Study of totally contact umbilical hemi-slant lightlike submanifolds of indefinite Sasakian manifolds

Rashmi Sachdeva, Rachna Rani, Rakesh Kumar *and* Satvinder Singh Bhatia

Abstract. In this paper, we prove that there do not exist totally contact umbilical hemi-slant lightlike submanifolds of indefinite Sasakian manifolds and of indefinite contact space forms other than totally contact geodesic hemi-slant lightlike submanifolds. Consequently, we prove that the induced connection on a totally contact umbilical hemi-slant lightlike submanifold of an indefinite Sasakian manifold is a metric connection. Finally we derive some characterization theorems for minimal hemi-slant lightlike submanifolds of an indefinite Sasakian manifold.

M.S.C. 2010: 53C25, 53C40, 53C50.
Key words: hemi-slant lightlike submanifolds; indefinite Sasakian manifolds; minimal lightlike submanifolds; totally contact umbilical lightlike submanifolds.

1 Introduction

The geometry of submanifolds with degenerate (lightlike) metric is difficult and strikingly different from the geometry of submanifolds with non-degenerate metric because of the fact that their (of degenerate submanifolds) normal vector bundle intersects with the tangent bundle. This means that we cannot use the classical theory of submanifolds to define induced objects on a lightlike submanifold. Since the geometry of lightlike submanifolds is needed to fill a gap in the general theory of submanifolds and have significant applications in general theory of relativity. Therefore Duggal and Bejancu [4] introduced and studied the geometry of lightlike submanifolds of semi-Riemannian manifolds extensively and further studied by many authors. On the other hand significant uses of the contact geometry in differential equations, optics and phase spaces of a dynamical system [7, 8] and very limited specific information available on its lightlike case motivated the authors to do work on the geometry of lightlike submanifolds of indefinite Sasakian manifolds.

As a generalization of invariant and totally real (anti-invariant) submanifolds of almost contact metric manifolds, slant submanifolds of Sasakian manifolds were introduced by Lotta [6] and further studied by Cabrerizo et al. [3]. In 2012, Sahin and

Balkan Journal of Geometry and Its Applications, Vol.22, No.1, 2017, pp. 70-80.

Yildirim [12] introduced slant lightlike submanifolds of indefinite Sasakian manifolds. Recently, authors have studied the geometry of totally umbilical slant and totally umbilical hemi-slant lightlike submanifolds when the ambient manifold is an indefinite almost complex or an indefinite almost contact manifold (see [9, 10, 11]). In this paper, we prove that there do not exist totally contact umbilical hemi-slant lightlike submanifolds of indefinite Sasakian manifolds other than totally contact geodesic hemi-slant lightlike submanifolds. Finally we also derive some characterization theorems for minimal hemi-slant lightlike submanifolds of an indefinite Sasakian manifold.

2 Lightlike submanifolds

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\bar{M}$ and g be the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M then M is called a lightlike submanifold of $\bar{M}$. For a degenerate metric g on M, $TM^{\perp}$ is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $Rad(T_xM) = T_xM \cap T_xM^{\perp}$ which is known as the radical (null) subspace. If the mapping $Rad(TM) : x \in M \longrightarrow Rad(T_xM)$, defines a smooth distribution on M of rank $r > 0$ then the submanifold M of $\bar{M}$ is called an r-lightlike submanifold and $Rad(TM)$ is called the radical distribution on M (for detail see [4]).

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, that is, $TM = Rad(TM) \perp S(TM)$ and $S(TM^{\perp})$ is a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and to $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ respectively. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}). \tag{2.1}$$

$$T\bar{M}\mid_M = TM \oplus tr(TM) = (Rad(TM) \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{2.2}$$

For quasi-orthonormal fields of frames, we have

Theorem 2.1. *([4]). Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exist a complementary vector bundle $ltr(TM)$ of $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM)\mid_{\mathcal{U}})$ consisting of smooth sections $\{N_i\}$ of $S(TM^{\perp})^{\perp}\mid_{\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of M, such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \quad \textit{for any} \quad i, j \in \{1, 2, .., r\}, \tag{2.3}$$

where $\{\xi_1, ..., \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. Then, according to the decomposition (2.2), the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \tag{2.4}$$

for $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X,Y), \nabla^{\perp}_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$, known as the second fundamental form and A_U is a linear operator on M, known as the shape operator. According to (2.1), considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively then (2.4) becomes

