \text{if } \Phi < 0. \end{cases}$$

For the resonance case, $\Phi = \lambda_1 = (\pi/l)^2$, the stationary problem is solvable if and only if $\tilde\mu_0 = \tilde\mu_1 = 0$, and in this case the solutions are $\tilde\varphi(x) = C\sin(\pi x/l)$, where $C > 0$ is constant. By Theorem 4.6, if $\Phi > (\pi/l)^2$ then g_t diverge as $t \to \infty$, otherwise g_t converge to the limit metric $g_\infty = \mathrm{d}x^2 + \varphi^{\,2}_{\infty}(x)\mathrm{d}\,\theta^2$ with $K(g_\infty) = \Phi$.

Certainly, if $\Phi = (\pi/l)^2$ and, in addition, see Theorem 4.6(iii),

$$\int_0^\infty (|\mu_j(\tau)| + |\mu_j'(\tau)|)\,\mathrm{d}\tau < \infty \quad (j = 0,1), \tag{4.7}$$

then $\varphi_\infty = (v_1^0 + \int_0^\infty f_1(\tau)\,\mathrm{d}\tau)\,\sin(\pi\,x/l)$, and if $\Phi < (\pi/l)^2$ then $\varphi_{\,\infty} = \tilde\varphi$. If a solution $\varphi(x,t)$ $(t \ge 0)$ is known and $|\varphi_{,\,x}| \le 1$, then we get $\psi(t,x) = \psi(t,0) + \int_0^x \sqrt{1 - (\varphi_{,\,x})^2}\,\mathrm{d}x$. Remark that rotation surfaces in $\mathbb{R}^3$ of constant Gaussian curvature are locally classified.

Remark 4.3. The flow (4.1) is the 'Yamabe-type' analogue of the *partial Ricci flow*

$$\partial_t g = -2\,\mathrm{Ric}^\perp(g) + 2\,\Phi\,g^\perp \tag{4.8}$$

on foliations, see [18]. Here $\Phi : M \to \mathbb{R}$ is a leaf-wise constant. The flow (4.8) was proposed as the tool to prescribe the partial Ricci tensor (2.2) and the constancy of the mixed sectional curvature $K_{\rm mix}$. It was conjectured in [18]: *Let $\mathcal{F}$ be a p-dimensional totally geodesic foliation of a closed Riemannian manifold (M^{n+p}, g), and $K_{\rm mix}$ be sufficiently close to a positive constant, then* (4.8) *evolves the metric g to a limit metric, whose $K_{\rm mix}$ is a positive function of a point.* It was proven local existence and uniqueness of solution, and for the warped product initial metric it was shown that solution metrics of (4.8) converge, as $t \to \infty$, to the metric with $\mathrm{Ric}^\perp = \Phi\,g^\perp$. Theorem 4.6 confirms the conjecture for some warped product metrics.

5 Prescribing $\mathrm{S}_{\rm mix}$ using $\mathcal{D}$-conformal metrics

This section uses $\mathcal{D}$-conformal change of metrics and is supported by results about stable stationary solutions to the non-linear equation (1.8) associated with the elliptic PDE on a leaf F – a closed Riemannian manifold, and compactness in $C(F)$ of the set of all such solutions. Our approach to **Q2** using a $\mathcal{D}$-conformal metrics is based on using spectral parameters of the Schrödinger operator on compact leaves, and exploring stable solutions of the elliptic PDE (1.5), that are stable stationary solutions of (1.8), one of them (i.e., for $\mathrm{S}_{\,\mathfrak{T}^\top} \equiv 0$) corresponds to the pseudo-Riemannian case. As promised in the introduction we examine two formulations of the problem of prescribing leafwise constant $\bar{\mathrm{S}}_{\rm mix}$. Let $\mathrm{S}^{\,\prime}_{\rm mix} = n\Phi$ be the mixed scalar curvature after transformation (1.2). Assume the following 'regularity' properties:

(a) *either* $\mathcal{D}$ *is nowhere integrable or* $\mathcal{D}$ *is integrable* (i.e., $\tilde T\equiv 0$),

(b) *either* $\mathrm{S}_{\mathfrak{T}^\top}$ *is nowhere vanishes or* $\mathrm{S}_{\mathfrak{T}^\top}\equiv 0$, see definition in (2.7).

For a compact leaf F, introduce the quantities

$$
(5.1)\quad K_1=\frac{\psi_3^+}{4\psi_2^-}\,\max\{18\psi_1^+\psi_2^+,\ 4(\psi_1^+)^3+27(\psi_2^+)^2\psi_3^+\},
$$

$$
K_2=\frac{\max\big\{36\psi_1^+\psi_2^+\psi_3^-(\psi_3^-+\psi_3^+),27\psi_3^+(\psi_2^+)^2(\psi_3^+)^2+3(\psi_3^-)^2+(\psi_1^+)^3(\psi_3^++3\psi_3^-)^2\big\}}{8\psi_2^+(3\psi_3^--\psi_3^+)},
$$

where $\psi_k^+=\max_F|\Psi_k|$, $\psi_k^-=\min_F|\Psi_k|$ $(k=1,2)$, $\psi_3^+=\max_F|\mathrm{S}_{\mathfrak{T}^\top}|/n$, $\psi_3^-=\min_F|\mathrm{S}_{\mathfrak{T}^\top}|/n$. For positive $f\in C(F)$ define

$$
\delta(f):=(\min{}_F f)/(\max{}_F f)\in(0,\,1].
$$

The following assumptions are helpful:

$$
(5.2)\qquad 27\,(\psi_2^+)^2\psi_3^+(\psi_1^-)^{-3}<\delta^{\,8}(e_0)\quad\text{when } \mathrm{S}_{\mathfrak{T}^\top}>0,
$$

