Invariant connections on Lie groups (I)

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Abstract. In this paper, we study the sets of the left invariant and of the bi-invariant connections on Lie groups, endowed with some additional properties: symmetry, flatness, Ricci-flatness, etc. Moreover, we give some new examples in low dimensions for some special types of affine connections.

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1 Introduction

On Lie groups, the invariant geometries are an important tool for testing conjectures and for classifying different differential/affine/metric objects. In particular, affine connections that are left or bi-invariant with respect to translations were considered in many papers ([8], [7], [1], [14],[15], [16], etc).For compact Lie groups, the set of bi-invariant connections was classified by Laquer ([5],[6]). We are not aware of a similar result, in the non-compact case.

For a n-dimensional Lie group G, the left-invariant connections are completely modelled as (1,2)-tensors on the Lie algebra $L(G)$, thus their set may be identified with $\mathbb{R}^{n^3}$. When additional properties are considered, this set reduces; our aim is to determine how and why.

There exist similar studies for specific families of affine connections on differentiable manifolds, but the techniques and results are of a completely different nature ([2], [3],[4]).

In this paper, we study (some sub-) sets of invariant connections and to what extent they may classify the Lie algebras or the Lie groups. In §2 we determine the sets of symmetric left-invariant connections, of the flat connections and of the symmetric and flat connections, respectively; examples are given in dimensions 2 and 3; we suggest two new conjectures.

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In §3 we define the mixed flat affine differential manifolds and we give examples of such (left-invariants) structures on Lie groups of low dimensions; the set of all the mixed flat left-invariant connections is determined in the general case.

In §4 we characterize the sets of left-invariant connections which are: Ricci-flat, symmetric and Ricci-flat, Ricci-symmetric, symmetric and Ricci-symmetric respectively.

In §5 we determine the sets of bi-invariant connections on the 2-dimensional non-commutative Lie group, which are symmetric or flat; there exists a unique symmetric and flat connection. On the Heisenberg group, we find the sets of all the bi-invariant connections, as well as the subset of symmetric ones.

2 The setting

Consider G a n-dimensional Lie group and $L(G)$ its Lie algebra. For each $a \in G$, we denote by L_a and R_a the left and right translations on G associated to a, given by $L_a(x) = ax$ and $R_a(x) = xa$, for all $x \in G$. An affine connection ∇ on G is left-invariant if, for any vector fields X and Y on G,

$$\nabla_{(L_a)_* X}(L_a)_* Y = (L_a)_* \nabla_X Y.$$

The right-invariant connections are defined in a similar manner. A connection is called bi-invariant if it is simultaneously left and right-invariant.

We denote by $\mathcal{C}(G)$, $\mathcal{C}(G)_l$, $\mathcal{C}(G)_{sl}$, $\mathcal{C}(G)_b$, $\mathcal{C}(G)_{sb}$ the sets of (affine) connections, of left-invariant, of symmetric left-invariant, of bi-invariant and of symmetric bi-invariant ones.

Let fix a basis $\{E_1, ..., E_n\}$ of $L(G)$. Each $\nabla \in \mathcal{C}(G)_l$ may be writen as $\nabla_{E_i} E_j = \Gamma_{ij}^k E_k$, for every $i, j = \overline{1,n}$, with real coefficients Γ_{ij}^k.

The set $\mathcal{C}(G)_l$ is in one-to-one correspondence with the set of (1,2)-tensor fields on $L(G)$, so may be identified with the real vector space $\mathbb{R}^{n^3}$. All the connections studied hereafter will belong to this "ambient space".

Proposition 2.1. *The set $\mathcal{C}(G)_{sl}$ is an affine subspace in $\mathcal{C}(G)_l$, of affine dimension $n^2(n+1)/2$. Moreover, $\mathcal{C}(G)_{sl}$ is a linear subspace in $\mathcal{C}(G)_l$ if and only if G is commutative.*

Proof. Consider a basis $\{E_1, ..., E_n\}$ of $L(G)$ with structural constants c_{ij}^k, for $i, j, k = \overline{1,n}$. Write $\nabla \in \mathcal{C}(G)_{sl}$ as $\nabla_{E_i} E_j = \Gamma_{ij}^k E_k$, for every $i, j = \overline{1,n}$, with real coefficients Γ_{ij}^k, such that

$$\Gamma_{ij}^k - \Gamma_{ji}^k - c_{ij}^k = 0. \tag{2.1}$$

We have here a system of n^3 affine equations in the unknowns Γ_{ij}^k. It follows that the coefficients Γ_{ij}^k parameterize $\mathcal{C}(G)_{sl}$ as an affine subspace in $\mathbb{R}^{n^3}$, of affine dimension $n^2(n+1)/2$. The second property is obvious, as all c_{ij}^k vanish. We point out also that these two properties do not depend on the choice of the basis in $L(G)$. □

Proposition 2.2. (i) *The set of flat connections in $\mathcal{C}(G)_l$ is the (non-void) intersection of $n^2(n^2-1)/3$ hyperquadrics in $\mathbb{R}^{n^3}$, for $n \geq 1$. Moreover, all these hyperquadrics have a center in the origin if and only if G is commutative.*

