Orlicz mixed affine quermassintegrals

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Abstract. The main aim of this paper is to generalize the mixed affine quermassintegrals to Orlicz space. Under the framework of Orlicz-Brunn-Minkowski theory, we introduce a new affine geometric quantity by calculating the Orlicz first order variation of the mixed affine quermassintegrals, and call it *Orlicz mixed affine quermassintegrals*. The fundamental notions and conclusions of the mixed affine quermassintegrals and the related isoperimetric inequalities are extended to an Orlicz setting. The concepts and inequalities for Orlicz quermassintegrals of convex bodies are also included in our conclusions. The new Orlicz isperimetric inequalities in special case which yield the Orlicz Minkowski inequalities and Orlicz Brunn-Minkowski inequalities for the quermassintegrals, the affine quermassintegrals and the Orlicz affine quermassintegrals.

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1 Introduction

One of the most important operations in geometry is vector addition. As an operation between sets K and L, defined by

$$K + L = \{x + y : x \in K, y \in L\},$$

it is usually called Minkowski addition and, combined with volume, plays an important role in the Brunn-Minkowski theory. During the last few decades, the theory has been extended to L_p-Brunn-Minkowski theory. The set, called L_p addition, was introduced by Firey in [6] and [7]. The operation, denoted by $+_p$, for $1 \leq p \leq \infty$, is defined by

$$h(K +_p L, x)^p = h(K, x)^p + h(L, x)^p, \tag{1.1}$$

for all $x \in \mathbb{R}^n$ and for K and L, being compact convex sets in $\mathbb{R}^n$ containing the origin. When $p = \infty$, the above equality is interpreted as $h(K +_\infty L, x) = \max\{h(K, x), h(L, x)\}$, as is customary. Here the functions are the support functions. If K is a nonempty closed (not necessarily bounded) convex set in $\mathbb{R}^n$, then

$$h(K, x) = \max\{x \cdot y : y \in K\},$$

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for $x \in \mathbb{R}^n$, defines the support function $h(K,x)$ of K. A nonempty closed convex set is uniquely determined by its support function. L_p addition and inequalities are the fundamental and core content in the L_p Brunn-Minkowski theory. For recent important results and more information from this theory, we refer to [11], [12], [13], [14], [20], [22], [25], [26], [27], [28], [29], [32], [33], [37], [38], [39] and the references therein. In recent years, a new extension of L_p-Brunn-Minkowski theory is to Orlicz-Brunn-Minkowski theory, initiated by Lutwak, Yang, and Zhang [30] and [31]. Gardner, Hug and Weil [9] constructed a general framework for the Orlicz-Brunn-Minkowski theory, and made clear for the first time the relation to Orlicz spaces and norms. The Orlicz addition of convex bodies was introduced, and the Orlicz-Brunn-Minkowski inequality was obtained (see [40]). The Orlicz centroid inequality for star bodies was introduced in [49] which is an extension from convex to star bodies. Advances in the theory can be found in [10], [16], [17], [19], [34], [42], [43], [44], [45], [46], [47], [48] and [50]. In 2014, Gardner, Hug and Weil ([9]) introduced the Orlicz addition $K +_\varphi L$ of compact convex sets K and L in $\mathbb{R}^n$ containing the origin, implicitly, by

$$h(K +_\varphi L, u)) = \inf\left\{\lambda > 0 : \varphi\left(\frac{h(K,u)}{\lambda}\right) + \varphi\left(\frac{h(L,u)}{\lambda}\right) \leq 1\right\}, \tag{1.2}$$

where $\varphi : [0,\infty) \to (0,\infty)$ is a convex and increasing function such that $\varphi(1) = 1$ and $\varphi(0) = 0$. Let Φ denote the set of convex functions $\varphi : [0,\infty) \to [0,\infty)$ that is increasing and satisfies $\varphi(0) = 0$ and $\varphi(1) = 1$. When $p \geq 1$ and $\varphi(t) = t^p$, the Orlicz addition $K +_\varphi L$ becomes the L_p-addition $K +_p L$. Orlicz mixed quermassintegrals with respect to the Orlicz addition, $W_{\varphi,i}(K,L)$, defined by

$$\begin{aligned} W_{\varphi,i}(K,L) &:= \frac{\varphi'_-(1)}{n-i} \lim_{\varepsilon\to 0^+} \frac{W_i(K +_\varphi \varepsilon \cdot L) - W_i(K)}{\varepsilon} \\ &= \frac{1}{n}\int_{S^{n-1}} \varphi\left(\frac{h(L,u)}{h(K,u)}\right) h(K,u) dS_i(K,u), \end{aligned} \tag{1.3}$$

for $\varphi \in \Phi$, $0 \leq i \leq n$ and K and L and are convex bodies containing the origin in their interiors in $\mathbb{R}^n$, and $W_i(K)$ is the usual quermassintegral of a convex body K, and $S_i(K,u)$ denotes the ith mixed surface area measure of K, and $\varphi'_-(1)$ denotes the value of the left derivative of convex function φ at point 1 (see [41] and [43]).

Lutwak [23] proposed to define the affine quermassintegrals for a convex body K, $\Phi_0(K)$, $\Phi_1(K)$, $\ldots$, $\Phi_n(K)$, by taking $\Phi_0(K) := V(K), \Phi_n(K) := \omega_n$ and for $0 < j < n$,

$$\Phi_{n-j}(K) := \omega_n \left[\int_{G_{n,j}} \left(\frac{\mathrm{vol}_j(K|\xi)}{\omega_j}\right)^{-n} d\mu_j(\xi)\right]^{-1/n}, \tag{1.4}$$

where $G_{n,j}$ denotes the Grassman manifold of j-dimensional subspaces in $\mathbb{R}^n$, and μ_j denotes the gauge Haar measure on $G_{n,j}$, and $\mathrm{vol}_j(K|\xi)$ denotes the j-dimensional volume of the positive projection of K on j-dimensional subspace $\xi \subset \mathbb{R}^n$ and ω_j denotes the volume of j-dimensional unit ball. Lutwak showed the Brunn-Minkowski inequality for the affine quermassintegrals. If K and L are convex bodies and $0 < j < n$, then

$$\Phi_j(K+L)^{1/(n-j)} \geq \Phi_j(K)^{1/(n-j)} + \Phi_j(L)^{1/(n-j)}. \tag{1.5}$$

Lutwak [24] conjectured that

$$\omega_n^j \Phi_i(K)^{n-j} \leq \omega_n^i \Phi_j(K)^{n-i},$$

for $0 \leq i < j < n$ and K is a convex body. In analogy to (1.4), one may also define mixed affine quermassintegrals, $\Phi_{n-j,i}(K)$, by (see Section 3)

$$\Phi_{n-j,i}(K) := \omega_n \left[\int_{G_{n,j}} \left(\frac{\mathrm{vol}_i^{(j)}(K|\xi)}{\omega_j} \right)^{-(n-i)} d\mu_j(\xi) \right]^{-1/(n-i)}, \tag{1.6}$$

where $0 \leq i < j \leq n$, and $\mathrm{vol}_i^{(j)}(K|\xi)$ denotes the j-dimensional mixed volume $V^{(j)}(\underbrace{K|\xi, \ldots, K|\xi}_{j-i}, \underbrace{B_j, \ldots, B_j}_{i})$ and B_j denotes the j-dimensional unit ball, and by letting $\Phi_{0,i}(K) := W_i(K)$, Obviously, when $i = 0$, $\mathrm{vol}_i^{(j)}(K|\xi)$ becomes the above j-dimensional volume $\mathrm{vol}_j(K|\xi)$.

In the paper, our main aim is to generalize the mixed affine quermassintegrals to Orlicz space. Under the framework of Orlicz-Brunn-Minkowski theory, we introduce a new affine geometric quantity call it Orlicz mixed affine quermassintegrals. The fundamental notions and conclusions of the mixed affine quermassintegrals and the Minkoswki and Brunn-Minkowski inequalities for the mixed affine quermassintegrals are extended to an Orlicz setting. The new Orlicz Minkowski and Brunn-Minkowski inequalities in special case which yield the Orlicz Minkowski inequalities and Orlicz Brunn-Minkowski inequalities for the quermassintegrals, the affine quermassintegrals and the Orlicz affine quermassintegrals, and yield also the L_p Minkowski inequality and Brunn-Minkowski inequalities for the affine quermassintegrals.