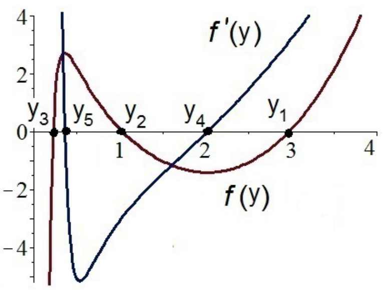
$$
(5.3)\qquad \delta(|\mathrm{S}_{\mathfrak{T}^\top}/n|)\,\delta^{\,2}(e_0)>1/3\quad\text{when } \mathrm{S}_{\mathfrak{T}^\top}<0.
$$

The equality $\mathrm{S}_{\mathfrak{T}^\top}\equiv 0$ appears in the case of pseudo-Riemannian manifolds.

Define polynomials $P_{\phi+}(z)$ and $P_{\phi-}(z)$ with constant coefficients,

$$
P_{\phi\pm}(z)=\left\{\begin{array}{ll}
\psi_3^\pm z^3-\lambda_0 z^2+\psi_1^\pm z-\psi_2^\mp & \text{if } \mathrm{S}_{\mathfrak{T}^\top}>0,\ \|h\|^2_g>b_{\mathfrak{T}} \text{ and (5.2)},\\
-\psi_3^\mp z^3-\lambda_0 z^2-\psi_1^\mp z-\psi_2^\mp & \text{if } \mathrm{S}_{\mathfrak{T}^\top}<0 \text{ and } \|h\|^2_g<b_{\mathfrak{T}},\\
-\lambda_0 z^2+\psi_1^\mp z-\psi_2^\pm & \text{if } \mathrm{S}_{\mathfrak{T}^\top}\equiv 0 \text{ and } \|h\|^2_g>b_{\mathfrak{T}}.
\end{array}\right.
$$

For $\mathrm{S}_{\mathfrak{T}^\top}>0$, positive roots of $f_\pm(y):=P_{\phi\pm}(y^2)$ and of $f'_\pm(y)$ are ordered as $y_3^-<y_5^-<y_2^-<y_4^-<y_1^-$ and $y_3^+<y_5^+<y_2^+<y_4^+<y_1^+$, see Figure 1; for $\mathrm{S}_{\mathfrak{T}^\top}\le 0$, the roots are ordered as $y_1^-<y_3^-<y_2^-$ and $y_1^+<y_3^+<y_2^+$, Figure 2.



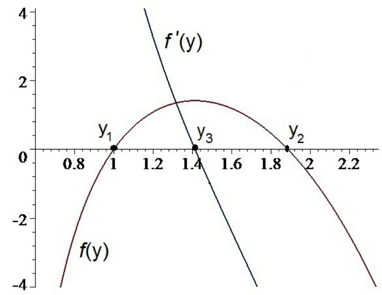


Figure 1: Graphs of functions $f_\pm(y)=P_{\phi\pm}(y^2)/y^3$ and f'.
(a) $\mathrm{S}_{\mathfrak{T}^\top}>0$, $\Psi_1>0$ and $\Psi_2>0$; (b) $\mathrm{S}_{\mathfrak{T}^\top}<0$, $\Psi_1<0$ and $\Psi_2>0$.

Theorem 5.1 ([25])**.** *Let* $(M,g,\bar\nabla)$ *be a foliated Riemann-Cartan manifold with conditions* $\|\tilde T\|^2_g>0$ *and* (1.3) *on a space-like compact leaf* F*. Given* $\Phi\in C^\infty(M)$ *obeying*

$$
\Phi_{\,|\,F}\in\left\{\begin{array}{ll}
(-\infty,\ -\beta^\top) & \text{if } \mathrm{S}_{\mathfrak{T}^\top}>0,\ \|h\|^2_g>b_{\mathfrak{T}} \text{ and (5.2)},\\
(-\beta^\top+1+\delta^{\,-4}(e_0)\sqrt{K_1},\ \infty) & \text{if } \mathrm{S}_{\mathfrak{T}^\top}<0 \text{ and } \|h\|^2_g<b_{\mathfrak{T}},\\
(-\beta^\top-\delta^{\,4}(e_0)\,(\psi_1^-)^2/(4\,\psi_2^+),\ -\beta^\top) & \text{if } \mathrm{S}_{\mathfrak{T}^\top}\equiv 0 \text{ and } \|h\|^2_g>b_{\mathfrak{T}},
\end{array}\right.
$$