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \tag{2.5}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \tag{2.6}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W). \tag{2.7}$$

As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued respectively, therefore they are called as the lightlike second fundamental form and the screen second fundamental form on M. Using (2.1)-(2.2) and (2.5)-(2.7), we obtain

$$\bar{g}(h^s(X,Y),W) + \bar{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{2.8}$$

$$\bar{g}(h^l(X,Y),\xi) + \bar{g}(Y, h^l(X,\xi)) + g(Y, \nabla_X \xi) = 0, \tag{2.9}$$

for any $X, Y \in \Gamma(TM)$, $W \in \Gamma(S(TM^{\perp}))$ and $\xi \in \Gamma(Rad(TM))$. Let $\bar{P}$ is a projection of TM on $S(TM)$ then we can write

$$\nabla_X \bar{P}Y = \nabla^*_X \bar{P}Y + h^*(X, \bar{P}Y), \quad \nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla^*_X \bar{P}Y, A^*_\xi X\}$ and $\{h^*(X, \bar{P}Y), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$, respectively. Here ∇^* and ∇^{*t}_X are linear connections on $S(TM)$ and $Rad(TM)$, respectively. By using (2.6), (2.7) and (2.10), we obtain

$$\bar{g}(h^l(X, \bar{P}Y), \xi) = g(A^*_\xi X, \bar{P}Y), \quad \bar{g}(h^*(X, \bar{P}Y), N) = \bar{g}(A_N X, \bar{P}Y). \tag{2.11}$$

Definition 2.1. An odd-dimensional semi-Riemannian manifold $\bar{M}$ is said to be an indefinite almost contact metric manifold if there exist structure tensors $\{\phi, V, \eta, \bar{g}\}$, where ϕ is a $(1,1)$ tensor field, V a vector field, called characteristic vector field, η a 1-form and $\bar{g}$ is the semi-Riemannian metric on $\bar{M}$ satisfying (see [2])

$$\phi^2 X = -X + \eta(X)V, \quad \eta \circ \phi = 0, \quad \phi V = 0, \quad \eta(V) = 1, \tag{2.12}$$

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X,Y) - \eta(X)\eta(Y), \quad \bar{g}(X,V) = \eta(X), \tag{2.13}$$

for $X, Y \in \Gamma(T\bar{M})$. An indefinite almost contact metric manifold $\bar{M}$ is called an indefinite Sasakian manifold if (see [5]),

$$(\bar{\nabla}_X \phi)Y = -\bar{g}(X,Y)V + \eta(Y)X, \ \textit{and} \ \bar{\nabla}_X V = \phi X, \tag{2.14}$$

for any $X, Y \in \Gamma(T\bar{M})$, where $\bar{\nabla}$ denote the Levi-Civita connection on $\bar{M}$.

3 Hemi-slant lightlike submanifolds

Let M be an r-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index q such that V is tangent to M. Then V does not belong to $Rad(TM)$ as if we assume that $V \in \Gamma(Rad(TM))$, then there exists a vector field $N \in \Gamma(ltr(TM))$ such that $g(N, V) = 1$, but using (2.3), (2.12) and (2.13), $g(N, V) = g(\phi N, \phi V) = 0$, this leads to a contradiction. Let the radical distribution be such that $\phi(Rad(TM)) = ltr(TM)$; then we have a local quasi-orthonormal field of frames on $\bar{M}$ along M as $\{X_a, V, \xi_i, N_i, W_\alpha\}$, where $\{\xi_i\}_{i=1}^r$ and $\{N_i\}_{i=1}^r$ are lightlike basis of $Rad(TM)$ and $ltr(TM)$, respectively and $\{X_a\}_{a=1}^k$ and $\{W_\alpha\}_{\alpha=1}^l$ are orthonormal basis of $S(TM)$ (except $\{V\}$) and $S(TM^\perp)$, respectively. From the lightlike basis $\{\xi_1, ..., \xi_r, N_i, ..., N_r\}$ of $Rad(TM) \oplus ltr(TM)$, we can construct an orthonormal basis $\{U_1, ..., U_{2r}\}$ as

$$U_1 = \frac{1}{\sqrt{2}}(\xi_1 + N_1) \qquad U_2 = \frac{1}{\sqrt{2}}(\xi_1 - N_1),$$

$$U_3 = \frac{1}{\sqrt{2}}(\xi_2 + N_2) \qquad U_4 = \frac{1}{\sqrt{2}}(\xi_2 - N_2),$$

$$\cdots \qquad \cdots$$

$$\cdots \qquad \cdots$$

$$U_{2r-1} = \frac{1}{\sqrt{2}}(\xi_r + N_r) \qquad U_{2r} = \frac{1}{\sqrt{2}}(\xi_r - N_r).$$