Comply with the basic spirit of Aleksandrov [2], Fenchel and Jensen [5] introduction of mixed quermassintegrals, and introduction of Lutwak's L_p-mixed quermassintegrals (see [21]), we are based on the study of the first order Orlicz variational of the mixed affine quermassintegrals. We prove that the Orlicz first order variation of the mixed affine quermassintegrals can be expressed as: If K and L are convex bodies containing the origin in their interiors, $\varphi \in \Phi$, $\varepsilon > 0$ and $0 \leq i < j \leq n$, then

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \Phi_{n-j,i}(K +_\varphi \varepsilon \cdot L) = \frac{j-i}{\varphi'_-(1)} \Phi_{n-j,i}(K)^{1+n-i} \Phi_{\varphi,n-j,i}(K,L)^{i-n}. \tag{1.7}$$

For $j = n$, (1.7) becomes the well-known result about Orlicz quermassintegral of K and L.

$$\lim_{\varepsilon \to 0^+} \frac{W_i(K +_\varphi \varepsilon \cdot L) - W_i(K)}{\varepsilon} = \frac{n-i}{\varphi'_-(1)} W_{\varphi,i}(K,L).$$

In this first order variational equation (1.7), we find a new geometric quantity. Based on this, we extract the required geometric quantity, denotes $\Phi_{\varphi,n-j,i}(K,L)$ and call it Orlicz mixed affine quermassintegral of convex bodies K and L containing the origin in their interiors, defined by

$$\Phi_{\varphi,n-j,i}(K,L) := \left(\frac{\varphi'_-(1)}{(j-i) \cdot \Phi_{n-j,i}(K)^{1+n-i}} \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \Phi_{n-j,i}(K +_\varphi \varepsilon \cdot L) \right)^{1/(i-n)}, \tag{1.8}$$

where $\varphi \in \Phi$ and $0 \leq i < j \leq n$. We prove the new affine geometric quantity, $\Phi_{\varphi,n-j,i}(K,L)$, has an integral representation.

$$\Phi_{\varphi,n-j,i}(K,L) = \omega_n \left[\int_{G_{n,j}} \frac{W_{\varphi,i}^{(j)}(K|\xi, L|\xi)}{\mathrm{vol}_i^{(j)}(K|\xi)} \left(\frac{\mathrm{vol}_i^{(j)}(K|\xi)}{\omega_j}\right)^{-(n-i)} d\mu_j(\xi)\right]^{-1/(n-i)}, \quad (1.9)$$

where $W_{\varphi,i}^{(j)}(K|\xi, L|\xi)$ denotes the j-dimensional Orlicz mixed quermassintegral of $K|\xi$ and $L|\xi$. We apply the integral geometry technique on Grassmann manifolds to prove the affine invariance of the Orlicz mixed affine quermassintegrals.

$$\Phi_{\varphi,n-j,i}(gK, gL) = \Phi_{\varphi,n-j,i}(K,L), \quad (1.10)$$

where K, L are convex bodies containing the origin in their interiors, $0 \leq i < j \leq n$, $\varphi \in \Phi$ and $g \in \mathrm{SL(n)}$.

Because the Orlicz mixed affine quermassintegrals is an extension of the affine quermassintegrals, a very natural question is raised: is there a Minkowski type isoperimetric inequality for the Orlicz mixed affine quermassintegrals? in the Section 4, we give a positive answer to this question and establish the Orlicz Minkoswki inequality for the new affine geometric quantity. If K and L are convex bodies containing the origin in their interiors, $\varphi \in \Phi$ and $0 \leq i < j \leq n$, then the Orlicz Minkowski inequality for the Orlicz mixed affine quermassintegrals is established.

$$\left(\frac{\Phi_{n-j,i}(K)}{\Phi_{\varphi,n-j,i}(K,L)}\right)^{n-i} \geq \varphi\left(\left(\frac{\Phi_{n-j,i}(L)}{\Phi_{n-j,i}(K)}\right)^{1/(j-i)}\right). \quad (1.11)$$

If φ is strictly convex, equality holds if and only if K and L are homothetic. For $j = n$, (1.11) becomes the following Orlicz Minkoswki inequality for the quermassintegrals of convex bodies (see [41] and [43]).

$$W_{\varphi,i}(K,L) \geq W_i(K) \cdot \varphi\left(\left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}\right). \quad (1.12)$$

If φ is strictly convex, equality holds if and only if K and L are homothetic. It is worth mentioning here that Zou [50] established the following inequality, which is the special case of (1.11). If K and L are convex bodies containing the origin in their interiors, $\varphi \in \Phi$ and $0 < j \leq n$, then

$$\left(\frac{\Phi_{\varphi,n-j}(K,L)}{\Phi_{n-j}(K)}\right)^{-n} \geq \varphi\left(\left(\frac{\Phi_{n-j}(L)}{\Phi_{n-j}(K)}\right)^{1/j}\right).$$

If φ is strictly convex, equality holds if and only if K and L are homothetic. Unfortunately, inequality (1.12) cannot be obtained from Zou's result.

In the Section 5, on the basis of the Minkoswki inequality for the Orlicz mixed affine quermassintegrals, we establish an Orlicz-Brunn-Minkoswki inequality for the mixed affine quermassintegrals. If K, L are convex bodies containing the origin in their interiors, $0 \leq i < j \leq n$ and $\varphi \in \Phi$, then for nay $\varepsilon > 0$

$$1 \geq \varphi\left(\left(\frac{\Phi_{n-j,i}(K)}{\Phi_{n-j,i}(K +_\varphi \varepsilon \cdot L)}\right)^{1/(j-i)}\right) + \varepsilon \cdot \varphi\left(\left(\frac{\Phi_{n-j,i}(L)}{\Phi_{n-j,i}(K +_\varphi \varepsilon \cdot L)}\right)^{1/(j-i)}\right). \quad (1.13)$$

If φ is strictly convex, equality holds if and only if K and L are homothetic. For $j=n$ and $\varepsilon=1$, (1.13) becomes the following Orlicz-Brunn-Minkoswki inequality for quermassintegrals (see [41] and [43]). If K, L are convex bodies containing the origin in their interiors, $0 \le i < n$ and $\varphi \in \Phi$, then

$$1 \ge \varphi\left(\left(\frac{W_i(K)}{W_i(K +_\varphi L)}\right)^{1/(n-i)}\right) + \varphi\left(\left(\frac{W_i(L)}{W_i(K +_\varphi L)}\right)^{1/(n-i)}\right). \tag{1.14}$$

If φ is strictly convex, equality holds with if and only if K and L are homothetic. It is worth mentioning here that Zou [50] established the following inequality, which is the special case of (1.13). If K, L are convex bodies containing the origin in their interiors, $0 < j \le n$ and $\varphi \in \Phi$, then

$$1 \ge \varphi\left(\left(\frac{\Phi_{n-j}(K)}{\Phi_{n-j}(K +_\varphi L)}\right)^{1/j}\right) + \varphi\left(\left(\frac{\Phi_{n-j}(L)}{\Phi_{n-j}(K +_\varphi L)}\right)^{1/j}\right).$$

If φ is strictly convex, equality holds if and only if K and L are homothetic. Unfortunately, inequality (1.14) cannot be obtained from Zou's result. Moreover, putting $\varepsilon = 1$ and $\varphi(t) = t^p$ in (1.13), where $1 \le p < \infty$, (1.13) becomes the L_p-Minkoswki inequality for the mixed affine quermassintegrals. If K, L are convex bodies containing the origin in their interiors, $0 \le i < j \le n$ and $1 \le p < \infty$, then

$$\Phi_{n-j,i}(K +_p L)^{p/(j-i)} \ge \Phi_{n-j,i}(K)^{p/(j-i)} + \Phi_{n-j,i}(L)^{p/(j-i)}, \tag{1.15}$$

with equality if and only if K and L are homothetic.

2 Preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. A body in $\mathbb{R}^n$ is a compact set equal to the closure of its interior. A set K is called a convex body, if it is compact and convex subsets with non-empty interiors. Let $\mathcal{K}^n$ denote the class of convex bodies containing the origin in their interiors in $\mathbb{R}^n$. We reserve the letter $u \in S^{n-1}$ for unit vectors, and the letter B for the unit ball centered at the origin. The surface of B is S^{n-1}. For a compact set K, we write $V(K)$ for the (n-dimensional) Lebesgue measure of K and call this the volume of K. If K is a nonempty closed (not necessarily bounded) convex set, then

$$h(K, x) = \sup\{x \cdot y : y \in K\},$$

for $x \in \mathbb{R}^n$, defines the support function of K, where $x \cdot y$ denotes the usual inner product x and y in $\mathbb{R}^n$. A nonempty closed convex set is uniquely determined by its support function. The support function is homogeneous of degree 1, that is,

$$h(K, rx) = rh(K, x),$$

for all $x \in \mathbb{R}^n$ and $r \ge 0$ (see e.g. [3]). Let d denote the Hausdorff metric on $\mathcal{K}^n$, i.e., for $K, L \in \mathcal{K}^n$,

$$d(K, L) = |h(K, u) - h(L, u)|_\infty,$$

where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$. Let $K \subset \mathbb{R}^n$ be a nonempty closed convex set. If ξ is a subspace of $\mathbb{R}^n$, then it is easy to show that

$$h(K|\xi, x) = h(K, x|\xi),$$

for $x \in \mathbb{R}^n$. The formula

$$h(AK, x) = h(K, A^t x), \tag{2.1}$$

for $x \in \mathbb{R}^n$ (see [8, p.18]), and a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$, gives the change in a support function under A, where A^t denotes the transpose of A. Equation (2.1) is proved in [8, p.18] for compact sets and $A \in GL(n)$, but the proof is the same if K is unbounded or A is singular.