(ii) *The set of local Euclidean (i.e. symmetric and flat) connections in $\mathcal{C}(G)_{sl}$ is the intersection of $n^2(n^2-1)/3$ hyperquadrics with $n^2(n-1)/2$ affine hyperplanes in $\mathbb{R}^{n^3}$, for $n \geq 1$. Moreover, this set contains the origin if and only if G is commutative.*

Proof. (i) Fix a basis $\{E_1, ..., E_n\}$ of $L(G)$; denote by c^i_{jk} the structural constants, by Γ^k_{ij} the coefficients of an arbitrary flat left invariant connection on G and by R^i_{jkl} the components of its curvature tensor field R. The vanishing of R yields to

$$\Gamma^s_{ki}\Gamma^l_{js} - \Gamma^s_{ji}\Gamma^l_{ks} - c^s_{jk}\Gamma^l_{si} = 0, \tag{2.2}$$

for all $i, j, k, l = \overline{1, n}$. This system of (apparently) n^4 quadratic equations depends on the n^3 unknowns Γ^i_{jk}; in fact, only $n^2(n^2-1)/3$ equations are effective, because exactly $n^2(n^2-1)/3$ coefficients R^i_{jkl} are independent, and the others may be deduced from them (see, for example [18]). Each equation in (2.2) defines an affine hyperquadric in $\mathbb{R}^{n^3}$; the quadratic part is independent of G (depends only on n), but the linear part depends on G, through the structural constants.

The linear part $c^s_{jk}\Gamma^l_{si}$ vanishes, for all $i, j, k, l = \overline{1, n}$ and for all coefficients of the connections if and only if all the structural constants vanish.

The system (2.2) is compatible, as it admits the trivial solution (which corresponds to the Cartan-Schouten connection ∇^-).

(ii) From (2.1) we deduce the condition involving the $n^2(n-1)/2$ affine hyperplanes in $\mathbb{R}^{n^3}$, as well as the characterization concerning the case when all the structural constants vanish. □

Remark 2.3. Suppose $n = 2$ and G commutative.

(i) We may reduce the parameterizing set of flat connections in $\mathcal{C}(G)_l$ to the set S of solutions of the ("minimal") system of equations

$$R^1_{112} = 0 \ , \ R^1_{212} = 0 \ , \ R^2_{112} = 0 \ , \ R^2_{212} = 0.$$

In other words,

$$\Gamma^2_{21}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22} = 0$$

$$\Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} - \Gamma^1_{12}\Gamma^1_{21} - \Gamma^2_{12}\Gamma^1_{22} = 0$$

$$\Gamma^2_{12}\Gamma^2_{21} + \Gamma^1_{21}\Gamma^2_{11} - \Gamma^1_{11}\Gamma^2_{21} - \Gamma^2_{11}\Gamma^2_{22} = 0$$

$$\Gamma^1_{22}\Gamma^2_{11} - \Gamma^1_{12}\Gamma^2_{21} = 0, \qquad \text{(redundant)}$$

in the eight unknowns $\Gamma^1_{11}, \Gamma^1_{12}, \Gamma^1_{21}, \Gamma^1_{22}, \Gamma^2_{11}, \Gamma^2_{12}, \Gamma^2_{21}, \Gamma^2_{22}$.

For simplicity, we denote the variables $x^1 = \Gamma^1_{11}$, $x^2 = \Gamma^1_{12}$, $x^3 = \Gamma^1_{22}$, $x^4 = \Gamma^2_{11}$, $x^5 = \Gamma^2_{12}$, $x^6 = \Gamma^2_{22}$, $x^7 = \Gamma^1_{21}$, $x^8 = \Gamma^2_{21}$. The set of flat left invariant connections in G is parameterized by the set S of the solutions of the following system of quadratic equations in $\mathbb{R}^8$:

$$x^8x^2 - x^4x^3 = 0$$
$$x^3x^1 + x^6x^2 - x^2x^7 - x^5x^3 = 0$$
$$x^1x^8 + x^6x^4 - x^5x^8 - x^7x^4 = 0.$$

An elementary calculation determines S, as the union of the following submanifolds in $\mathbb{R}^8$, of dimension 6,5,5 and 4, respectively:

$$\{(x^1, x^2, x^8x^2(x^4)^{-1}, x^4, x^5, x^7+x^8(x^5-x^1)(x^4)^{-1}, x^7, x^8) \mid x^1, x^2, x^4, x^5, x^7, x^8 \in \mathbb{R}, x^4 \neq 0\}$$
$$\{(x^1, x^2, x^3, 0, x^5, x^7 + x^3(x^5 - x^1)(x^2)^{-1}, x^7, 0) \mid x^1, x^2, x^3, x^5, x^7 \in \mathbb{R}, x^2 \neq 0\}$$

$$\{(x^1, 0, x^3, 0, x^1, x^6, x^7, x^8) \mid x^1, x^3, x^6, x^7, x^8 \in \mathbb{R}\} \ \ (\text{a } 5-\text{plane})$$
$$\{(x^1, 0, 0, 0, x^5, x^6, x^7, 0) \mid x^1, x^5, x^6, x^7 \in \mathbb{R}\} \ \ (\text{a } 4-\text{plane})$$