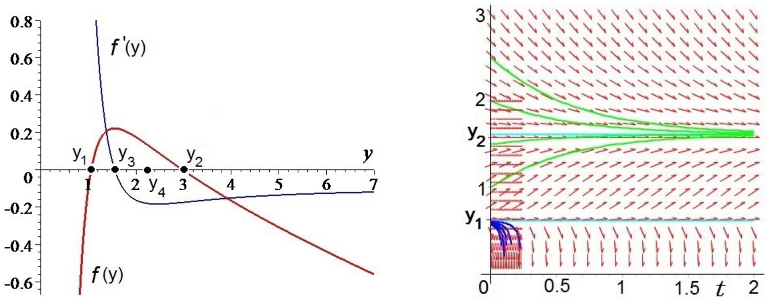


Figure 2: $f_\pm(y) = P_{\phi\pm}(y^2)/y^3$, with $\beta < 0$ and $4\,|\beta|\Psi_2 < \Psi_1^2$.
(a) Graphs of $f_\pm$ and $f'_\pm$ for $\mathrm{S}_{\mathfrak{T}^\top} \equiv 0$, $\Psi_1 > 0$, $\Psi_2 > 0$. (b) y_1 is unstable, y_2 is stable.

there exists a positive function $u_* \in C^\infty(M)$ *obeying* (1.5) *such that* M *with a Riemann-Cartan structure* $(g' = g^\top \oplus u_*^2\, g^\perp, \mathfrak{T}' = u_*^2\, \mathfrak{T}^\top \oplus \mathfrak{T}^\perp)$ *has* $\bar{\mathrm{S}}'_{\rm mix} = n\Phi$ *on* F*; moreover,* $y_2^- \le u_*/e_0 \le y_2^+$*, and the set* $\{u_{*|F}\}$ *of such functions is compact in* $C(F)$*.*

Proof. The required conformal factor u should satisfy on F the nonlinear elliptic PDE (1.5) with the Schrödinger operator (1.6) and $\beta^\top, \Psi_i$ $(i = 1, 2, 3)$ given in (3.4). We are looking for such solutions of (1.5) are stationary solutions of (1.8). Denote by $f(x, y)$ $(x \in F)$ the rhs of (1.8). Then $f_\pm(y) = P_{\phi\pm}(y^2)/y^3$ are the majorizing and minorizing functions for f. By assumptions, the order of roots of $f(x, y)$ with any $x \in F$ is the same as for $P_{\phi\pm}$. Consider three cases according to sign of $\mathrm{S}_{\mathfrak{T}^\top}$.

Case 1. Let $\mathrm{S}_{\mathfrak{T}^\top} > 0$, $\|h\|_g^2 > b_{\mathfrak{T}}$ and (5.2) holds. Then $\lambda_0 > 0$ (the least eigenvalue of $\mathcal{H}$ on F); hence, each of bicubic polynomials $P_{\phi+}(y^2)$ and $P_{\phi-}(y^2)$ has three positive roots: $y_3^- < y_2^- < y_1^-$ and $y_3^+ < y_2^+ < y_1^+$, which can be expressed by Cardano or trigonometric formulas. Since (1.7) and (5.2) yield $(\psi_1^-)^3 > 27\,(\psi_2^+)^2\psi_3^+$, there is $u_* \in C^\infty(M)$ obeying (1.5), i.e., (u_*, Φ) solves **Q2** on F, and $y_2^- \le u_*/e_0 \le y_2^+$ holds.

Case 2. Let $\mathrm{S}_{\mathfrak{T}^\top} < 0$ and $\|h\|_g^2 < b_{\mathfrak{T}}$. Then, each of polynomials $P_{\phi+}(y^2)$ and $P_{\phi-}(y^2)$ has two positive roots: $y_1^- < y_2^-$ and $y_1^+ < y_2^+$, Figure 1(b). By (5.1) there exists $u_* \in C^\infty(M)$ obeying (1.5).

Case 3. Let $\mathrm{S}_{\mathfrak{T}^\top} \equiv 0$ and $\|h\|_g^2 > b_{\mathfrak{T}}$. The problem amounts to finding a positive solution of the elliptic PDE (1.5) with $\Psi_3 = 0$. For $\Psi_1 > 0$ and $\Psi_2 \ne 0$ each of polynomials $P_{\phi+}(y^2)$, $P_{\phi-}(y^2)$ has two positive roots $y_1^- < y_2^-$ and $y_1^+ < y_2^+$, Figure 2. By (1.7), there is a function $u_* \in C^\infty(M)$ obeying (1.5). $\square$

Remark 5.1. Under stronger geometric conditions along F, the solution $u_{*|F}$ in Case 2 of Theorem 5.1, is unique in the set $\{\tilde u \in C(F) : y_3^- < \tilde u/e_0 < y_1^+\}$, and if $\Phi > -\beta^\top + 1 + \delta^{-4}(e_0)\sqrt{K_2}$ then the solution $u_{*|F}$ is unique in $\mathcal{U}_1 = \{\tilde u \in C(F) : \tilde u/e_0 > y_1^-\}$. In Case 3 of Theorem 5.1, the solution $u_{*|F}$ is unique in $\mathcal{U}_1$.

In the next theorem [25], we consider two cases: $\mathrm{S}_{\mathfrak{T}^\top} < 0$ and $\mathrm{S}_{\mathfrak{T}^\top} \equiv 0$. We omit the case $\mathrm{S}_{\mathfrak{T}^\top} > 0$, having technical explicit conditions for uniqueness of a solution.