2.1 Quermassintegrals

If $K_i \in \mathcal{K}^n$ $(i = 1, 2, \ldots, r)$ and λ_i $(i = 1, 2, \ldots, r)$ are nonnegative real numbers, then of fundamental importance is the fact that the volume of $\sum_{i=1}^{r} \lambda_i K_i$ is a homogeneous polynomial in λ_i given by (see e.g. [35])

$$V(\lambda_1 K_1 + \cdots + \lambda_n K_n) = \sum_{i_1,\ldots,i_n} \lambda_{i_1} \ldots \lambda_{i_n} V_{i_1 \ldots i_n}, \tag{2.2}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding r. The coefficient $V_{i_1 \ldots i_n}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$ and is uniquely determined by (2.2), it is called the mixed volume of $K_i, \ldots, K_{i_n}$, and is written as $V(K_{i_1}, \ldots, K_{i_n})$. Let $K_1 = \ldots = K_{n-i} = K$ and $K_{n-i+1} = \ldots = K_n = L$, then the mixed volume $V(K_1, \ldots, K_n)$ is written as $V(K[n-i], L[i])$. If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = B$ The mixed volumes $V_i(K[n-i], B[i])$ is written as $W_i(K)$ and called as quermassintegrals (or ith mixed quermassintegrals) of K. We write $W_i(K, L)$ for the mixed volume $V(K[n-i-1], B[i], L[1])$ and call as mixed quermassintegrals. Aleksandrov [1] and Fenchel and Jessen [5] (also see Busemann [4] and Schneider [35]) have shown that for $K \in \mathcal{K}^n$, and $i = 0, 1, \ldots, n-1$, there exists a regular Borel measure $S_i(K, \cdot)$ on S^{n-1}, such that the mixed quermassintegrals, $W_i(K, L)$, has the following representation:

$$W_i(K, L) = \frac{1}{n-i} \lim_{\varepsilon \to 0^+} \frac{W_i(K + \varepsilon \cdot L) - W_i(K)}{\varepsilon} = \frac{1}{n} \int_{S^{n-1}} h(L, u) dS_i(K, u). \tag{2.3}$$

Associated with $K_1, \ldots, K_n \in \mathcal{K}^n$ is a Borel measure $S(K_1, \ldots, K_{n-1}, \cdot)$ on S^{n-1}, called the mixed surface area measure of $K_1, \ldots, K_{n-1}$, which has the property that for each $K \in \mathcal{K}^n$ (see e.g. [8], p.353),

$$V(K_1, \ldots, K_{n-1}, K) = \frac{1}{n} \int_{S^{n-1}} h(K, u) dS(K_1, \ldots, K_{n-1}, u). \tag{2.4}$$

In fact, the measure $S(K_1, \ldots, K_{n-1}, \cdot)$ can be defined by the propter that (2.4) holds for all $K \in \mathcal{K}^n$. Let $K_1 = \ldots = K_{n-i-1} = K$ and $K_{n-i} = \ldots = K_{n-1} = L$, then the mixed surface area measure $S(K_1, \ldots, K_{n-1}, \cdot)$ is written as $S(K[n-i], L[i], \cdot)$. When $L = B$, $S(K[n-i], L[i], \cdot)$ is written as $S_i(K, \cdot)$ and called as ith mixed surface area measure. A fundamental inequality for mixed quermassintegrals states that: If $K, L \in \mathcal{K}^n$ and $0 \le i < n-1$, then

$$W_i(K,L)^{n-i} \ge W_i(K)^{n-i-1} W_i(L), \tag{2.5}$$

with equality if and only if K and L are homothetic and $L = \{o\}$. Good general references for this material are [4] and [19].

2.2 p-mixed quermassintegrals

Mixed quermassintegrals are the first variation of the ordinary quermassintegrals, with respect to Minkowski addition. The p-mixed quermassintegrals $W_{p,0}(K,L)$, $W_{p,1}(K,L), \ldots, W_{p,n-1}(K,L)$, as the first variation of the ordinary quermassintegrals, with respect to Firey addition: For $K, L \in \mathcal{K}^n$, and real $p \ge 1$, defined by (see e.g. [21])

$$W_{p,i}(K,L) = \frac{p}{n-i} \lim_{\varepsilon \to 0^+} \frac{W_i(K +_p \varepsilon \cdot L) - W_i(K)}{\varepsilon}. \tag{2.6}$$

The mixed p-quermassintegrals $W_{p,i}(K,L)$, for all $K, L \in \mathcal{K}^n$, has the following integral representation:

$$W_{p,i}(K,L) = \frac{1}{n} \int_{S^{n-1}} h(L,u)^p dS_{p,i}(K,u), \tag{2.7}$$

where $S_{p,i}(K,\cdot)$ denotes the Boel measure on S^{n-1}. The measure $S_{p,i}(K,\cdot)$ is absolutely continuous with respect to $S_i(K,\cdot)$, and has Radon-Nikodym derivative

$$\frac{dS_{p,i}(K,\cdot)}{dS_i(K,\cdot)} = h(K,\cdot)^{1-p}, \tag{2.8}$$

where $S_i(K,\cdot)$ is a regular Boel measure on S^{n-1}. The measure $S_{n-1}(K,\cdot)$ is independent of the body K, and is just ordinary Lebesgue measure, S, on S^{n-1}. $S_i(B,\cdot)$ denotes the i-th surface area measure of the unit ball in $\mathbb{R}^n$. In fact, $S_i(B,\cdot) = S$ for all i. The surface area measure $S_0(K,\cdot)$ just is $S(K,\cdot)$. When $i = 0$, $S_{p,i}(K,\cdot)$ is written as $S_p(K,\cdot)$ (see [26] and [27]). A fundamental inequality for mixed p-quermassintegrals stats that: For $K, L \in \mathcal{K}^n, p > 1$ and $0 \le i < n-1$,

$$W_{p,i}(K,L)^{n-i} \ge W_i(K)^{n-i-p} W_i(L)^p, \tag{2.9}$$

with equality if and only if K and L are homothetic. L_p-Brunn-Minkowski inequality for the quermassintegrals established by Lutwak [21]. If $K, L \in \mathcal{K}^n$ and $p \ge 1$ and $0 \le i \le n$, then

$$W_i(K +_p L)^{p/(n-i)} \ge W_i(K)^{p/(n-i)} + W_i(L)^{p/(n-i)}, \tag{2.10}$$

with equality if and only if K and L are homothetic or $L = \{o\}$. Obviously, putting $i = 0$ in (2.7), the mixed p-quermassintegrals $W_{p,i}(K,L)$ become the well-known L_p-mixed volume $V_p(K,L)$, defined by (see e.g. [27])

$$V_p(K,L) = \frac{1}{n} \int_{S^{n-1}} h(L,u)^p dS_p(K,u). \tag{2.11}$$

2.3 Orlicz addition and Orlicz linear combination

Definition 2.1 Let $m \geq 2, \varphi \in \Phi$, $K_j \in \mathcal{K}^n$ and $j = 1, \ldots, m$, define the Orlicz addition of $K_1, \ldots, K_m$, denoted by $K_1 +_\varphi \cdots +_\varphi K_m$, defined by

$$h(K_1 +_\varphi \cdots +_\varphi K_m, u) = \inf\left\{\lambda > 0 : \sum_{j=1}^{m} \varphi\left(\frac{h(K_j, x)}{\lambda}\right) \leq 1\right\}, \tag{2.12}$$

for all $x \in \mathbb{R}^n$ (see [9] and [40]).

Equivalently, the Orlicz addition $K_1 +_\varphi \cdots +_\varphi K_m$ can be defined implicitly by

$$\varphi\left(\frac{h(K_1, x)}{h(K_1 +_\varphi \cdots +_\varphi K_m, x)}\right) + \cdots + \varphi\left(\frac{h(K_m, x)}{h(K_1 +_\varphi \cdots +_\varphi K_m, x)}\right) = 1, \tag{2.13}$$

for all $x \in \mathbb{R}^n$.

The Orlicz linear combination on the case $m = 2$ is defined.

Definition 2.2 Orlicz linear combination $+_\varphi(K, L, \alpha, \beta)$ for $K, L \in \mathcal{K}^n$, $\varphi \in \Phi$, and $\alpha, \beta \geq 0$ (not both zero), defined by

$$\alpha \cdot \varphi\left(\frac{h(K, x)}{h(+_\varphi(K, L, \alpha, \beta), x)}\right) + \beta \cdot \varphi\left(\frac{h(L, x)}{h(+_\varphi(K, L, \alpha, \beta), x)}\right) = 1, \tag{2.14}$$

for all $x \in \mathbb{R}^n$ (see [9] and [40]).