(ii) We may reduce the parameterizing set of symmetric flat connections in $\mathcal{C}(G)_l$ to the set of solutions of the ("minimal") system of equations

$$R^1_{112} = 0 \ \ , \ \ R^1_{212} = 0 \ \ , \ \ R^2_{112} = 0 \ \ , \ \ R^2_{212} = 0 \ \ , \ \ \Gamma^1_{12} = \Gamma^1_{21} \ \ , \ \ \Gamma^2_{12} = \Gamma^2_{21}.$$

In other words

$$\Gamma^2_{12}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22} = 0$$

$$\Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} - (\Gamma^1_{12})^2 - \Gamma^2_{12}\Gamma^1_{22} = 0$$

$$\Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - (\Gamma^2_{12})^2 - \Gamma^1_{12}\Gamma^2_{11} = 0$$

$$\Gamma^1_{22}\Gamma^2_{11} - \Gamma^1_{12}\Gamma^2_{12} = 0 \qquad (\text{redundant})$$

in the six unknowns $\Gamma^1_{11}, \Gamma^1_{12}, \Gamma^1_{22}, \Gamma^2_{11}, \Gamma^2_{12}, \Gamma^2_{22}$.

For simplicity, we denote the variables $x^1 = \Gamma^1_{11}$, $x^2 = \Gamma^1_{12}$, $x^3 = \Gamma^1_{22}$, $x^4 = \Gamma^2_{11}$, $x^5 = \Gamma^2_{12}$, $x^6 = \Gamma^2_{22}$. The set of symmetric flat left invariant connections in G is parameterized by the set S of the solutions of the following system of quadratic equations in $\mathbb{R}^6$:

$$x^5x^2 - x^4x^3 = 0$$
$$x^3x^1 + x^6x^2 - (x^2)^2 - x^5x^3 = 0$$
$$x^5x^1 + x^6x^4 - (x^5)^2 - x^2x^4 = 0$$

An elementary calculation determines S, as the union of the following submanifolds in $\mathbb{R}^6$, of dimension 4,3,3 and 2, respectively:

$$\{(x^1, x^2, x^5x^2(x^4)^{-1}, x^4, x^5, x^2 + x^5(x^5 - x^1)(x^4)^{-1}) \mid x^1, x^2, x^4, x^5 \in \mathbb{R}, x^4 \neq 0\}$$

$$\{(x^1, x^2, x^3, 0, 0, x^2 - x^3x^1(x^2)^{-1}) \mid x^1, x^2, x^3 \in \mathbb{R}, x^2 \neq 0\}$$

$$\{(x^1, 0, x^3, 0, x^1, x^6) \mid x^1, x^3, x^6 \in \mathbb{R}\} \ \ (\text{a } 3-\text{plane})$$

$$\{(x^1, 0, 0, 0, 0, x^6) \mid x^1, x^6 \in \mathbb{R}\} \ \ (\text{a } 2-\text{plane})$$

(Another method consists in particularizing $x^7 := x^2$ and $x^8 := x^5$ in (i).)

Remark 2.4. Suppose $n = 2$ and G non-commutative. We may choose a basis $\{E_1, E_2\}$ of $L(G)$ such that $[E_1, E_2] = E_1$.

(i) We may reduce the parameterizing set of flat connections in $\mathcal{C}(G)_l$ to the set of solutions of the ("minimal") system of equations

$$R^1_{112} = 0 \ \ , \ \ R^1_{212} = 0 \ \ , \ \ R^2_{112} = 0 \ \ , \ \ R^2_{212} = 0$$

In other words

$$\Gamma^2_{21}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22} - \Gamma^1_{11} = 0$$

$$\Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} - \Gamma^1_{12}\Gamma^1_{21} - \Gamma^2_{12}\Gamma^1_{22} - \Gamma^1_{12} = 0$$

$$\Gamma^2_{12}\Gamma^2_{21} + \Gamma^1_{21}\Gamma^2_{11} - \Gamma^1_{11}\Gamma^2_{21} - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^2_{11} = 0$$

$$\Gamma^1_{22}\Gamma^2_{11} - \Gamma^1_{12}\Gamma^2_{21} - \Gamma^2_{12} = 0$$

in the eight unknowns $\Gamma^1_{11}, \Gamma^1_{12}, \Gamma^1_{21}, \Gamma^1_{22}, \Gamma^2_{11}, \Gamma^2_{12}, \Gamma^2_{21}, \Gamma^2_{22}$.

For simplicity, we denote the variables $x^1 = \Gamma^1_{11}$, $x^2 = \Gamma^1_{12}$, $x^3 = \Gamma^1_{22}$, $x^4 = \Gamma^2_{11}$, $x^5 = \Gamma^2_{12}$, $x^6 = \Gamma^2_{22}$, $x^7 = \Gamma^1_{21}$, $x^8 = \Gamma^2_{21}$. The set of flat left invariant connections in G is parameterized by the set S of the solutions of the following system of quadratic equations in $\mathbb{R}^8$:

$$x^8x^2 - x^4x^3 - x^1 = 0$$

$$x^3x^1 + x^6x^2 - x^2x^7 - x^5x^3 - x^2 = 0$$

$$x^1x^8 + x^6x^4 - x^5x^8 - x^7x^4 + x^4 = 0$$

$$x^4x^3 - x^8x^2 - x^5 = 0.$$

We obtain S as the union of the following submanifolds in $\mathbb{R}^8$, of dimension 4,4,3 and 3 respectively:

$$\{(\pm\sqrt{-x^2x^4}, x^2, \mp\frac{x^2(x^6-x^7-1)}{2\sqrt{-x^2x^4}}, x^4, \mp\sqrt{-x^2x^4}, x^6,$$

$$x^7, \mp\frac{x^4(x^6-x^7+1)}{2\sqrt{-x^2x^4}}) \mid x^2, x^4, x^6, x^7 \in \mathbb{R}, x^2x^4 < 0\}$$

$$\{(0,0,x^3,0,0,x^6,x^7,x^8) \mid x^3,x^6,x^7,x^8 \in \mathbb{R}\}$$

$$\{(0,x^2,x^3,0,0,x^7+1,x^7,0) \mid x^2,x^3,x^7 \in \mathbb{R} \;\; , \;\; x^2 \neq 0\}$$

$$\{(0,0,0,x^4,0,x^6,x^6+1,x^8) \mid x^4,x^6,x^8 \in \mathbb{R}.\}$$

(ii) We want to parameterize the set of symmetric flat connections in $\mathcal{C}(G)_l$, so we suppose, in addition to (i), that (1.1) holds. As the only non-null structural constants are $c_{12}^1 = -c_{21}^1 = 1$, we deduce

$$\Gamma_{12}^1 - \Gamma_{21}^1 - 1 = 0 \;\; , \;\; \Gamma_{12}^2 = \Gamma_{21}^2 .$$

We obtain S as the union of the following submanifolds in $\mathbb{R}^8$, of dimension 2,2,1 and 1 respectively:

$$\{(\pm\sqrt{-x^2x^4}, x^2, \pm\frac{x^2(x^2+1)}{\sqrt{-x^2x^4}}, x^4, \mp\sqrt{-x^2x^4}, -x^2-2,$$

$$x^2-1, \mp\sqrt{-x^2x^4}) \mid x^2, x^4 \in \mathbb{R}, x^2x^4 < 0\}$$

$$\{(0,0,x^3,0,0,x^6,-1,0) \mid x^3,x^6 \in \mathbb{R}\}$$

$$\{(0,-1,x^3,0,0,-1,-2,0) \mid x^3 \in \mathbb{R}\}$$

$$\{(0,0,0,x^4,0,-2,-1,0) \mid x^4 \in \mathbb{R}\}.$$

Remark 2.5. Suppose $n = 3$ and G commutative.

(i) We may reduce the parameterizing set of flat connections in $\mathcal{C}(G)_l$ to the set S of solutions of the (”minimal”) system of 24 equations

$$R_{112}^i = 0 \;\; , \;\; R_{113}^i = 0 \;\; , \;\; R_{123}^i = 0 \;\; , \;\; R_{212}^i = 0$$

$$R_{213}^i = 0 \;\; , \;\; R_{223}^i = 0 \;\; , \;\; R_{313}^i = 0 \;\; , \;\; R_{323}^i = 0$$

for $i \in \{1,2,3\}$ in the 27 unknowns Γ_{jk}^i. In other words

$$\Gamma_{11}^1\Gamma_{21}^i + \Gamma_{11}^2\Gamma_{22}^i + \Gamma_{11}^3\Gamma_{23}^i - \Gamma_{21}^1\Gamma_{11}^i - \Gamma_{21}^2\Gamma_{12}^i - \Gamma_{21}^3\Gamma_{13}^i = 0,$$

for $i \in \{1,2,3\}$ and another 21 similar equations (each of them defining a quadratic variety in $\mathbb{R}^{27}$). The system is compatible, as it admits the trivial solution ∇^-, with all the coefficients null.

(ii) We may now reduce the parameterizing set of symmetric flat connections in $\mathcal{C}(G)_l$, by adding the 9 equations $\Gamma_{jk}^i = \Gamma_{kj}^i$ for $i,j,k \in \{1,2,3\}$, $j < k$, to the system from the previous remark. This new system is also compatible, as it admits the solution ∇^-.

Remark 2.6. Suppose $n = 3$ and G non-commutative. The study becomes more complex(and, due to the lack of space, will be carried out elsewhere), as we must take into account the classification of the 3-dimensional Lie algebras ([8], [12], [18]),for the following non-commutative Lie groups: the Heisenberg group H^3,the orthogonal group $O(3)$, the special linear group $SL(2,\mathbb{R})$, the Lorentz group $O(1,2)$, the Euclidean motions group $E(2)$ and the Minkowski motions group $E(1,1)$.

We sketch here the case of the Heisenberg group H^3. Fix a basis $\{E_1, E_2, E_3\}$ of $L(H^3)$ such that $[E_1, E_2] = E_3$.

The set of flat connections in $\mathcal{C}(H^3)_l$ is given by a system of 24 equations in the 27 unknowns Γ^i_{jk}.