Thus $Span\{\xi_i, N_i\}$ is a non-degenerate space of constant index r implies $Rad(TM) \oplus ltr(TM)$ is non-degenerate and of constant index r on $\bar{M}$. Therefore

$$index\ (T\bar{M}) = index\ (Rad(TM) \oplus ltr(TM)) + index\ (S(TM) \perp S(TM^\perp)),$$

implies that $q = r + index\ (S(TM) \perp S(TM^\perp))$. If $r = q$, then $S(TM) \perp S(TM^\perp)$ is Riemannian and hence $S(TM)$ is Riemannian. Thus we have the following lemma.

Lemma 3.1. *Let M be an r-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index q such that the characteristic vector field V is tangent to M. Assume that the radical distribution $Rad(TM)$ is a distribution such that $\phi(Rad(TM)) = ltr(TM)$. If $r = q$ then the screen distribution $S(TM)$ is Riemannian.*

To define slant submanifolds, we need angle between two vector fields of the submanifold. The radical distribution is totally lightlike so it is not possible to define an angle between two of its vector fields. From Lemma 3.1, the screen distribution is Riemannian and it is possible to define an angle between two of its vector fields. Thus we define hemi-slant lightlike submanifolds of indefinite Sasakian manifolds as:

Definition 3.1. Let M be a q-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index q with characteristic vector field V tangent to M. Then M is said to be a hemi-slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:
(i) $Rad(TM)$ is a distribution on M such that $\phi(Rad(TM)) = ltr(TM)$.
(ii) For all $x \in \mathcal{U} \subset M$ and for each non-zero vector field X tangent to $S(TM) = D^\theta \perp V$, if X and V are linearly independent, then the angle $\theta(X)$ between ϕX and the vector space S(TM) is constant, where D^θ is complementary distribution to V in screen distribution $S(TM)$.

A hemi-slant lightlike submanifold is said to be proper if $D^\theta \neq 0$ and $\theta \neq 0, \pi/2$. Hence by using the definition of hemi-slant lightlike submanifolds, the tangent bundle TM of M is decomposed as $TM = S(TM) \perp Rad(TM) = D^\theta \perp \{V\} \perp Rad(TM)$.

Example 3.2. Let M be a lightlike submanifold of a semi-Euclidean space $(\mathbf{R}^9_2, \bar{g})$ and defined by $x_1 = s$, $x_2 = t$, $x_3 = u\ sinv$, $x_4 = sinu$, $y_1 = t$, $y_2 = s$, $y_3 = u\ cosv$, $y_4 = cosu$, where $u, v \in (0, \pi/2)$ and $\mathbf{R}^9_2$ is of signature $(-,+,+,+,-,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}$. Then the local frame of fields $\{\xi_1, \xi_2, Z_1, Z_2, V\}$ of TM is given by $\xi_1 = \partial x_1 + \partial y_2$, $\xi_2 = \partial x_2 + \partial y_1$, $Z_1 = sinv\partial x_3 + cosu\partial x_4 + cosv\partial y_3 - sinu\partial y_4$, $Z_2 = u\ cosv\partial x_3 - u\ sinv\partial y_3$, $V = \partial z$. Hence M is a 2-lightlike submanifold with $Rad(TM) = span\{\xi_1, \xi_2\}$ and $S(TM) = span\{Z_1, Z_2\} \perp V$, which is Riemannian. It can be easily seen that $S(TM)$ is a slant distribution with slant angle $\theta = \pi/4$. Further the screen transversal bundle $S(TM^\perp)$ is spanned by $W_1 = sinu\partial x_4 + cosu\partial y_4$, $W_2 = sinv\partial x_3 - cosu\partial x_4 + cosv\partial y_3 + sinu\partial y_4$. The transversal lightlike bundle $ltr(TM)$ is spanned by $N_1 = -\frac{1}{2}(-\partial x_1 - \partial y_2)$, $N_2 = \frac{1}{2}(\partial x_2 - \partial y_1)$. Clearly $\phi\xi_1 = 2N_2, \phi\xi_2 = -2N_1$. Hence M is a hemi-slant lightlike submanifold of $\mathbf{R}^9_2$.