When $\varphi(t) = t^p$ and $p \geq 1$, then the Orlicz linear combination $+_\varphi(K, L, \alpha, \beta)$ changes to the L_p linear combination $\alpha \cdot K +_p \beta \cdot L$. Moreover, we shall write $K +_\varphi \varepsilon \cdot L$ instead of $+_\varphi(K, L, 1, \varepsilon)$, for $\varepsilon \geq 0$ and assume throughout that this is defined by (2.14), where $\alpha = 1, \beta = \varepsilon$ and $\varphi \in \Phi$. It is easy that $+_\varphi(K, L, 1, 1) = K +_\varphi L$.

3 Orlicz mixed affine querlmassintegrals

In order to define the Orlicz mixed affine querlmassintegrals, we need define the mixed affine quermassintegrals and recall the Orlicz quermassintegrals.

Definition 3.1 For $K, L \in \mathcal{K}^n$, $\varphi \in \Phi$ and $0 \leq i < n$, the Orlicz quermassintegral of K and L, $W_{\varphi,i}(K, L)$, defined by

$$W_{\varphi,i}(K, L) := \frac{1}{n} \int_{S^{n-1}} \varphi\left(\frac{h(L, u)}{h(K, u)}\right) h(K, u) dS_i(K, u). \tag{3.1}$$

The definition is introduced in the literatures [41] and [43].

Lemma 3.1 *If $K, L \in \mathcal{K}^n$, $0 \leq i < n$, $\varepsilon > 0$ and $\varphi \in \Phi$, then*

$$W_i(K +_\varphi \varepsilon \cdot L) = W_{\varphi,i}(K +_\varphi \varepsilon \cdot L, K) + \varepsilon \cdot W_{\varphi,i}(K +_\varphi \varepsilon \cdot L, L). \tag{3.2}$$

Proof From (2.3), (3.1) and (2.14), we have for any $Q \in \mathcal{K}^n$

$$\begin{aligned} &W_{\varphi,i}(Q, K) + \varepsilon \cdot W_{\varphi,i}(Q, L) \\ &\quad = \frac{1}{n} \int_{S^{n-1}} \left(\varphi\left(\frac{h(K, u)}{h(Q, u)}\right) + \varepsilon \cdot \varphi\left(\frac{h(L, u)}{h(Q, u)}\right)\right) h(Q, u) dS_i(Q, u) \end{aligned}$$

$$= \frac{1}{n}\int_{S^{n-1}} h(Q,u)dS_i(Q,u)$$
$$= W_i(Q). \tag{3.3}$$

Putting $Q = K +_\varphi \varepsilon \cdot L$ in (3.3), (3.2) easy follows. □

Lemma 3.2 *If $K, L \in \mathcal{K}^n$ and $\varphi \in \Phi$, then for $\varepsilon > 0$*

$$K +_\varphi \varepsilon \cdot L \to K, \tag{3.4}$$

in the Hausdorff metric as $\varepsilon \to 0^+$.

In [9], Lemma 3.2 is first given. Next, we give a direct proof.

Proof From (2.14), we have

$$h(K +_\varphi \varepsilon \cdot L, u) = \frac{h(K,u)}{\varphi^{-1}\left(1 - \varepsilon\varphi\left(\frac{h(L,u)}{h(K +_\varphi \varepsilon \cdot L, u)}\right)\right)}.$$

Since φ^{-1} is continuous, φ is bounded and in view of $\varphi^{-1}(1) = 1$, we have

$$\varphi^{-1}\left(1 - \varepsilon\varphi\left(\frac{h(L,u)}{h(K +_\varphi \varepsilon \cdot L, u)}\right)\right) \to 1,$$

as $\varepsilon \to 0^+$.

This yields

$$h(K +_\varphi \varepsilon \cdot L, u) \to h(K,u)$$

as $\varepsilon \to 0^+$. □

Lemma 3.3 *If $\varphi \in \Phi$, $0 \le i < n$ and $K, L \in \mathcal{K}^n$, then for $\varepsilon > 0$*

$$\frac{\varphi'_-(1)}{n-i} \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} W_i(K +_\varphi \varepsilon \cdot L) = \frac{1}{n}\int_{S^{n-1}} \varphi\left(\frac{h(L,u)}{h(K,u)}\right) h(K,u)^{n-i} dS_i(K,u). \tag{3.5}$$

Lemma 3.3 is proved in the literatures [41] and [43].

Definition 3.2 (Mixed affine querlmassintegrals) The mixed affine quermassintegral of convex body K, $\Phi_{n-j,i}(K)$, defined by

$$\Phi_{n-j,i}(K) := \omega_n \left[\int_{G_{n,j}} \left(\frac{\mathrm{vol}_i^{(j)}(K|\xi)}{\omega_j}\right)^{-(n-i)} d\mu_j(\xi)\right]^{-1/(n-i)}, \tag{3.6}$$

where $0 \le i < j \le n$ and by letting $\Phi_{0,i}(K) := W_i(K)$ and $\Phi_{n,0}(K) = \Phi_n(K) = \omega_n$.

When $i = 0$, $\mathrm{vol}_i^{(j)}(K|\xi)$ becomes the well-known j-dimensional volume $\mathrm{vol}_j(K|\xi)$. Obviously, when $i = 0$, $\Phi_{n-j,i}(K) = \Phi_{n-j,0}(K) = \Phi_{n-j}(K)$, when $i = 0$ and $j = n$, $\Phi_{n-j,i}(K) = \Phi_{0,0}(K) = V(K)$.

Lemma 3.4 [9] *If $K, L \in \mathcal{K}^n$, $\varepsilon > 0$ and $\varphi \in \Phi$, then*

$$(K +_\varphi \varepsilon \cdot L)|\xi = K|\xi +_\varphi \varepsilon \cdot L|\xi. \tag{3.7}$$

In order to define the Orlicz mixed affine querlmassintegrals, we still need calculate the first order variation of the mixed affine querlmassintegrals.

Lemma 3.5 *If $\varphi \in \Phi$, $0 \le i < j \le n$ and $K, L \in \mathcal{K}^n$, then for any $\varepsilon > 0$*

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \Phi_{n-j,i}(K +_\varphi \varepsilon \cdot L) = \frac{j-i}{\varphi'_-(1)} \Phi_{n-j,i}(K)^{1+n-i} \Phi_{\varphi,n-j,i}(K,L)^{i-n}. \tag{3.8}$$

Proof On the one hand, from Lemma 3.3, we have

$$\begin{aligned}
&\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \int_{G_{n,j}} \mathrm{vol}_i^{(j)}((K +_\varphi \varepsilon \cdot L)|\xi)^{i-n} d\mu_j(\xi) \\
&= \lim_{\varepsilon \to 0^+} \int_{G_{n,j}} \frac{\mathrm{vol}_i^{(j)}((K +_\varphi \varepsilon \cdot L)|\xi)^{i-n} - \mathrm{vol}_i^{(j)}(K|\xi)^{i-n}}{\varepsilon} d\mu_j(\xi) \\
&= (i-n) \int_{G_{n,j}} \left(\mathrm{vol}_j(K|\xi)^{i-n-1} \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \int_{G_{n,j}} \mathrm{vol}_i^{(j)}((K +_\varphi \varepsilon \cdot L)|\xi) d\mu_j(\xi) \right) d\mu_j(\xi) \\
&= \frac{(i-n)(j-i)}{\varphi'_-(1)} \int_{G_{n,j}} \mathrm{vol}_i^j(K|\xi)^{i-n-1} W_{\varphi,i}^{(j)}(K|\xi, L|\xi) d\mu_j(\xi).
\end{aligned} \tag{3.9}$$

and on the other hand, from (1.9), (3.6) and (3.9), we obtain

$$\begin{aligned}
\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \Phi_{n-j,i}(K +_\varphi \varepsilon \cdot L) &= \frac{\omega_n}{\omega_j} \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \left[\int_{G_{n,j}} \mathrm{vol}_i^{(j)}((K +_\varphi \varepsilon \cdot L)|\xi)^{i-n} d\mu_j(\xi) \right]^{1/(i-n)} \\
&= \frac{\omega_n}{(i-n)\omega_j} \left(\int_{G_{n,j}} \mathrm{vol}_i^{(j)}(K|\xi)^{i-n} d\mu_j(\xi) \right)^{(1+n-i)/(i-n)} \\
&\quad \times \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \int_{G_{n,j}} \mathrm{vol}_i^{(j)}((K +_\varphi \varepsilon \cdot L)|\xi)^{i-n} d\mu_j(\xi) \\
&= \frac{j-i}{\varphi'_-(1)} \frac{\omega_n}{\omega_j} \left(\int_{G_{n,j}} \mathrm{vol}_i^{(j)}(K|\xi)^{i-n} d\mu_j(\xi) \right)^{(1+n-i)/(i-n)} \\
&\quad \times \int_{G_{n,j}} \frac{W_{\varphi,i}^{(j)}(K|\xi, L|\xi)}{\mathrm{vol}_i^{(j)}(K|\xi)} \mathrm{vol}_i^{(j)}(K|\xi)^{i-n} d\mu_j(\xi) \\
&= \frac{j-i}{\varphi'_-(1)} \Phi_{n-j,i}(K)^{1+n-i} \Phi_{\varphi,n-j,i}(K,L)^{i-n}. \qquad \square
\end{aligned}$$