Theorem 5.2. *Let* $(M, g, \bar\nabla)$ *be a foliated closed Riemann-Cartan space with space-like leaves,* $\|\tilde T\|_g^2 > 0$*,* (1.1) *and* (1.4)*. Given a smooth leafwise constant function*

$$\Phi \in \begin{cases} (1 - \beta^\top + \delta^{-4}(e_0)\sqrt{K_2},\ \infty) & \text{if } \mathrm{S}_{\mathfrak{T}^\top} < 0,\ \|h\|_g^2 < b_{\mathfrak{T}} \text{ and (5.3)}, \\ (-\beta^\top - \delta^{4}(e_0)(\psi_1^-)^2/(4\,\psi_2^+),\ -\beta^\top) & \text{if } \mathrm{S}_{\mathfrak{T}^\top} \equiv 0 \text{ and } \|h\|_g^2 > b_{\mathfrak{T}}, \end{cases}$$

there exists a leafwise smooth $u_* \in C(M)$ *obeying* (1.5)*, unique in* $\mathcal{U}_1 = \{\tilde u \in C(M) : \tilde u/e_0 > y_1^-\}$*, such that* M *with the structure* $(g' = g^\top \oplus u_*^2\, g^\perp,\ \mathfrak{T}' = u_*^2\, \mathfrak{T}^\top \oplus \mathfrak{T}^\perp)$ *has* $\bar{\mathrm{S}}'_{\rm mix} = n\Phi$*; moreover,* $y_2^- \le u_*/e_0 \le y_2^+$*.*

Proof. Consider two cases for (1.5) and (1.8). Case 1. Let $\mathrm{S}_{\mathfrak{T}^\top} \equiv 0$ and $\langle h^\top, h^\top\rangle_g > b_{\mathfrak{T}}$. We apply Theorem 6.2 and then Proposition 5.3. Case 2. Let $\mathrm{S}_{\mathfrak{T}^\top} < 0$, $\langle h^\top, h^\top\rangle_g < b_{\mathfrak{T}}$ and (5.3) holds. As in the proof in Case 2 of Theorem 5.1, we apply [25, Theorem 6] (omitted in this survey) and then Proposition 5.3. □

Let $F\times\mathbb{R}^n$ be the product with a compact leaf F, and $g(\cdot,q) = g_{ij}(x,q)$ a leafwise Riemannian metric (that is on $F_q = F\times\{q\}$ for $q\in\mathbb{R}^n$) such that the volume form of the leaves $d\,\mathrm{vol}_F = |g|^{1/2}\,\mathrm{d}x$ depends on $x\in F$ only (for example, the leaves are harmonic submanifolds). This assumption simplifies arguments used in the proof of Proposition 5.3 below (we consider products $\mathbb{B} = L^2\times\mathbb{R}^n$ and $\mathbb{B}_k = H^k\times\mathbb{R}^n$ instead of vector bundles over $\mathbb{R}^n$), on the other hand, it is sufficient for proving the geometric results. The leafwise Laplacian in a local chart (U,x) on (F,g) is written as $\Delta\, u = \nabla_i(g^{ij}\,\nabla_j\, u) = |g|^{-1/2}\partial_i(|g|^{1/2}g^{ij}\partial_j\, u)$, see [2]. This defines a self-adjoint elliptic operator $-\Delta_q = -g^{ij}(x,q)\partial^2_{ij} - b^j(x,q)\partial_j$, where $q\in\mathbb{R}^n$ is a parameter and $\Delta_0 = \Delta$. Here $b^j = |g|^{-\frac12}\partial_i(|g|^{\frac12}g^{ij})$ are smooth functions in $U\times\mathbb{R}^n$. Thus, the Schrödinger operator $\mathcal{H}_q = -\Delta_q - \beta^\top(x,q)\,\mathrm{id}$ $(q\in\mathbb{R}^n)$, acts in the Hilbert space L^2 with the domain H^2, and it is self-adjoint.

Proposition 5.3 (Smooth dependence of a solution on a transversal parameter, [22]). *Let* $\lambda_0(q)$ *be the least eigenvalue of* $\mathcal{H}_q$, $q\in\mathbb{R}^n$*. If* $\beta\in C^\infty(F\times\mathbb{R}^n)$ *then* $\lambda_0\in C^\infty(\mathbb{R}^n)$ *and there exists a unique* $e_0\in C^\infty(F\times\mathbb{R}^n)$ *such that* $e_0(\cdot,q)>0$ *is an eigenfunction of* $\mathcal{H}_q$ *related to* $\lambda_0(q)$ *with* $\|e_0(\cdot,q)\|_{L^2} = 1$*.*

6 Attractor for the nonlinear heat equation

Note that 'stable' solutions of (1.5) are point attractors of (1.8). By [2, Theorem 4.51], (1.8) has a unique smooth solution in $\mathcal{C}_{t_0}$ for some $t_0>0$. Here $\mathcal{C}_t = F\times[0,t)$, $(0<t\le\infty)$ is a cylinder with the base F (a compact leaf). Substitute $u = e_0 w$ into (1.8), and using $\Delta(e_0w) = e_0\Delta w + w\Delta e_0 + \langle 2\,\nabla e_0, \nabla w\rangle$ and $\Delta e_0 + \beta e_0 = -\lambda_0 e_0$, obtain the Cauchy's problem for $w(x,t)$,

$$(6.1)\qquad \partial_t w = \Delta w + \langle 2\,\nabla\log e_0,\, \nabla w\rangle + f(w,\,\cdot\,),\quad w(\cdot\,,0) = u_0/e_0 > 0,$$

where $f(w,\,\cdot\,) = -\lambda_0\, w + (\Psi_1 e_0^{-2})\, w^{-1} - (\Psi_2 e_0^{-4})\, w^{-3} + (\Psi_3 e_0^2)\, w^3$. By (6.1),

$$(6.2)\qquad \phi_-(w) \le \partial_t w - \Delta w - \langle 2\nabla\log e_0, \nabla w\rangle \le \phi_+(w),$$

where the functions ϕ_- and ϕ_+ are defined for each case separately.