Denote the projection morphisms from TM on D^θ and $Rad(TM)$ by P and Q respectively, then any X tangent to M can be written as $X = PX + \eta(X)V + QX$. On applying ϕ to both sides and then using the definition of hemi-slant lightlike submanifolds with $\phi V = 0$, we can write

$$\phi X = TPX + FPX + FQX, \tag{3.1}$$

where $TPX \in \Gamma(D^\theta)$, $FPX \in \Gamma(tr(TM))$ and $FQX \in \Gamma(ltr(TM))$. Similarly, for any $U \in \Gamma(tr(TM))$, we can write $\phi U = BU + CU$, where BU and CU are tangential and transversal components of ϕU, respectively.

Lemma 3.2. *Let M be a hemi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ then $FPX \in \Gamma(S(TM^\perp))$, for any $X \in \Gamma(D^\theta)$.*

Proof. Using (2.1) and (2.3), it is clear that $FPX \in \Gamma(S(TM^\perp))$, if and only if, $\bar{g}(FPX, \xi) = 0$, for any $\xi \in \Gamma(Rad(TM))$. Using (2.12), (2.13) and (3.1), we have $\bar{g}(FPX, \xi) = \bar{g}(\phi PX - TPX, \xi) = \bar{g}(\phi PX, \xi) = -\bar{g}(PX, \phi\xi) = 0$, for any $X \in \Gamma(D^\theta)$ and hence the result follows. □

Thus using the Lemma 3.2 with (2.14), it follows that $F(S(TM))$ is a subspace of $S(TM^\perp)$. Therefore there exists an invariant subspace μ of $S(TM^\perp)$ such that

$$S(TM^\perp) = F(S(TM)) \perp \mu, \tag{3.2}$$

hence $T_p\bar{M} = S(T_pM) \perp \{Rad(T_pM) \oplus ltr(T_pM)\} \perp \{F(S(T_pM)) \perp \mu_p\}$.

Differentiating (3.1) and using (2.5) to (2.7), for any $X, Y \in \Gamma(TM)$, we have

$$(\nabla_X T)PY = A_{FPY}X + A_{FQY}X + Bh^l(X,Y) + Bh^s(X,Y) - g(X,Y)V + \eta(Y)X,$$

$$(\nabla_X F)PY = Ch^s(X,Y) - h^s(X,TPY) - D^s(X,FQY), \tag{3.3}$$

$$(\nabla_X F)QY = -h^l(X,TPY) - D^l(X,FPY), \tag{3.4}$$

where $(\nabla_X T)PY = \nabla_X TPY - TP\nabla_X Y$, $(\nabla_X F)PY = \nabla^s_X FPY - FP\nabla_X Y$, and $(\nabla_X F)QY = \nabla^l_X FQY - FQ\nabla_X Y$.

Theorem 3.3. *Let M be a q-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index q. Then M is a hemi-slant lightlike submanifold if and only if $\phi(ltr(TM))$ is a distribution on M and for any vector field X tangent to M, there exists a constant $\lambda \in [-1, 0]$ such that $(TP)^2X = \lambda PX$, where $\lambda = -\cos^2\theta$.*

Proof. Assume that M is a hemi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then $\phi(Rad(TM)) = ltr(TM)$, that is, $\phi(ltr(TM)) = Rad(TM)$, which proves (a). In order to prove (b), let $X \in \Gamma(TM)$ which is linearly independent of the characteristic vector field V and $PX \in \Gamma(D^\theta)$ then using (2.12) and (2.13), we have

$$cos\theta(PX) = \frac{\bar{g}(\phi PX, TPX)}{|\phi PX||TPX|} = -\frac{\bar{g}(PX, \phi TPX)}{|PX||TPX|} = -\frac{\bar{g}(PX, TPTPX)}{|PX||TPX|}. \tag{3.5}$$

On the other hand, $cos\theta(PX) = \frac{|TPX|}{|\phi PX|} = \frac{|TPX|}{|PX|}$, combining this with (3.5), we obtain $cos^2\theta(PX) = -\frac{\bar{g}(PX,(TP)^2X)}{|PX|^2}$. Taking into account that $\theta(PX)$ is constant on $\Gamma(D^\theta)$, from above equation, we infer that $(TP)^2X = \lambda PX$, where $\lambda \in [-1, 0]$. Converse part follows directly from (a) and (b). □