From the proof of Lemma 3.5, we find a new affine geometric quantity, which is defined by:

Definition 3.3 If $\varphi \in \Phi$, $0 \le i < j < n$ and $K, L \in \mathcal{K}^n$, then Orlicz mixed affine querlmassintegral of K and L, $\Phi_{\varphi,n-j,i}(K,L)$, defined by

$$\Phi_{\varphi,n-j,i}(K,L) := \omega_n \left[\int_{G_{n,j}} \frac{W_{\varphi,i}^{(j)}(K|\xi, L|\xi)}{\mathrm{vol}_i^{(j)}(K|\xi)} \left(\frac{\mathrm{vol}_i^{(j)}(K|\xi)}{\omega_j} \right)^{-(n-i)} d\mu_j(\xi) \right]^{-1/(n-i)}. \tag{3.10}$$

Specifically, for $j = n$, we agreed:

$$\Phi_{\varphi,0,i}(K,L) = \left(\frac{W_i(K)}{W_{\varphi,i}(K,L)} \right)^{1/(n-i)} W_i(K).$$

Lemma 3.6 *If $K, L \in \mathcal{K}^n$, $0 \le i < j \le n$ and $\varphi \in \Phi$, then*

$$\Phi_{\varphi,n-j,i}(K,K) = \frac{1}{\varphi(1)^{1/(n-i)}} \Phi_{n-j,i}(K). \tag{3.11}$$

Proof The definition of the Orlicz mixed affine quermassintegrals, together with (3.6) and (3.10), (3.11) easy follows. □

Remark 3.1 When $\varphi(t) = t^p, 1 < p < \infty$, we write $\Phi_{\varphi,n-j,i}(K,L)$ as $\Phi_{p,n-j,i}(K,L)$, and call it L_p mixed affine quermassintegral of K and L, and

$$\Phi_{p,n-j,i}(K,L) = \omega_n \left[\int_{G_{n,j}} \frac{W_{p,i}^{(j)}(K|\xi, L|\xi)}{\mathrm{vol}_i^{(j)}(K|\xi)} \left(\frac{\mathrm{vol}_i^{(j)}(K|\xi)}{\omega_j} \right)^{-(n-i)} d\mu_j(\xi) \right]^{-1/(n-i)}.$$

When $i = 0$, write $\Phi_{p,n-j,i}(K,L)$ as $\Phi_{p,n-j,0}(K,L) = \Phi_{p,n-j}(K,L)$ and call it L_p affine quermassintegral of K and L, and

$$\Phi_{p,n-j}(K,L) = \omega_n \left[\int_{G_{n,j}} \frac{V_p^{(j)}(K|\xi, L|\xi)}{\mathrm{vol}_j(K|\xi)} \left(\frac{\mathrm{vol}_j(K|\xi)}{\omega_j} \right)^{-n} d\mu_j(\xi) \right]^{-1/n},$$

where $V_p^{(j)}(K|\xi, L|\xi)$ denotes the j-dimensional L_p mixed volumne of $K|\xi$ and $L|\xi$. When $\varphi(t) = t$, write $\Phi_{\varphi,n-j,i}(K,L)$ as $\Phi_{1,n-j,i}(K,L)$, and call the i-th mixed affine quermassintegral of K and L, and

$$\Phi_{1,n-j,i}(K,L) = \omega_n \left[\int_{G_{n,j}} \frac{W_i^{(j)}(K|\xi, L|\xi)}{\mathrm{vol}_i^{(j)}(K|\xi)} \left(\frac{\mathrm{vol}_i^{(j)}(K|\xi)}{\omega_j} \right)^{-(n-i)} d\mu_j(\xi) \right]^{-1/(n-i)},$$

where $W_i^{(j)}(K|\xi, L|\xi)$ denotes the j-dimensional mixed quermassintegral of $K|\xi$ and $L|\xi$. Obviously, when $K = L$, $\Phi_{1,n-j,i}(K,L)$ becomes the mixed affine quermassintegrals $\Phi_{n-j,i}(K)$.

Lemma 3.7 [9] *If $K, L \in \mathcal{K}^n$, $\varphi \in \Phi$ and any $g \in \mathrm{SL(n)}$, then for $\varepsilon > 0$*

$$g(K +_\varphi \varepsilon \cdot L) = (gK) +_\varphi \varepsilon \cdot (gL). \tag{3.12}$$

In the following, we will prove that Orlicz mixed affine querlmassintegral $\Phi_{\varphi,n-j,i}(K,L)$ is invariant under simultaneous unimodular centro-affine transformation.

Lemma 3.8 *If $K,L\in\mathcal{K}^n$, $0\leq i<j\leq n$, $\varphi\in\Phi$ and any $g\in \mathrm{SL(n)}$, then*

$$\Phi_{\varphi,n-j,i}(gK,gL)=\Phi_{\varphi,n-j,i}(K,L).$$

Proof Suppose $\xi\in G_{n,j}$ and $S^{j-1}=S^{n-1}|\xi$. For any $g\in \mathrm{SL(n)}$, $u\in S^{j-1}$ and $Q\in\mathcal{S}^{n-1}$, we have

$$h(gQ,u)=h(gQ|\xi,u). \tag{3.13}$$

From (2.1), (3.13) and the Definition 3.1, we obtain

$$\begin{aligned}
&W^{(j)}_{\varphi,i}(gK|\xi,gL|\xi)\\
&\quad=\frac{1}{j}\int_{S^{n-1}|\xi}\varphi\left(\frac{h(gL|\xi,u)}{h(gK|\xi,u)}\right)h(gK|\xi,u)dS_i(gK|\xi,u)\\
&\quad=\frac{1}{j}\int_{S^{n-1}}\varphi\left(\frac{h(L,g^tu)}{h(K,g^tu)}\right)h(K,g^tu)dS_i(K,g^tu)\\
&\quad=\frac{1}{j}\int_{S^{n-1}|\xi}\varphi\left(\frac{h(L|\xi,g^tu)}{h(K|\xi,g^tu)}\right)h(K|\xi,g^tu)dS_i(K|\xi,g^tu)\\
&\quad=W^{(j)}_{\varphi,i}(K|\xi,L|\xi).
\end{aligned}\tag{3.14}$$

On the other hand, from (3.10) and (3.14), we have

$$\begin{aligned}
&\Phi_{\varphi,n-j,i}(gK,gL)\\
&\quad=\omega_n\left[\int_{G_{n,j}}\frac{W^{(j)}_{\varphi,i}(gK|\xi,gL|\xi)}{\mathrm{vol}^{(j)}_i(gK|\xi)}\left(\frac{\mathrm{vol}^{(j)}_i(gK|\xi)}{\omega_j}\right)^{-(n-i)}d\mu_j(\xi)\right]^{-1/(n-i)}\\
&\quad=\omega_n\left[\int_{G_{n,j}}\frac{W^{(j)}_{\varphi,i}(K|\xi,L|\xi)}{\mathrm{vol}^{(j)}_i(K|\xi)}\left(\frac{\mathrm{vol}^{(j)}_i(K|\xi)}{\omega_j}\right)^{-(n-i)}d\mu_j(\xi)\right]^{-1/(n-i)}\\
&\quad=\Phi_{\varphi,n-j,i}(K,L).
\end{aligned}$$

$\square$

Next, we give another direct proof of Lemma 3.8.

Second proof From Lemma 3.5 and Lemma 3.7, we have for $g\in \mathrm{SL(n)}$,

$$\begin{aligned}
&\Phi_{\varphi,n-j,i}(gK,gL)\\
&\quad=\left(\frac{\varphi'_{-}(1)}{(j-i)\Phi_{n-j,i}(gK)^{1+n-i}}\cdot\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+}\Phi_{n-j,i}(gK+_\varphi\varepsilon\cdot gL)\right)^{-1/(n-i)}\\
&\quad=\left(\frac{\varphi'_{-}(1)}{(j-i)\Phi_{n-j,i}(gK)^{1+n-i}}\cdot\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+}\Phi_{n-j,i}(g(K+_\varphi\varepsilon\cdot L))\right)^{-1/(n-i)}
\end{aligned}$$

$$= \left(\frac{\varphi'_-(1)}{(j-i)\Phi_{n-j,i}(K)^{1+n-i}} \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \Phi_{n-j,i}(K +_\varphi \varepsilon \cdot L) \right)^{-1/(n-i)}$$
$$= \Phi_{\varphi,n-j,i}(K,L). \qquad \square$$

We need also the following Lemma to prove our main results.