For the set of symmetric and flat connections in $\mathcal{C}(H^3)_l$, we must add the following 8 equations

$$\Gamma^3_{12} - \Gamma^3_{12} - 1 = 0\ ,\ \Gamma^1_{12} = \Gamma^1_{21}\ ,\ \Gamma^1_{13} = \Gamma^1_{31}\ ,\ \Gamma^1_{23} = \Gamma^1_{32}$$

$$\Gamma^2_{12} = \Gamma^2_{21}\ ,\ \Gamma^2_{13} = \Gamma^2_{31}\ ,\ \Gamma^2_{23} = \Gamma^2_{32}\ ,\ \Gamma^3_{12} = \Gamma^3_{21}\ ,\ \Gamma^3_{13} = \Gamma^3_{31},$$

and we obtain a system of 33 equations in the 27 unknowns Γ^i_{jk}.

The set of Ricci-flat connections in $\mathcal{C}(H^3)_l$ is given by a system of 9 equations in the 27 unknowns Γ^i_{jk}; for the symmetric and Ricci-flat connections we obtain a system of 18 equations.

The set of symmetric and Ricci-flat connections in $\mathcal{C}(H^3)_l$ is given by a system of 9 equations in the 27 unknowns Γ^i_{jk}; for the symmetric and Ricci-flat connections we obtain a system of 18 equations.

The set of Ricci-symmetric connections in $\mathcal{C}(H^3)_l$ is given by a system of 3 equations in the 27 unknowns Γ^i_{jk}; for the symmetric and Ricci-symmetric connections we obtain a system of 12 equations.

The set of symmetric and Ricci-symmetric connections in $\mathcal{C}(H^3)_l$ is given by a system of 3 equations in the 27 unknowns Γ^i_{jk}; for the symmetric and Ricci-symmetric connections we obtain a system of 12 equations.

Remark 2.7. Suppose $n \geq 4$. The system (2.2) admits always a non-trivial solution (hence a line of solutions), for example for the Cartan-Schouten connection ∇^0, which is half of the Lie bracket on $L(G)$.

(i) Instead, it is not evident at all if the system (2.1)+(2.2) admits always a solution, as for growing n it becomes overdetermined (with $n^2(n-1)(2n+5)/6$ equations vs. n^3 unknowns). This is an important topic, which gave rise to the study of the left structures (affine structures) and suggested, for example, the Auslander-Milnor's conjecture ([9]):"*On each solvable Lie group there exists a symmetric and flat left-invariant connection*". This conjecture was refuted through counterexamples constructed on filiform algebras. We may weaken the Auslander-Milnor conjecture, through the following two conjectures:

Conjecture 1. On each Lie group there exists a symmetric and Ricci-flat left-invariant connection.

Conjecture 2. On each Lie group there exists a symmetric and Ricci-symmetric left-invariant connection.

Obviously, Conjecture 1 true implies Conjecture 2 true. (For the Conjecture 1, the 2-dimensional case is not interesting, as (in this case) every Ricci-flat connection is flat.)

(iii) Another interesting problem is to determine all the non-flat left-invariant connections which are Ricci-flat. The Riemannian case is solved (there are none !); in the indefinite case, there exist such connections. In the following we shall deal with the pure affine case.

Remark 2.8. A left-invariant connection $\nabla \in \mathcal{C}(G)_l$ is called Cartan connection ([13]) if $\nabla_X X = 0$ for any $X \in L(G)$. It is easy to prove that any such connection is of the form

$$\nabla_X Y = \frac{1}{2}([X,Y] + T(X,Y)), \tag{2.3}$$

for any $X, Y \in L(G)$ and any skew-symmetric tensor field T of type (1,2) on $L(G)$. (It follows that extending T gives precisely the torsion field of ∇.) As examples we have the classical Cartan-Schouten connections ∇^-, ∇^+, ∇^0 and any other one collinear with them. Obviously, we have: (i) the set of left-invariant Cartan connections may be parameterized by $\mathbb{R}^{\frac{n^2(n-1)}{2}}$, where $n = dimG$; (ii) there exists a unique symmetric left-invariant Cartan connection, namely ∇^0 (in fact it is even bi-invariant);(iii) the Propositions 2.1 and 2.2,(i) may be re-written accordingly; (iv) as a Cartan connection satisfies $\nabla_X Y + \nabla_Y X = 0$, for every $X, Y \in L(G)$, we see that such connection is the skew-symmetric analogue of symmetric (i.e. torsion-free) one.

3 Mixed flat connections

Let M be a n-dimensional differentiable manifold and ∇ a linear (affine) connection on M. Denote R, Ric and T the curvature, the Ricci and the torsion tensor fields of ∇, respectively. We define a (1,3)-tensor field on M, by $U(X,Y)Z := R(X,Y)Z - (n-1)^{-1}\{Ric(Y,Z)X - Ric(X,Z)Y\}$. We call U *the Riemann-Ricci tensor field*; it is skew-symmetric in the first two variables and traceless in the third variable.

Definition 3.1. The affine differentiable manifold (M, ∇) is called *mixed flat* if U identically vanishes.

In the following, we make some comments and give some examples. More details about the geometry of mixed flat manifolds as well as their applications in the Theory of Relativity will appear elsewhere ([15]).

Remark 3.2. (i) Obviously, every flat affine connection on a differentiable manifold is mixed flat; every mixed flat and Ricci flat affine connection must be flat.

(ii) In dimension 2, any affine differentiable manifold (M, ∇) is mixed flat.