Define the parallelepiped $\mathcal{P} = \prod_{k=1}^3[\Psi_k^-, \Psi_k^+] \subset \mathbb{R}^3_+$, where

$$\begin{aligned}\Psi_k^+ &= \max{}_F\,(|\Psi_k|\, e_0^{-2k}),\quad \Psi_k^- = \min{}_F\,(|\Psi_k|\, e_0^{-2k})\quad (k=1,2),\\ \Psi_3^+ &= \max{}_F\,(|\Psi_3|\, e_0^2),\qquad \Psi_3^- = \min{}_F\,(|\Psi_3|\, e_0^2).\end{aligned}$$

Then $\mathcal{P}_0 = \{(\Psi_1(x), \Psi_2(x), \Psi_3(x)) : x\in F\}$ is a closed subset of $\mathcal{P}$.

In this section, we consider stabilization of solutions of (1.8) when $\Psi_3 = 0$. The technical cases $\Psi_3 > 0$ and $\Psi_3 < 0$ are studied similarly, see [25], and we omit them.

Let $\underline{\Psi_3 = 0,\ \Psi_1 > 0 \text{ and } \Psi_2 > 0}$, see Section 7, point ($\mathrm{c}_1$). Then we have elliptic PDE (1.5) and associated Cauchy's problem (1.8) with $\Psi_3 = 0$. In this case,

$$\begin{aligned} \phi_+(y) &= P_{\phi_+}(y^2)/y^3, \quad \text{where} \quad P_{\phi_+}(z) = -\lambda_0 z^2 + \Psi_1^- z - \Psi_2^+, \\ \phi_-(y) &= P_{\phi_-}(y^2)/y^3, \quad \text{where} \quad P_{\phi_-}(z) = -\lambda_0 z^2 + \Psi_1^+ z - \Psi_2^-, \end{aligned}$$

and $f(w,\cdot)=(\Psi_1 e_0^{-2})w^{-1}-(\Psi_2 e_0^{-4})w^{-3}-\lambda_0 w$ obeys $\partial_w f(w,x) \le \partial_w \phi_-(w)$. Let

$$0 < \lambda_0 < (\Psi_1^-)^2/(4\Psi_2^+). \tag{6.3}$$

Each of functions $\phi_-(y)$ and $\phi_+(y)$ has two positive roots; moreover, $y_1^- < y_2^-$ and $y_1^+ < y_2^+$. Since $\phi_-(y) < \phi_+(y)$ for $y > 0$, we also have $y_2^- < y_2^+$ and $y_1^- > y_1^+$. Denote by $y_3^- \in (y_1^-, y_2^-)$ a unique positive root of $\partial_y \phi_-(y) = -\lambda_0 - \Psi_1^- y^{-2} + 3\,\Psi_2^+ y^{-4}$. Notice that $\phi_-(y) > 0$ for $y \in (y_1^-, y_2^-)$ and $\phi_-(y) < 0$ for $y \in (0, \infty) \setminus [y_1^-, y_2^-]$; moreover, $\phi_-(y)$ increases in $(0, y_3^-)$ and decreases in (y_3^-, ∞). The line $z = -\lambda_0\, y$ is asymptotic for the graph of $\phi_-(y)$ when $y \to \infty$, and $\lim_{y\downarrow 0} \phi_-(y) = -\infty$. The function $\partial_y \phi_-(y)$ decreases in $(0, y_4^-)$ and increases in (y_4^-, ∞), where $y_4^- := (6\,\Psi_2^+/\Psi_1^-)^{1/2} > y_3^-$, and $\lim_{y\to\infty} \partial_y \phi_-(y) = -\lambda_0$, see Figure 2. We conclude that $y_1^+ < y_1^- < y_3^- < y_2^- < y_2^+$. Hence, the function $\mu^+(\sigma) := -\sup_{y \ge y_2^- - \sigma} \partial_y \phi_-(y) = \min\{|\partial_y \phi_-(y_2^- - \sigma)|, \lambda_0\}$ is positive for $\sigma \in (0, y_2^- - y_3^-)$. Define closed in $C(F)$ nonempty sets

$$\mathcal{U}^{\,\varepsilon,\eta} = \{\tilde u \in C(F):\ y_2^- - \varepsilon \le \tilde u/e_0 \le y_2^+ + \eta\},\ \varepsilon \in (0,\, y_2^- - y_1^-),\ \eta \in (0,\infty].$$

We have $\mathcal{U}_0 \subset \mathcal{U}^{\,\varepsilon,\eta} \subset \mathcal{U}^{\,\varepsilon,\infty} \subset \mathcal{U}_1$, where the set $\mathcal{U}_1 = \{\tilde u \in C(F) : \tilde u/e_0 > y_1^-\}$ is open, and $\mathcal{U}_0 = \{\tilde u \in C(F) : y_2^- \le \tilde u/e_0 \le y_2^+\}$. Let $S_t : C(F) \to C(F)$ be the map which relates to each initial value $u_0 \in C(F)$ the value of the classical solution of (1.8) at the moment $t \in [0, T)$ (if this solution exists and is unique). Since the rhs of (1.8) does not depend explicitly on t, the family $\{S_t\}_{0\le t<T}$ has the semigroup property, and it is a semigroup ($T = \infty$) when (1.8) has a global solution for any $u_0(x) \in C(F)$.