Lemma 3.9 (Jensen's inequality) *Let μ be a probability measure on a space X and $g : X \to I \subset \mathbb{R}$ is a μ-integrable function, where I is a possibly infinite interval. If $\phi : I \to \mathbb{R}$ is a convex function, then*

$$\int_X \phi(g(x))d\mu(x) \geq \phi\left(\int_X g(x)d\mu(x)\right). \tag{3.15}$$

If ϕ is strictly convex, equality holds if and only if $g(x)$ is constant for μ-almost all $x \in X$ (see [15, p.165]).

4 Orlicz Minkowski inequality for the Orlicz mixed affine quermassintegrals

Theorem 4.1 (Orlicz-Minkowski inequality) *If $\varphi \in \Phi$, $0 \leq i < j \leq n$ and $K, L \in \mathcal{K}^n$, then*

$$\left(\frac{\Phi_{n-j,i}(K)}{\Phi_{\varphi,n-j,i}(K,L)}\right)^{n-i} \geq \varphi\left(\left(\frac{\Phi_{n-j,i}(L)}{\Phi_{n-j,i}(K)}\right)^{1/(j-i)}\right). \tag{4.1}$$

If φ is strictly convex, equality holds if and only if K and L are homothetic.

Proof When $j = n$, (4.1) becomes the Orlicz Minkowski inequality (1.12) for the Orlicz quermassintegrals, hence we assume $0 \leq i < j < n$. Since

$$\int_{G_{n,j}} d\nu(\xi) = \int_{G_{n,j}} \frac{\mathrm{vol}_i^j(K|\xi)^{-(n-i)}}{\int_{G_{n,j}} \mathrm{vol}_i^j(K|\xi)^{-(n-i)} d\mu_j(\xi)} d\mu_j(\xi) = 1,$$

so the above equation defines a Borel probability measure ν on $G_{n,j}$, nemely:

$$d\nu(\xi) = \frac{\mathrm{vol}_i^j(K|\xi)^{-(n-i)}}{\int_{G_{n,j}} \mathrm{vol}_i^j(K|\xi)^{-(n-i)} d\mu_j(\xi)} d\mu_j(\xi). \tag{4.2}$$

From (1.12), (3.6), (3.10), (4.2), Jensen integral inequality and Hölder integral inequality, we obtain

$$\left(\frac{\Phi_{n-j,i}(K)}{\Phi_{\varphi,n-j,i}(K,L)}\right)^{n-i} = \frac{\displaystyle\int_{G_{n,j}} \frac{W_{\varphi,i}^{(j)}(K|\xi, L|\xi)}{\mathrm{vol}_i^{(j)}(K|\xi)} \left(\frac{\mathrm{vol}_i^{(j)}(K|\xi)}{\omega_j}\right)^{-(n-i)} d\mu_j(\xi)}{\displaystyle\int_{G_{n,j}} \left(\frac{\mathrm{vol}_i^{(j)}(K|\xi)}{\omega_j}\right)^{-(n-i)} d\mu_j(\xi)}$$

$$
\begin{aligned}
&= \int_{G_{n,j}} \frac{W^{(j)}_{\varphi,i}(K|\xi,L|\xi)}{\mathrm{vol}^{(j)}_i(K|\xi)} d\nu \\
&\geq \int_{G_{n,j}} \varphi\left(\left(\frac{\mathrm{vol}^{(j)}_i(L|\xi)}{\mathrm{vol}^{(j)}_i(K|\xi)}\right)^{1/(j-i)}\right) d\nu \geq \varphi\left(\int_{G_{n,j}} \left(\frac{\mathrm{vol}_j(L|\xi)}{\mathrm{vol}_j(K|\xi)}\right)^{1/(j-i)} d\nu\right) \\
&= \varphi\left(\frac{\int_{G_{n,j}} \mathrm{vol}^{(j)}_i(K|\xi)^{(-(j-i)(n-i)-1)/(j-i)} \mathrm{vol}^{(j)}_i(L|\xi)^{1/(j-i)} d\mu_j(\xi)}{\int_{G_{n,j}} \mathrm{vol}^{(j)}_i(K|\xi)^{-(n-i)} d\mu_j(\xi)}\right) \\
&\geq \varphi\left(\frac{\left(\int_{G_{n,j}} \mathrm{vol}^{(j)}_i(K|\xi)^{i-n} d\mu_j(\xi)\right)^{\frac{(j-i)(n-i)+1}{(j-i)(n-i)}}}{\int_{G_{n,j}} \mathrm{vol}_j(K|\xi)^{i-n} d\mu_j(\xi) \left(\int_{G_{n,j}} \mathrm{vol}^{(j)}_i(L|\xi)^{i-n} d\mu_j(\xi)\right)^{\frac{1}{(j-i)(n-i)}}}\right) \\
&= \varphi\left(\left(\frac{\Phi_{n-j,i}(L)}{\Phi_{n-j,i}(K)}\right)^{1/(j-i)}\right).
\end{aligned}
$$

Next, we discuss the equal condition of (4.1). If φ is strictly convex, suppose that K and L are homothetic, i. e. there exist $\lambda > 0$ such that $L = \lambda K$. Hence

$$
\begin{aligned}
\left(\frac{\Phi_{n-j,i}(K)}{\Phi_{\varphi,n-j,i}(K,L)}\right)^{n-i} &= \left(\frac{\Phi_{\varphi,n-j,i}(K,\lambda K)}{\Phi_{n-j,i}(K)}\right)^{-(n-i)} \\
&= \left(\frac{\varphi(\lambda)^{-1/(n-i)}\Phi_{n-j,i}(K)}{\Phi_{n-j,i}(K)}\right)^{-(n-i)} \\
&= \varphi(\lambda) \\
&= \varphi\left(\left(\frac{\Phi_{n-j,i}(\lambda K)}{\Phi_{n-j,i}(K)}\right)^{1/(j-i)}\right) \\
&= \varphi\left(\left(\frac{\Phi_{n-j,i}(L)}{\Phi_{n-j,i}(K)}\right)^{1/(j-i)}\right).
\end{aligned}
$$

This implies the equality in (4.1) holds.

On the other hand, suppose the equality holds in (4.1), then these three inequalities in the above proof must satisfy the equal sign. Since the first inequality in the above proof is following:

$$
\frac{W^{(j)}_{\varphi,i}(K|\xi,L|\xi)}{\mathrm{vol}^{(j)}_i(K|\xi)} \geq \varphi\left(\left(\frac{\mathrm{vol}^{(j)}_i(L|\xi)}{\mathrm{vol}^{j}_i(K|\xi)}\right)^{1/(j-i)}\right).
$$

When φ is strictly convex, if the equality holds, form the equality condition of Orlicz-Minkowski inequality (1.12), yields that $K|\xi$ and $L|\xi$ must be homothetic. The second inequality in the above proof is following:

$$
\int_{G_{n,j}} \varphi\left(\left(\frac{\mathrm{vol}^{(j)}_i(L|\xi)}{\mathrm{vol}^{(j)}_i(K|\xi)}\right)^{1/(j-i)}\right) d\nu \geq \varphi\left(\int_{G_{n,j}} \left(\frac{\mathrm{vol}^{(j)}_i(L|\xi)}{\mathrm{vol}^{(j)}_i(K|\xi)}\right)^{1/(j-i)} d\nu\right).
$$

When φ is strictly convex, if the equality holds, form the equality condition of Jensen inequality (3.15), then $\frac{\mathrm{vol}^{(j)}_i(L|\xi)}{\mathrm{vol}^{(j)}_i(K|\xi)}$ must be a constant, this yields that $K|\xi$ and $L|\xi$ must be homothetic. In this proof, the third inequality is obtained by applying the Hölder inequality. Form the equality condition of Hölder inequality, yields that equality holds $\mathrm{vol}_i^{(j)}(K|\xi)$ and $\mathrm{vol}_i^{(j)}(L|\xi)$ must be proportional, namely $K|\xi$ and $L|\xi$ be homothetic.

Combinations of these equal conditions, it follows that equality in (4.1) holds, if φ is strictly convex, equality holds if and only if K and L are homothetic. □

Corollary 4.1 (L_p Minkowski inequality) *If $K, L \in \mathcal{K}^n$, $1 \leq p < \infty$ and $0 \leq i < j \leq n$, then*

$$\left(\frac{\Phi_{n-j,i}(K)}{\Phi_{p,n-j,i}(K,L)}\right)^{n-i} \geq \left(\frac{\Phi_{n-j,i}(L)}{\Phi_{n-j,i}(K)}\right)^{p/(j-i)}. \tag{4.3}$$

If φ is strictly convex, equality holds if and only if K and L are homothetic.

Proof This follows immediately from (4.1) with $\varphi(t) = t^p$, $1 \leq p < \infty$. □

Putting $j = n$ in (4.3), (4.3) becomes the L_p Minkowski inequality (2.9) for the quermassintegrals. Putting $i = 0$ and $j = n$ in (4.3), (4.3) becomes the well-known L_p Minkowski inequality for volumes.