(iii) Consider a mixed flat affine differentiable manifold (M, ∇). If $T = 0$, then Ric is symmetric and cyclic-parallel.

(iv) Let (N, g) be a semi-Riemannian manifold of dimension greater than 3 and ∇ its Levi-Civita connection. Then (N, ∇) is mixed flat if and only if (N, g) has constant sectional curvature. It follows that this new notion is irrelevant for the semi-Riemannian geometry. Nonetheless, the notion of mixed flatness is important in the non-Riemannian case, as it extends the notion of constant sectional curvature beyond the frontiers of metric theories.

(v) Let G be a n-dimensional Lie group and ∇ a left-invariant connection on G. Altogether with R and Ric, the tensor field U is also left-invariant.

The study of mixed flat connections on Lie groups is particularly relevant, because the examples of Lie groups, which admit left-invariant Riemannian metrics with constant sectional curvature, are quite rare (cf. [8]).

Consider the Cartan-Schouten connections ∇^-, ∇^+, ∇^0. We remark that ∇^- and ∇^+ are always flat, hence they are also mixed flat. The connection ∇^0 is mixed flat if and only if

$$(n-1)[[X, Y], Z] = B(Y, Z)X - B(X, Z)Y, \tag{3.1}$$

for every $X, Y, Z \in L(G)$, where B is the Killing form on G. Of course, this relation holds for every Levi-Civita connection of a bi-invariant metric with constant sectional curvature, as previously pointed out.

In the sequel we provide an example when this condition holds, in the affine differential setting.

Proposition 3.3. (i) *The set of mixed flat connections in $\mathcal{C}(G)_l$ is the (non-void) intersection of $n^2(n^2-1)/3$ hyperquadrics in $\mathbb{R}^{n^3}$, for $n \geq 1$. Moreover, if G is commutative, then all these hyperquadrics have a center in the origin.*

The proof is similar to that of Proposition 2.2.

4 On the symmetry of the Ricci tensor

Let M be a n-dimensional differentiable manifold and ∇ a linear (affine) connection on M. Denote R, Ric and T the curvature, the Ricci and the torsion tensor fields of ∇, respectively. In the affine differentiable setting, the symmetry of the Ricci tensor is a quite subtle property. It is known ([11]) that, for a symmetric connection ∇, the Ricci-symmetry is equivalent with the local existence of a volume form which is ∇-parallel (i.e. ∇ is locally equiaffine). We shall extend this result for arbitrary connections.

We define a 2-form t on M, by

$$t(Y, Z) = trace\{X \to \sum [T(T(X, Y), Z) + (\nabla_X T)(Y, Z)]\},$$

with cyclic sum after $X, Y, Z \in \mathcal{X}(M)$.

Remark 4.1. (i) We distinguish the special cases when the 2-form t is exact or closed. Each such property defines a new interesting family of affine differential manifolds.

(ii) Suppose there exists a one-form α on M such that $T(Y,Z) = \alpha(Z)Y - \alpha(Y)Z$ (i.e. ∇ is semi-symmetric). Then

$$t(Y,Z) = (\nabla_Y \alpha)Z - (\nabla_Y \alpha)Z$$

and $t = 0$ if and only if α is closed.

From the first Bianchi identity, by contracting, we get

$$Ric(Y,Z) - Ric(Z,Y) = -traceR(Y,Z) + t(Y,Z).$$

We get the following

Lemma 4.2. *The Ricci tensor is symmetric if and only if $traceR(Y,Z) = t(Y,Z)$, for every vector fields Y, Z.*

Let ω be a local volume element on M. Then, there exists a one-form τ such that $\nabla_Y \omega = \tau(Y)\omega$ (i.e. ω is a ∇-recurrent n-form, with recurrency factor τ). One knows (cf. [11]) that

$$R(Y,Z)\omega = -[traceR(Y,Z)]\omega.$$

On another hand, we derive

$$R(Y,Z)\omega = 2[d\tau(Y,Z)]\omega + \tau(T(Y,Z))\omega.$$

The last two relations lead to

$$traceR(Y,Z) + 2[d\tau(Y,Z)] + \tau(T(Y,Z)) = 0. \tag{4.1}$$

Theorem 4.3. *Let ∇ be an affine connection on the differentiable manifold M, with $t = 0$.*

(i) *If there exists a local ∇-parallel volume element, then Ric is symmetric.*

(ii) *Suppose Ric is symmetric and suppose there exists a (local) volume element with recurrency factor τ such that $ImT \subset ker\tau$. Then there exists a local ∇-parallel volume element on M.*

Proof. From Lemma 4.2, *Ric* is symmetric if and only if $traceR(Y,Z) = 0$ for every vector fields Y, Z.

(i) Let ω be a local ∇-parallel volume element on M, so its recurrency factor $\tau = 0$. From (4.1) it follows that $traceR(Y,Z) = 0$, which proves the symmetry of *Ric*.

(ii) Suppose *Ric* is symmetric and ω is the volume element with the required property. It follows that $traceR(Y,Z) = 0$, for every vector fields Y, Z. Relation (4.1) implies that

$$2[d\tau(Y,Z)] + \tau(T(Y,Z)) = 0.$$

From the hypothesis, $\tau(T(Y,Z)) = 0$, for every vector fields Y, Z. It follows that τ is an exact one-form; thus there exists a function f such that $\tau = -dlnf$. The volume element $f\omega$ is ∇-parallel. □

Remark 4.4. (i) The theorem 4.3.,(ii) generalizes the quoted result from [11], which can be recovered for $T = 0$.