Proposition 6.1. *Let* (6.3) *holds. Then* (*i*) *for any* $u_0 \in \mathcal{U}^{\,\varepsilon,\eta}$, *Cauchy's problem* (1.8) *with* $\Psi_3 = 0$ *admits a unique global solution. Moreover,* $\mathcal{U}^{\,\varepsilon,\eta}$ *are invariant sets for associated semigroup* $\mathcal{S}_t : u_0 \to u(\cdot\,, t)$ $(t \ge 0)$ *in* $\mathcal{C}_\infty$; (*ii*) *for any* $\sigma \in (0, \varepsilon)$ *there exists* $t_1 > 0$ *such that* $\mathcal{S}_t(\mathcal{U}^{\,\varepsilon,\infty}) \subseteq \mathcal{U}^{\,\sigma,\infty}$ *for all* $t \ge t_1$.

Proposition 6.1 supports the following.

Theorem 6.2. (*i*) *If* (6.3) *holds then* (1.5) *with* $\Psi_3 = 0$ *has in* $\mathcal{U}_1 \cap C^\infty(F)$ *a unique solution* u_* *satisfying* $y_1^- \le u_*/e_0 \le y_1^+$; *moreover,* $u_* = \lim_{t\to\infty} u(\cdot, t)$, *where* u *solves* (1.8) *with* $\Psi_3 = 0$ *and* $u_0 \in \mathcal{U}_1$, *and for any* $\sigma \in (0,\, y_2^- - y_3^-)$ *the set* $\mathcal{U}^{\,\sigma,\infty}$ *is attracted by associated semigroup* $\mathcal{S}_t$ *exponentially fast to* u_* *in* C*-norm:*

$$\|u(\cdot\,,t) - u_*\|_{C(F)} \le \delta^{-1}(e_0)\, e^{-\mu^+(\sigma)\, t} \|u_0 - u_*\|_{C(F)} \quad (t > 0,\ u_0 \in \mathcal{U}^{\,\sigma,\infty}).$$

(*ii*) *Let* $\beta^\top, \Psi_1, \Psi_2$ *be smooth functions on the product* $F \times \mathbb{R}^n$ *with a smooth metric* $\langle\cdot, q\rangle$. *If* (6.3) *holds for any* $F \times \{q\}$ $(q \in \mathbb{R}^n)$ *then* u_* *is smooth on* $F \times \mathbb{R}^n$.

7 Comparison ODE

If Ψ_i $(i = 1, 2, 3)$ are real constants then (1.8) belongs to the class of *reaction-diffusion equations*, which are well understood and whose solutions can be written explicitly. Namely, leafwise constant solutions of (1.8) obey the Cauchy's problem:

$$y' = f(y) = P(y^2)/y^3, \qquad y(0) = y_0 > 0, \tag{7.1}$$

with $P(z) = \Psi_3\, z^3 + \beta\, z^2 + \Psi_1\, z - \Psi_2$. For $\Psi_3 \neq 0$ the polynomial $P(z)$ has three different real roots if and only if the discriminant $D_P := -\mathrm{Res}(P, P')/\Psi_3$ (a cubic polynomial in β) is positive, where $\mathrm{Res}(P, P')$ is the resultant of two polynomials. Consequently, $P(z)$ has one real root if and only if $D_P < 0$. In a sense, $\Psi_3 = 0$ is the bifurcation point for (7.1). We are looking for stable stationary solutions of (7.1), those are roots of P. If P has a real root $\tilde y > 0$ such that $f'(\tilde y) < 0$ then $y = \tilde y$ is attractor for the semigroup associated to (7.1). The basin of attractor is determined by other two positive roots which surround $\tilde y$, Figures 1–2.

(a) Let $\Psi_3 > 0$. Thus, $P(z)$ obeys $P(0) = -\Psi_2 < 0,\ P(\infty) = \infty$ and $P(-\infty) = -\infty$. If $D_P > 0$, then all three real roots $z_3 < z_2 < z_1$ of $P(z)$ are positive. If $(\beta^\top)^2 - 3\Psi_1\Psi_3 > 0,\ \beta^\top < 0$ and $\Psi_1 > 0$, then both real roots $z_4 > z_5$ of $P'(z)$ are positive. Thus, conditions $\Psi_1 > 0,\ \Psi_2 > 0,\ \Psi_3 > 0,\ \beta < 0$ and $D_P > 0$ guarantee existence of a stable stationary solution $y_2 = z_2^2 > 0$ of (7.1), Figure 1(a). The basin of a single-point attractor $y = y_2$ for the semigroup (7.1) is the invariant set of continuous functions $y(t)$, with values in $(y_3,\, y_1)$.