Corollary 4.2 (Orlicz Minkowski inequality) *If $K, L \in \mathcal{K}^n$ and $\varphi \in \Phi$, then*

$$V_\varphi(K,L) \geq V(K)\varphi\left(\left(\frac{V(L)}{V(K)}\right)^{1/n}\right). \tag{4.4}$$

If φ is strictly convex, equality holds if and only if K and L are homothetic (see [9] and [40]).

Proof This follows immediately from (4.1) with $i = 0$ and $j = n$. □

The following uniqueness is a direct consequence of the Orlicz-Minkoswki inequality for the Orlicz mixed affine quermassintegrals.

Theorem 4.2 *If $\varphi \in \Phi$ and is strictly convex, $0 \leq i < j \leq n$ and $\mathcal{M} \subset \mathcal{K}^n$ such that $K, L \in \mathcal{M}$. If*

$$\Phi_{\varphi,n-j,i}(M,K) = \Phi_{\varphi,n-j,i}(M,L), \quad \textit{for all } M \in \mathcal{M} \tag{4.5}$$

or

$$\frac{\Phi_{\varphi,n-j,i}(K,M)}{\Phi_{n-j,i}(K)} = \frac{\Phi_{\varphi,n-j,i}(L,M)}{\Phi_{n-j,i}(L)}, \quad \textit{for all } M \in \mathcal{M} \tag{4.6}$$

then $K = L$.

Proof Suppose (4.5) hold. Taking K for M, then from Lemma 3.6 and Theorem 4.1, we obtain

$$\varphi(1)\Phi_{n-j,i}(K)^{-(n-i)} = \Phi_{\varphi,n-j,i}(K,L)^{-(n-i)} \geq \Phi_{n-j,i}(K)^{-(n-i)}\varphi\left(\left(\frac{\Phi_{n-j,i}(L)}{\Phi_{n-j,i}(K)}\right)^{1/(j-i)}\right),$$

with equality if and only if K and L are homothetic. Hence

$$\varphi(1) \geq \varphi\left(\left(\frac{\Phi_{n-j,i}(L)}{\Phi_{n-j,i}(K)}\right)^{1/(j-i)}\right),$$

with equality if and only if K and L are homothetic. Since φ is increasing function on $(0,\infty)$, this follows that

$$\Phi_{n-j,i}(K) \geq \Phi_{n-j,i}(L),$$

with equality if and only if K and L are homothetic. On the other hand, if taking L for M, we similar get $\Phi_{n-j,i}(K) \leq \Phi_{n-j,i}(L)$, with equality if and only if K and L are homothetic. Hence $\Phi_{n-j,i}(K) = \Phi_{n-j,i}(L)$, and K and L are homothetic, it follows that K and L must be equal.

Suppose (4.6) hold. Taking L for M, then from Lemma 3.6 and Theorem 4.1, we obtain

$$\varphi(1) = \frac{\Phi_{n-j,i}(K)^{n-i}}{\Phi_{\varphi,n-j,i}(K,L)^{n-i}} \geq \varphi\left(\left(\frac{\Phi_{n-j,i}(L)}{\Phi_{n-j,i}(K)}\right)^{1/(j-i)}\right),$$

with equality if and only if K and L are homothetic. Hence

$$\varphi(1) \geq \varphi\left(\left(\frac{\Phi_{n-j,i}(L)}{\Phi_{n-j,i}(K)}\right)^{1/(j-i)}\right),$$

with equality if and only if K and L are homothetic. Since φ is decreasing function on $(0,\infty)$, this follows that

$$\Phi_{n-j,i}(K) \geq \Phi_{n-j,i}(L),$$

with equality if and only if K and L are homothetic. On the other hand, if taking L for M, we similar get $\Phi_{n-j,i}(K) \leq \Phi_{n-j,i}(L)$, with equality if and only if K and L are homothetic. Hence $\Phi_{n-j,i}(K) = \Phi_{n-j,i}(L)$, and K and L are homothetic, it follows that K and L must be equal. □

Corollary 4.3 *If $\varphi \in \Phi$ and is strictly convex, $0 < j \leq n$ and $\mathcal{M} \subset \mathcal{K}^n$ such that $K, L \in \mathcal{M}$. If*

$$\Phi_{\varphi,n-j}(M,K) = \Phi_{\varphi,n-j}(M,L), \quad for\ all\ M \in \mathcal{M}$$

or

$$\frac{\Phi_{\varphi,n-j}(K,M)}{\Phi_{n-j}(K)} = \frac{\Phi_{\varphi,n-j}(L,M)}{\Phi_{n-j}(L)}, \quad for\ all\ M \in \mathcal{M}$$

then $K = L$.

Proof This follows immediately from Theorem 4.2 with $i = 0$. □

Corollary 4.4 *If $\varphi \in \Phi$ and is strictly convex, $0 \leq i \leq n$ and $\mathcal{M} \subset \mathcal{K}^n$ such that $K, L \in \mathcal{M}$. If*

$$W_{\varphi,i}(M,K) = W_{\varphi,i}(M,L), \quad for\ all\ M \in \mathcal{M}$$

or

$$\frac{W_{\varphi,i}(K,M)}{W_i(K)} = \frac{W_{\varphi,i}(L,M)}{W_i(L)}, \quad for\ all\ M \in \mathcal{M}$$

then $K = L$.

Proof This follows immediately from Theorem 4.2 with $j = n$. □

Corollary 4.5 *If $\varphi \in \Phi$ and is strictly convex and $\mathcal{M} \subset \mathcal{K}^n$ such that $K, L \in \mathcal{M}$. If*

$$V_{\varphi}(M,K) = V_{\varphi}(M,L), \quad for\ all\ M \in \mathcal{M}$$

or

$$\frac{V_{\varphi}(K,M)}{V(K)} = \frac{V_{\varphi}(L,M)}{V(L)}, \quad for\ all\ M \in \mathcal{M}$$

then $K = L$.

Proof This follows immediately from Theorem 4.2 with $j = n$ and $i = 0$. □

5 Orlicz-Brunn-Minkoswki inequality for the Orlicz mixed affine quermassintegrals

Lemma 5.1 *If $K, L \in \mathcal{K}^n$, $0 \le i < j \le n$ and $\varphi \in \Phi$, then for any $\varepsilon > 0$*

$$1 = \left(\frac{\Phi_{n-j,i}(K +_\varphi \varepsilon \cdot L)}{\Phi_{\varphi,n-j,i}(K +_\varphi \varepsilon \cdot L, K)}\right)^{n-i} + \varepsilon \cdot \left(\frac{\Phi_{n-j,i}(K +_\varphi \varepsilon \cdot L)}{\Phi_{\varphi,n-j,i}(K +_\varphi \varepsilon \cdot L, L)}\right)^{n-i}. \tag{5.1}$$

Proof From (3.1), Lemma 3.1 and Lemma 3.4, we have

$$\begin{aligned} & W^{(j)}_{\varphi,i}((K +_\varphi \varepsilon \cdot L)|\xi, K|\xi) + \varepsilon W^{(j)}_{\varphi,i}((K +_\varphi \varepsilon \cdot L)|\xi, L|\xi) \\ &\quad = W^{(j)}_{\varphi,i}((K|\xi) +_\varphi \varepsilon \cdot (L|\xi), K|\xi) + \varepsilon W^{(j)}_{\varphi,i}((K|\xi) +_\varphi \varepsilon \cdot (L|\xi), L|\xi) \\ &\quad = W^{(j)}_{\varphi,i}((K|\xi) +_\varphi \varepsilon \cdot (L|\xi), (K|\xi) +_\varphi \varepsilon \cdot (L|\xi)) \\ &\quad = \mathrm{vol}^{(j)}_i((K|\xi) +_\varphi \varepsilon \cdot (L|\xi)) \\ &\quad = \mathrm{vol}^{(j)}_i((K +_\varphi \varepsilon \cdot L)|\xi). \end{aligned} \tag{5.2}$$

Let $Q = K +_\varphi \varepsilon \cdot L$, from (3.6), (3.10) and (5.2), we have

$$\begin{aligned} & \Phi_{\varphi,n-j,i}(Q, K)^{-(n-i)} + \varepsilon \cdot \Phi_{\varphi,n-j,i}(Q, L)^{-(n-i)} \\ &\quad = \frac{1}{\omega_n^{n-i}} \int_{G_{n,j}} \frac{W^{(j)}_{\varphi,i}(Q|\xi, K|\xi) + \varepsilon \cdot W^{(j)}_{\varphi,i}(Q|\xi, L|\xi)}{\mathrm{vol}^{(j)}_i(Q|\xi)} \left(\frac{\mathrm{vol}^{(j)}_i(Q|\xi)}{\omega_j}\right)^{-(n-i)} d\mu_j(\xi) \\ &\quad = \frac{1}{\omega_n^{n-i}} \int_{G_{n,j}} \left(\frac{\mathrm{vol}^j_i(Q|\xi)}{\omega_j}\right)^{-(n-i)} d\mu_j(\xi) \\ &\quad = \Phi_{n-j,i}(Q)^{-(n-i)}. \end{aligned}$$

The proof is complete. □

Lemma 5.2 [50] *Let $K, L \in \mathcal{K}^n$, $\varepsilon > 0$ and $\varphi \in \Phi$.*
(1) If K and L are homothetic, then K and $K +_\varphi \varepsilon \cdot L$ are homothetic.
(2) If K and $K +_\varphi \varepsilon \cdot L$ are homothetic, then K and L are homothetic.