(ii) The generalization is effective, as may be seen from the following example. Let consider a 2-dimensional non-commutative Lie group as in the Remark 2.4 with the formulas therein. Let ∇ be the left-invariant connection with all the components null, except $\Gamma^1_{11} = 1$ and $\Gamma^1_{12} = -1$. Then $Ric_{12} = Ric_{21}$ (hence Ric is symmetric); ∇ is not flat nor Ricci-flat, as $R^1_{112} = Ric_{12} = -1$. The connection ∇ is non-symmetric, as $T^1_{12} = -2$.

A short calculation proves that $t = 0$, so there exist non-trivial cases when the hypothesis in theorem 4.3.,(ii) applies.

(Here the connection ∇ is semi-symmetric, as in Remark 4.1., for the closed left-invariant one-form α such that $\alpha(E_2) = -2$ and $\alpha(E_1) = 0$).

(iii) For any Lie group G, the Cartan-Schouten connection ∇^- has the following properties: is bi-invariant; $T = -[,]$ on $L(G)$; $t = 0$. This is in agreement with theorem 4.3., as ∇^- is Ricci-flat (hence Ricci-symmetric) and *each* left-invariant volume element is ∇^--parallel.

This example shows that the Conjecture 1 is true if we replace the requirement $T = 0$ with $t = 0$.

5 The set of bi-invariant connections

Let G be a n-dimensional Lie group. The following (known) result characterizes the bi-invariant connections (see for example [13] for a partial sketch of proof).

Theorem 5.1. *For a left-invariant connection $\nabla \in \mathcal{C}_l(G)$ the following claims are equivalent:*

(i) ∇ *is bi-invariant;*

(ii) *for every $X, Y, Z \in L(G)$ we have*

$$[X, \nabla_Y Z] - \nabla_{[X,Y]} Z - \nabla_Y [X, Z] = 0; \tag{5.1}$$

(iii) ∇ *is ad-invariant, i.e., for every $X \in L(G)$ we have $ad_X \nabla = 0$;*

(iv) ∇ *is Ad-invariant, i.e., for every $X, Y \in L(G)$ and $a \in G$, we have $\nabla_{Ad_a X} Ad_a Y = Ad_a(\nabla_X Y)$.*

Remark 5.2. (i) It is well known that the set $\mathcal{C}(G)_b$ is always non-void, as it contains the Cartan-Schouten connections ∇^-, ∇^+ and ∇^0. (On $L(G)$, they act as the null operator, as the Lie bracket and half of the Lie bracket, respectively). The connection ∇^0 is symmetric, so $\mathcal{C}(G)_{sb}$ is also non-void. The connection ∇^- is flat, but in general it is not symmetric.

(ii) Using (2.3), we see that a (general) Cartan connection is bi-invariant if and only if the skew-symmetric tensor field T satisfies $[X, T(Y, Z)] - T([X, Y], Z) - T(Y, [X, Z]) = 0$ for every $X, Y, Z \in L(G)$ (i.e. T is *ad*-invariant). A non-trivial example is the following: consider ω a left-invariant one-form on G, vanishing on the derived algebra of $L(G)$ (i.e. $\omega([X, Y]) = 0$ for every $X, Y \in L(G)$).Define $T(X, Y) := \omega(X)Y - \omega(Y)X$.

(iii) In the following, we investigate the sets of all the bi-invariant connections, using the relation (3.1). We fix a basis $\{E_i \mid i = \overline{1,n}\}$ in $L(G)$ and denote c^i_{jk} the structural constants and by Γ^i_{jk} the coefficients of an arbitrary bi-invariant connection on G. Then (5.1) leads to the linear system

$$\Gamma^s_{ij} c^k_{sh} = \Gamma^k_{is} c^s_{jh} + \Gamma^k_{sj} c^s_{ih},$$

for every $i, j, k, h = \overline{1,n}$, with (at first sight) n^3 unknowns Γ^i_{jk} and n^4 equations.

(iv) For a commutative G, the sets $\mathcal{C}(G)_b$ and $\mathcal{C}(G)_l$ coincide, so, in the sequel of this paragraph we shall suppose G non-commutative.

Remark 5.3. Consider now a non-commutative 2-dimensional Lie group G as in Remark 2.4. We make the convention that the vanishing coefficients of the connections are not written anymore. A short calculation shows that:

(i) the bi-invariant connections verify

$$\nabla_{E_1} E_2 = aE_1 \;\; , \;\; \nabla_{E_2} E_1 = bE_1 \;\; , \;\; \nabla_{E_2} E_2 = (a+b)E_2,$$

for every $a, b \in \mathbb{R}$. For $a = b = 0$, $a = b = -1$ and $a = b = -\frac{1}{2}$ we find the Cartan-Schouten connections ∇^-, ∇^+ and ∇^0 respectively.