(b) Let $\Psi_3 < 0$. The cubic polynomial $P(z)$ obeys $P(0) = -\Psi_2 < 0,\ P(\infty) = -\infty$. Its maximal real root z_2 is an attractor for (7.1). The condition $D_P > 0$ and the fact that the maximal root z_0 of the derivative P' is positive provide $z_2 > 0$ (and $z_1 > 0$ is the minimal positive root of P). If $\beta > 0,\ \Psi_1 < 0$ and $\beta^2 - 3\,\Psi_1\Psi_3 > 0$ (the discriminant of P' is positive) then both roots of $P'(z) = 3\,\Psi_3 z^2 + 2\,\beta z + \Psi_1$ are real and the maximal root z_0 is positive. The condition $D_P > 0$ implies $\beta^2 - 3\,\Psi_1\Psi_3 > 0$. Thus, conditions $\Psi_1 < 0,\ \Psi_2 > 0,\ \Psi_3 < 0,\ \beta > 0$ and $D_P > 0$ guarantee existence of a stable stationary solution $y_2 = z_2^2 > 0$ of (7.1), see Figure 1(b). Note that $f(y)$ is concave for $y > 0$, and $f'(y)$ is monotone decreasing (with $f'(0+) = \infty$ and $f'(\infty) = -\infty$) and has one positive root. The basin of a single-point attractor $y = y_2$ for the semigroup of (7.1) is the (invariant) set of continuous functions $y(t) > y_1$.

(c) Let $\Psi_3 = 0$. Then $P(z) = \beta\, z^2 + \Psi_1\, z - \Psi_2$. A positive root $\tilde z$ of $P(z)$ yields a stationary solution $\tilde y = (\tilde z)^{1/2}$ of (7.1). If $P'(\tilde z) < 0$ then $\tilde y$ is a single-point attractor.

(c_1) Let $\beta < 0$. Then $P(0) = -\Psi_2 < 0$ and $P(\infty) = -\infty$. Thus, $P(z)$ has real roots if and only if $P(z_0) > 0$, where $z_0 = -\Psi_1/\beta$ is a root of $P'(z) = 0$. The inequality $P(z_0) > 0$ holds when $-(\Psi_1)^2/(4\Psi_2) < \beta < 0$. Maximal root y_2 of $f(y) = 0$ is asymptotically stable; $f'(y)$ has a unique positive root y_3, and $f'(y)$ takes minimum at y_4, Figure 2. If $-4\,\beta\Psi_2 = \Psi_1^2$ then (7.1) has one positive stationary solution, and has no stationary solutions if $-4\,\beta\Psi_2 > \Psi_1^2$.

(c_2) Let $\beta > 0$. Then $P(0) = -\Psi_2 < 0$ and $P(\infty) = \infty$. Thus, $P(z)$ has one positive root z_2, which corresponds to unstable stationary solution of (7.1), because $P'(z_2) > 0$. For $\beta = 0$, (7.1) has a unique positive stationary solution, it is unstable.

(c_3) Let $\Psi_2 = 0$. Then $f(y) = \beta\, y + \Psi_1 y^{-1}$. If $\beta \geq 0$ then (7.1) has no positive stationary solutions. If $\beta < 0$ and $\Psi_1 > 0$ then $f(y) = 0$ has one positive root $y_2 = (\Psi_1/|\beta|)^{1/2}$. The solution y_1 is stable (attractor) because $f'(y_2) < 0$.

Example 7.1. Let F be a circle S^1 of length l. Then (1.8) with $\Psi_3 = 0$ is the Cauchy's problem

$$u_{,t} = u_{,xx} + f(u), \quad u(x,0) = u_0(x) > 0 \quad (x \in S^1,\ t \geq 0),$$

where $f(u)=\beta u+\Psi_1 u^{-1}-\Psi_2 u^{-3}$. The stationary equation with $u(x)$ yields

$$u'' + f(u) = 0, \quad u(0) = u(l), \quad u'(0) = u'(l), \quad l > 0. \tag{7.2}$$

We rewrite (7.2) as the dynamical system

$$u' = v, \quad v' = -f(u) \quad (u > 0). \tag{7.3}$$

Periodic solutions of (7.2) correspond to solutions of (7.3) with the same period. The system (7.3) is Hamiltonian, since $\partial_u v = \partial_v f(u)$, its Hamiltonian $\mathrm{H}(u,v)$ (the first integral) solves the system $\partial_u \mathrm{H}(u,v) = f(u)$, $\partial_v \mathrm{H}(u,v) = v$. Thus, $\mathrm{H}(u,v) = \frac{1}{2}(v^2 + \beta u^2) + \Psi_1 \ln u + \frac{1}{2}\Psi_2\, u^{-2}$. The trajectories of (7.3) belong to level lines of $\mathrm{H}(u,v)$. Consider three cases.

Case 1. Let $\beta < 0$. Then (7.3) has two fixed points: $(y_i, 0)$ $(i = 1,2)$ with $y_1 > y_2$. To clear up the type of fixed points, we linearize (7.3) at $(y_i, 0)$, $\vec{\eta}\,' = A_i\,\vec{\eta}$, $A_i = \begin{pmatrix} 0 & 1 \\ -f'(y_i) & 0 \end{pmatrix}$. Since $f'(y_1) < 0$ and $f'(y_2) > 0$, the point $(y_1,\ 0)$ is a "saddle" and $(y_2,\ 0)$ is a "center". The separatrix is $\mathrm{H}(u,v) = \mathrm{H}(y_1, 0)$, i.e.,

$$v^2 = |\beta|(u^2 - y_1^2) - 2\,\Psi_1 \ln(u/y_1) - \Psi_2(u^{-2} - y_1^{-2}),$$

see Figure 3(a). The separatrix divides the half-plane $u > 0$ into three simply connected areas. Then $(y_2,\ 0)$ is a unique minimum point of H in $D = \{(u,\ v) :\ \mathrm{H}(u,v) < \mathrm{H}(y_1, 0),\ 0 < u < y_1\}$. The phase portrait of (7.3) in D consists of the cycles surrounding the fixed point $(y_2,\ 0)$, all correspond to non-constant solutions of (7.2) with various l. Other two areas do not contain cycles.