Corollary 3.4. *Let M be a hemi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index q. Then for any $X, Y \in \Gamma(TM)$, we have*

$$g(TPX, TPY) = \cos^2\theta\{g(PX, PY) - \eta(PX)\eta(PY)\}, \tag{3.6}$$

$$\bar{g}(FPX, FPY) = \sin^2\theta\{g(PX, PY) - \eta(PX)\eta(PY)\}. \tag{3.7}$$

4 Totally contact umbilical lightlike submanifolds

Definition 4.1. If the second fundamental form h of a submanifold, tangent to characteristic vector field V, of an indefinite Sasakian manifold $\bar{M}$ is of the form

$$h(X, Y) = \{g(X, Y) - \eta(X)\eta(Y)\}H + \eta(X)h(Y, V) + \eta(Y)h(X, V), \tag{4.1}$$

for any $X, Y \in \Gamma(TM)$, where H is a vector field transversal to M, then M is called a totally contact umbilical submanifold and moreover if $H = 0$ then submanifold is called a totally contact geodesic. The above definition also holds for a lightlike submanifold M. For a totally contact umbilical lightlike submanifold M, we have

$$h^l(X, Y) = \{g(X, Y) - \eta(X)\eta(Y)\}H^l + \eta(X)h^l(Y, V) + \eta(Y)h^l(X, V), \tag{4.2}$$

$$h^s(X, Y) = \{g(X, Y) - \eta(X)\eta(Y)\}H^s + \eta(X)h^s(Y, V) + \eta(Y)h^s(X, V), \tag{4.3}$$

where $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(S(TM^\perp))$.

From now on, we denote $\bar{M}$ as an indefinite Sasakian manifold unless otherwise stated.

Lemma 4.1. *Let M be a totally contact umbilical proper hemi-slant lightlike submanifold of $\bar{M}$ then $\nabla_X X \in \Gamma(S(TM))$, for any $X \in \Gamma(D^\theta)$.*

Proof. Let $X \in \Gamma(D^\theta)$ and $N \in \Gamma(ltr(TM))$ such that $\phi N = \xi$ then using (2.13) and (2.14), we obtain $\bar{g}(\nabla_X X, N) = \bar{g}(h^l(X, TX), \xi) + \bar{g}(D^l(X, FX), \xi)$. Further using (2.8), (4.2) and (4.3), we obtain $\bar{g}(\nabla_X X, N) = g(X, TX)g(H^l, \xi)$. Now using (2.12) and (2.13), we have $g(X, TX) = g(X, \phi X) = 0$, for any $X \in \Gamma(D^\theta)$ and hence $\bar{g}(\nabla_X X, N) = 0$. Thus using (2.3), the assertion follows. □

Theorem 4.2. *Every totally contact umbilical proper hemi-slant lightlike submanifold M of an indefinite Sasakian manifold $\bar{M}$ is totally contact geodesic.*

Proof. Let $X, Y \in \Gamma(D^\theta)$ then $FQX = FQY = 0$, on adding (3.3) and (3.4), we have $\nabla^s_X FPY - FP\nabla_X Y - FQ\nabla_X Y = Ch^s(X, Y) - h^s(X, TPY) - h^l(X, TPY) - D^l(X, FPY)$. Replace X by TX and Y by X, we get $FP\nabla_{TX}X + FQ\nabla_{TX}X - \nabla^s_{TX}FPX = -Ch^s(TX, X) + h^s(TX, TPX) + h^l(TX, TPX) + D^l(TX, FPX)$. Since M is a totally contact umbilical hemi-slant lightlike submanifold then using (3.6), (4.3) and the fact that $h^s(TX, X) = \bar{g}(\phi X, X)H^s = 0$ and $\eta(X) = 0$, for any $X \in \Gamma(D^\theta)$, we get $FP\nabla_{TX}X + FQ\nabla_{TX}X - \nabla^s_{TX}FPX = cos^2\theta g(PX, PX)H + D^l(TX, FPX)$. Now, on taking scalar product both sides with respect to $FPX \in (S(TM^\perp))$, we get $cos^2\theta g(X, X)\bar{g}(H^s, FPX) = \bar{g}(FP\nabla_{TX}X, FPX) - \bar{g}(\nabla^s_{TX}FPX, FPX)$, further using (3.7), we get

$$cos^2\theta g(X, X)\bar{g}(H^s, FPX) = sin^2\theta\bar{g}(P\nabla_{TX}X, PX) - \bar{g}(\nabla^s_{TX}FPX, FPX). \tag{4.4}$$