Theorem 5.1 (Orlicz-Brunn-Minkowski inequality) *If $K, L \in \mathcal{K}^n$, $\varepsilon > 0$, $0 \le i < j \le n$ and $\varphi \in \Phi$, then*

$$1 \ge \varphi\left(\left(\frac{\Phi_{n-j,i}(K)}{\Phi_{n-j,i}(K +_\varphi \varepsilon \cdot L)}\right)^{1/(j-i)}\right) + \varepsilon \cdot \varphi\left(\left(\frac{\Phi_{n-j,i}(L)}{\Phi_{n-j,i}(K +_\varphi \varepsilon \cdot L)}\right)^{1/(j-i)}\right). \tag{5.3}$$

If φ is strictly convex, equality holds if and only if K and L are homothetic.

Proof From Lemma 5.1 and Theorem 4.1, we obtain

$$\begin{aligned} 1 &= \left(\frac{\Phi_{n-j,i}(K +_\varphi \varepsilon \cdot L)}{\Phi_{\varphi,n-j,i}(K +_\varphi \varepsilon \cdot L, K)}\right)^{n-i} + \varepsilon \cdot \left(\frac{\Phi_{n-j,i}(K +_\varphi \varepsilon \cdot L)}{\Phi_{\varphi,n-j,i}(K +_\varphi \varepsilon \cdot L, L)}\right)^{n-i} \\ &\ge \varphi\left(\left(\frac{\Phi_{n-j,i}(K)}{\Phi_{n-j,i}(K +_\varphi \varepsilon \cdot L)}\right)^{1/(j-i)}\right) + \varepsilon \cdot \varphi\left(\left(\frac{\Phi_{n-j,i}(L)}{\Phi_{n-j,i}(K +_\varphi \varepsilon \cdot L)}\right)^{1/(j-i)}\right). \end{aligned}$$

If φ is strictly convex, from equality condition of the Orlicz-Minkowski inequality, the equality holds if and only if K and $K +_\varphi \varepsilon \cdot L$ are homothetic, and L and $K +_\varphi \varepsilon \cdot L$ are homothetic and combine with Lemma 5.2, this yields that if φ is strictly convex, equality holds in (5.3) if and only if K and L are homothetic. □

Corollary 5.1 (L_p- Brunn-Minkowski inequality) *If $K, L \in \mathcal{K}^n$, $1 \le p < \infty$, $\varepsilon > 0$ and $0 \le i < j \le n$, then*

$$\Phi_{n-j,i}(K +_p \varepsilon \cdot L)^{p/(j-i)} \ge \Phi_{n-j,i}(K)^{p/(j-i)} + \varepsilon \cdot \Phi_{n-j,i}(L)^{p/(j-i)}, \tag{5.4}$$

with equality if and only if K and L are homothetic.

Proof This follows immediately from (5.3) with $\varphi(t) = t^p, 1 \le p < \infty$. □

Putting $j = n$, $i = 0$ and $\varepsilon = 1$ in (5.4), (5.4) becomes Lutwak's L_p dual Brunn-Minkowski inequality (see [21])

$$V(K +_p L)^{p/n} \ge V(K)^{p/n} + V(L)^{p/n}, \tag{5.5}$$

with equality if and only if K and L are homothetic.

Corollary 5.2 (Orlicz Brunn-Minkowski inequality) *If $K, L \in \mathcal{K}^n$ and $\varphi \in \Phi$, then*

$$1 \ge \varphi\left(\left(\frac{V(K)}{V(K +_\varphi L)}\right)^{1/n}\right) + \left(\left(\frac{V(L)}{V(K \widehat{+}_\varphi L)}\right)^{1/n}\right). \tag{5.6}$$

If φ is strictly convex, equality holds if and only if K and L are homothetic (see [9]and [40]).

Proof This follows immediately from (5.3) with $\varepsilon = 1$, $i = 0$ and $j = n$. □

Corollary 5.3 *If $\varphi \in \Phi$, $0 \le i < j \le n$ and $K, L \in \mathcal{K}^n$, then*

$$\left(\frac{\Phi_{n-j,i}(K)}{\Phi_{\varphi,n-j,i}(K,L)}\right)^{n-i} \ge \varphi\left(\left(\frac{\Phi_{n-j,i}(L)}{\Phi_{n-j,i}(K)}\right)^{1/(j-i)}\right). \tag{5.7}$$

If φ is strictly convex, equality holds if and only if K and L are homothetic.

Proof Let

$$K_\varepsilon = K +_\varphi \varepsilon \cdot L.$$

From Lemma 3.2, Lemma 3.5 and (5.3), we obtain

$$\begin{aligned}
&\frac{j-i}{\varphi'_-(1)}\Phi_{n-j,i}(K)^{1+n-i}\Phi_{\varphi,n-j,i}(K,L)^{-(n-i)} = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+}\Phi_{n-j,i}(K_\varepsilon)\\
&= \lim_{\varepsilon\to 0^+}\frac{\Phi_{n-j,i}(K_\varepsilon) - \Phi_{n-j,i}(K)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0^+}\frac{1-\dfrac{\Phi_{n-j,i}(K)}{\Phi_{n-j,i}(K_\varepsilon)}}{\varphi(1)-\varphi\left(\left(\dfrac{\Phi_{n-j,i}(K)}{\Phi_{n-j,i}(K_\varepsilon)}\right)^{1/(j-i)}\right)} \cdot \frac{1-\varphi\left(\left(\dfrac{\Phi_{n-j,i}(K)}{\Phi_{n-j,i}(K_\varepsilon)}\right)^{1/(j-i)}\right)}{\varepsilon} \cdot \Phi_{n-j,i}(K_\varepsilon)\\
&= \lim_{\varepsilon\to 0^+}\frac{1-t}{\varphi(1)-\varphi\left(t^{1/(j-i)}\right)} \cdot \lim_{\varepsilon\to 0^+}\frac{1-\varphi\left(\left(\dfrac{\Phi_{n-j,i}(K)}{\Phi_{n-j,i}(K_\varepsilon)}\right)^{1/(j-i)}\right)}{\varepsilon} \cdot \lim_{\varepsilon\to 0^+}\Phi_{n-j,i}(K_\varepsilon)\\
&\ge \frac{j-i}{\varphi'_-(1)} \cdot \lim_{\varepsilon\to 0^+}\varphi\left(\left(\frac{\Phi_{n-j,i}(L)}{\Phi_{n-j,i}(K_\varepsilon)}\right)^{1/(j-i)}\right) \cdot \lim_{\varepsilon\to 0^+}\Phi_{n-j,i}(K_\varepsilon)
\end{aligned}$$

$$= \frac{j-i}{\varphi'_{-}(1)} \cdot \varphi\left(\left(\frac{\Phi_{n-j,i}(L)}{\Phi_{n-j,i}(K)}\right)^{1/(j-i)}\right) \cdot \Phi_{n-j,i}(K). \tag{5.8}$$

From (5.8), (5.7) easy follows.

Corollary 5.4 *If $\varphi \in \Phi$, $0 < j \leq n$ and $K, L \in \mathcal{K}^n$, then*

$$\left(\frac{\Phi_{n-j}(K)}{\Phi_{\varphi,n-j}(K,L)}\right)^n \geq \varphi\left(\left(\frac{\Phi_{n-j}(L)}{\Phi_{n-j}(K)}\right)^{1/j}\right)$$

$$\Leftrightarrow 1 \geq \varphi\left(\left(\frac{\Phi_{n-j}(K)}{\Phi_{n-j}(K+_\varphi L)}\right)^{1/j}\right) + \varphi\left(\left(\frac{\Phi_{n-j}(L)}{\Phi_{n-j}(K+_\varphi L)}\right)^{1/j}\right).$$

If φ is strictly convex, equality holds if and only if K and L are homothetic.

Corollary 5.5 *If $\varphi \in \Phi$, $0 \leq i \leq n$ and $K, L \in \mathcal{K}^n$, then*

$$\frac{W_{\varphi,i}(K,L)}{W_i(K)} \geq \varphi\left(\left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}\right)$$

$$\Leftrightarrow 1 \geq \varphi\left(\left(\frac{W_i(K)}{W_i(K+_\varphi L)}\right)^{1/(n-i)}\right) + \varphi\left(\left(\frac{W_i(L)}{W_i(K+_\varphi L)}\right)^{1/(n-i)}\right).$$

If φ is strictly convex, equality holds if and only if K and L are homothetic.

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