(ii) the symmetric bi-invariant connections verify

$$\nabla_{E_1} E_2 = (b+1)E_1 \;\; , \;\; \nabla_{E_2} E_1 = bE_1 \;\; , \;\; \nabla_{E_2} E_2 = (2b+1)E_2,$$

for every $b \in \mathbb{R}$. For $b = -\frac{1}{2}$ we find the Cartan-Schouten connection ∇^0.

(iii) the flat bi-invariant connections verify

$$\nabla_{E_2} E_1 = bE_1 \;\; , \;\; \nabla_{E_2} E_2 = bE_2$$

for every $b \in \mathbb{R}$. For $b = 0$ we find the Cartan-Schouten connection ∇^-.

(iv) there exists a unique flat and symmetric bi-invariant connection, given by

$$\nabla_{E_2} E_1 = -E_1 \;\; , \;\; \nabla_{E_2} E_2 = -E_2.$$

To our knowledge, this remarkable bi-invariant connection is a new one! We call it *the Euclidean connection* of the 2-dimensional non-commutative Lie groups. This connection cannot be a Levi-Civita connection of a bi-invariant metric on G, because it would imply that G is compact, and thus (due to the dimension 2) commutative.

One can deduce some simple properties of the differential affine manifold (G, ∇): the auto-parallel left invariant vector fields $X \in L(G)$ (i.e. with $\nabla_X X = 0$) are exactly those collinear with E_1; there exists no parallel left-invariant vector fields Y (i.e. with $\nabla_Z Y = 0$, for every $Z \in L(G)$).

Consider a realization of G, as the so-called "$ax + b$"-group, i.e. the affine group of transformations of the real line. We may identify it with the product $\mathbb{R}^* \times \mathbb{R}$, with coordinates (x^1, x^2) and with the multiplication

$$(a^1, a^2)(b^1, b^2) = (a^1 b^1, a^1 b^2 + a^2).$$

The Lie algebra of G admits a basis $\{E_1, E_2\}$, with $E_1 = x^1\partial_2$ and $E_2 = -x^1\partial_1$, such that $[E_1, E_2] = E_1$. In the (global) coordinates (x^1, x^2), the components of ∇ all vanish, which proves that the bi-invariant connection discovered above is exactly the canonical "Euclidean" connection induced on G. Its auto-parallel curves are the real lines of the plane, restricted to G; we remark that there exist non-complete auto-parallel curves (those which cannot pass through the origin).

Example 5.4. Consider now the non-commutative 3-dimensional Lie group H^3 as in Remark 2.6. A tedious calculation shows that, on H^3, we have the following properties.

(i) The set of bi-invariant connections is given by all the real numbers $\Gamma^i_{jk} \in \mathbb{R}^{27}$, with $i, j, k = \overline{1,3}$, such that $\Gamma^3_{11}, \Gamma^3_{12}, \Gamma^3_{21}, \Gamma^3_{22}$ are arbitrary and

$$\Gamma^1_{22} = \Gamma^1_{23} = \Gamma^1_{33} = \Gamma^1_{32} = \Gamma^1_{31} = \Gamma^1_{13} = \Gamma^2_{11} = \Gamma^2_{13} = \Gamma^2_{31} = \Gamma^2_{33} = \Gamma^2_{32} = \Gamma^2_{23} = \Gamma^3_{33} = 0$$

$$\Gamma^2_{12} = \Gamma^3_{13}\ ,\ \Gamma^1_{11} = \Gamma^3_{13} + \Gamma^3_{31}\ ,\ \Gamma^2_{22} = \Gamma^3_{23} + \Gamma^3_{32}\ ,\ \Gamma^1_{12} = \Gamma^3_{32}\ ,\ \Gamma^2_{21} = \Gamma^3_{31}\ ,\ \Gamma^1_{21} = \Gamma^3_{23}.$$

This set may be parameterized as a product $V \times \mathbb{R}^4$, where V is a 4-dimensional subspace in $\mathbb{R}^{10}$ (determined by the last previous relation). Hence, the set of bi-invariant connections may be modelled as a 8-dimensional subspace in $\mathbb{R}^{14}$.

(ii) The set of the symmetric bi-invariant connections is modelled by a 6-dimensional affine subspace in $\mathbb{R}^{14}$, of the form $W \times \mathbb{R}^2$, where W is a 4-dimensional affine subspace in $\mathbb{R}^{12}$, defined by

$$\Gamma^1_{21} = \Gamma^1_{12}\ ,\ \Gamma^2_{21} = \Gamma^2_{12}\ ,\ \Gamma^3_{31} = \Gamma^3_{13}\ ,\ \Gamma^3_{32} = \Gamma^3_{23}\ ,\ \Gamma^2_{12} = \Gamma^3_{13}\ ,$$

$$\Gamma^1_{11} = 2\Gamma^3_{13}\ ,\ \Gamma^2_{22} = 2\Gamma^3_{23}\ ,\ \Gamma^1_{12} = \Gamma^3_{23}\ ,\ \Gamma^3_{21} = \Gamma^3_{12} - 1$$

and arbitrary $\Gamma^3_{11}, \Gamma^3_{22}$. (The null components are the same as in (i)).

(iii) Similar computations may be made for the bi-invariant connections which are: flat, flat and symmetric, Ricci-flat, Ricci-flat and symmetric, Ricci-symmetric, Ricci-symmetric and symmetric, respectively.

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