Case 2. Let $\beta \geq 0$. Then (7.3) has one fixed point $(y_1, 0)$ and $f'(y_1) > 0$. Hence, $(y_1, 0)$ is a "center". Since $(y_1,\ 0)$ is a unique minimum of $\mathrm{H}(u,v)$ in the semiplane $u > 0$, the phase portrait of (7.3) consists of cycles surrounding $(y_1,\ 0)$, all correspond to non-constant solutions of (7.2) with various l, Figure 3(b).

For $\Psi_2 = 0$ and $\Psi_1 > 0$, the Hamiltonian of (7.3) is $\mathrm{H}(u,v) = \frac{1}{2}(v^2 + \beta\, u^2) + \Psi_1 \ln u$. Solving $\mathrm{H}(u,v) = C$ with respect to v and substituting to $(7.3)_1$, we get $u' = \sqrt{-\beta\, u^2 - 2\,\Psi_1 \ln u + 2\,C}$. If $\beta \geq 0$ then (7.3) has no cycles (since it has no fixed points); hence, (7.2) has no solutions. If $\beta < 0$, then the separatrix $\mathrm{H}(u,v) = \mathrm{H}(u_*, 0)$ is $v^2 = |\beta|(u^2 - u_*^2) - 2\,\Psi_1 \ln(u/u_*)$, (7.3) has a unique fixed point $(u_*, 0)$ which is a "saddle". The separatrix divides the half-plane $u > 0$ into four simply connected areas. Each one has no fixed points of (7.3), hence the system has no cycles. We conclude that u_* is a unique solution of (7.2).

Case 3. Consider (7.2) for $\Psi_1 = 0$, $\Psi_2 > 0$ and $l = 2\pi$. Set $p = u'$ and represent $p = p(u)$ as a function of u. Then $u'' = dp/du$ and

$$(p^2)' = -2\,\beta\, u + 2\,\Psi_2\, u^{-3} \ \Rightarrow\ (u')^2 = C_1 - \beta\, u^2 - \Psi_2\, u^{-2}.$$

After separation of variables and integration, we obtain

$$u = \begin{cases} \big(\frac{C_1}{2\beta} + \frac{1}{2\beta}\sqrt{C_1^2 - 4\beta\,\Psi_2}\sin(2\sqrt{\beta}(x + C_2))\big)^{1/2},\ (C_1^2 \geq 4\beta\Psi_2) & \text{for } \beta > 0, \\ \big(-\frac{C_1}{2|\beta|} + \frac{1}{2|\beta|}\sqrt{C_1^2 + 4|\beta|\Psi_2}\cosh(2\sqrt{|\beta|}(x + C_2))\big)^{1/2} & \text{for } \beta < 0, \\ \sqrt{\Psi_2/C_1 + C_1(x + C_2)^2} & \text{for } \beta = 0. \end{cases}$$

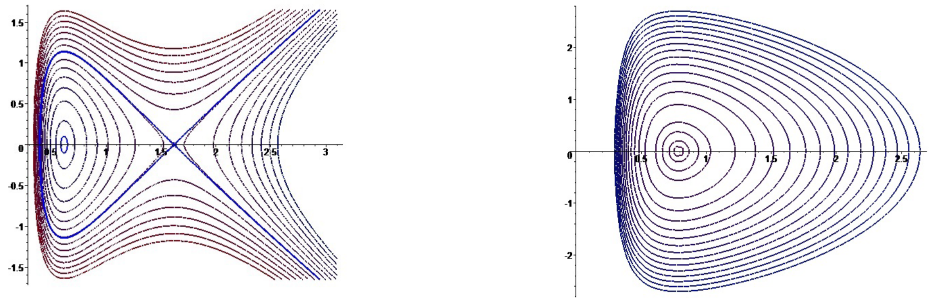


Figure 3: Example 7.1: 1. $\beta < 0$, 2. $\beta > 0$.

Thus, for $\beta \le 0$, (7.2) has no positive solutions, and for $\beta > 0$ the solution is 2π-periodic and positive only if

(a) $\beta \ne \frac{k^2}{4}$ $(k \in \mathbb{N})$ and $C_1 = 2\,(\beta\,\Psi_2)^{1/2}$; a solution $u_* = (\Psi_2/\beta)^{1/4}$ is unique, or

(b) $\beta = \frac{k^2}{4}$ $(k \in \mathbb{N})$; all solutions form a 2-dimensional manifold

$$u_0(C_1, C_2) = (1/k)\big(2C_1 + 2(C_1^2 - n^2\Psi_2)^{1/2}\sin(k(x + C_2))\big)^{1/2}.$$

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Author's address:

Vladimir Rovenski
Department of Mathematics, University of Haifa,
Mount Carmel, 31905 Haifa, Israel.
E-mail: vrovenski@univ.haifa.ac.il