Let $X \in \Gamma(D^\theta)$ then on taking covariant derivative of (3.7) with respect to $\bar{\nabla}_{TX}$, we get $\bar{g}(\nabla^s_{TX}FPX, FPX) = sin^2\theta g(\nabla_{TX}PX, PX)$ and then using this in (4.4), we obtain $cos^2\theta g(X, X)\bar{g}(H^s, FPX) = 0$. Since M is a proper hemi-slant lightlike submanifold and g is a Riemannian metric on $S(TM)$ then we have $\bar{g}(H^s, FPX) = 0$ and further using the Lemma 3.2 with (3.2), we obtain

$$H^s \in \Gamma(\mu). \tag{4.5}$$

Let $X, Y \in \Gamma(D^\theta)$ then using (2.14), we have $\bar{\nabla}_X \phi Y = \phi\bar{\nabla}_X Y - g(X, Y)V$ then using (2.5), (2.7) and (4.1), we have $\nabla_X TPY + g(X, TPY)H - A_{FPY}X + \nabla^s_X FPY + D^l(X, FPY) = T\nabla_X Y + F\nabla_X Y + g(X, Y)\phi H - g(X, Y)V$. On taking scalar product both sides with respect to ϕH^s and then using the invariant property of μ with (2.13) and (4.5), we obtain

$$\bar{g}(\nabla^s_X FPY, \phi H^s) = g(X, Y)\bar{g}(H^s, H^s). \tag{4.6}$$

Since μ is an invariant subspace therefore using the Sasakian character of $\bar{M}$, that is, using (2.14) for any $H^s \in \Gamma(\mu)$, we get $-A_{\phi H^s}X + \nabla^s_X \phi H^s + D^l(X, \phi H^s) = -TA_{H^s}X - FA_{H^s}X + B\nabla^s_X H^s + C\nabla^s_X H^s + \phi D^l(X, H^s)$. Taking the scalar product both sides with respect to FPY, we get

$$\bar{g}(\nabla^s_X \phi H^s, FPY) = -\bar{g}(FA_{H^s}X, FPY) + \bar{g}(C\nabla^s_X H^s, FPY). \tag{4.7}$$

We know that for any $U \in \Gamma(tr(TM))$, BU and CU are tangential and transversal components of ϕU, respectively. Therefore if $U \in \Gamma(ltr(TM))$ then $\phi U = BU \in \Gamma(Rad(TM))$ and $CU = 0$. Moreover $S(TM^\perp) = F(S(TM))\perp\mu$, implies that for any $U \in \Gamma(S(TM^\perp))$, $BU \in \Gamma(S(TM))$ and $CU \in \Gamma(\mu)$. Since $\nabla^s_X \alpha_S \in \Gamma(S(TM^\perp))$ therefore $C\nabla^s_X \alpha_S \in \Gamma(\mu)$. Hence from (4.7) and then using (3.7), we have

$$\bar{g}(\nabla^s_X \phi H^s, FPY) = -\bar{g}(FA_{H^s}X, FPY) = -sin^2\theta g(A_{H^s}X, PY). \tag{4.8}$$

Since $\bar{\nabla}$ is a metric connection therefore $(\bar{\nabla}_X g)(FPY, \phi H^s) = 0$, this further implies that $\bar{g}(\nabla^s_X FPY, \phi H^s) = \bar{g}(\nabla^s_X \phi H^s, FPY)$, therefore using (4.8), we obtain

$$\bar{g}(\nabla^s_X FPY, \phi H^s) = -sin^2\theta g(A_{H^s}X, PY). \tag{4.9}$$

From (4.6) and (4.9), we have $g(X, Y)g(H^s, H^s) = -sin^2\theta g(A_{H^s}X, PY)$, using (2.8), we get $g(X, Y)g(H^s, H^s) = -sin^2\theta\bar{g}(h^s(X, PY), H^s) = -sin^2\theta g(X, Y)g(H^s, H^s)$. This implies that $(1 + sin^2\theta)g(X, Y)g(H^s, H^s) = 0$. Since M is a proper hemi-slant lightlike submanifold and g is a Riemannian metric on D^θ then $g(H^s, H^s) = 0$, using the fact that $S(TM^\perp)$ is non-degenerate, we obtain

$$H^s = 0. \tag{4.10}$$

Furthermore, again using the Sasakian character of $\bar{M}$ for any $X \in \Gamma(D^\theta)$, we have $\nabla_X TX + h(X, TX) - A_{FX}X + \nabla^s_X FX + D^l(X, FX) = T\nabla_X X + F\nabla_X X + Bh(X, X) + Ch^s(X, X) - g(X, X)V$. Since M is a totally contact umbilical hemi-slant lightlike submanifold therefore using $h(X, TX) = 0$, for any $X \in \Gamma(D^\theta)$ and then comparing the tangential components, we obtain $\nabla_X TX - A_{FX}X = T\nabla_X X + Bh(X, X) - g(X, X)V$. Taking the scalar product both sides with respect to $N \in \Gamma(ltr(TM))$ such that $\phi N = \xi \in \Gamma(Rad(TM))$ and using the Lemma 4.1, we get

$$\bar{g}(A_{FX}X, N) = -\bar{g}(\phi h^l(X, X), N) = \bar{g}(h^l(X, X), \xi). \tag{4.11}$$

Now using (2.5), (2.7), (2.13), (2.14) and the Lemma 4.1, we have $\bar{g}(A_{FX}X, N) = \bar{g}(\bar{\nabla}_X \phi X, N) - \bar{g}(\bar{\nabla}_X TX, N) = -\bar{g}(\bar{\nabla}_X X, \xi) - \bar{g}(\nabla_X TX, N) = -\bar{g}(h^l(X, X), \xi)$, for any $X \in \Gamma(D^\theta)$. Hence using this in (4.11), we have $2\bar{g}(h^l(X, X), \xi) = 0$, as M is a totally contact umbilical proper hemi-slant lightlike submanifold therefore we have $g(X, X)\bar{g}(H^l, \xi) = 0$. Since g is a Riemannian metric on D^θ therefore $\bar{g}(H^l, \xi) = 0$, then using (2.3), we obtain that

$$H^l = 0. \tag{4.12}$$

Thus from (4.10) and (4.12), the proof is complete. □

Contrary to the classical theory of submanifolds, the induced connection ∇ on a lightlike submanifold M of a semi-Riemannian manifold $\bar{M}$ is not a metric connection. So as a consequence of above Theorem, we have the following important result.

Theorem 4.3. *Let M be a totally contact umbilical proper hemi-slant lightlike submanifold of $\bar{M}$. Then the induced connection ∇ is a metric connection on M.*

Proof. Since M is a totally contact umbilical proper hemi-slant lightlike submanifold $\bar{M}$ then using (4.2) and (4.12), we have

$$h^l(X, Y) = \eta(X)h^l(Y, V) + \eta(Y)h^l(X, V), \tag{4.13}$$

for any $X, Y \in \Gamma(TM)$. Let $X, Y \in \Gamma(D^\theta)$ and $\xi \in \Gamma(Rad(TM))$ then using $\eta(X) = \eta(Y) = \eta(\xi) = 0$ in (4.13), we have $h^l(X, Y) = 0$, $h^l(X, \xi) = 0$, $h^l(\xi, \xi) = 0$ and moreover from (2.14), we have $\bar{\nabla}_V V = 0$, implies that $h^l(V, V) = 0$. Put $X = V$ and $Y = \xi$ in (2.9), we have $\bar{g}(h^l(V, \xi), \xi) = 0$ and using (2.3) it leads to $h^l(V, \xi) = 0$. Let $X \in \Gamma(D^\theta)$ then second equation of (2.14) implies that $\nabla_X V + h^l(X, V) + h^s(X, V) = TPX + FPX$, using the Lemma 3.2 and then comparing the transversal components, we get $h^l(X, V) = 0$. Hence h^l vanishes identically on M and thus using Theorem 2.2 in [4], at page 159, the induced connection ∇ becomes a metric connection on M. □

Denote by $\bar{R}$ and R the curvature tensors of $\bar{\nabla}$ and ∇ respectively then by straightforward calculations ([2]), we have

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y \\ &\quad - A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z) \\ &\quad + D^l(X, h^s(Y,Z)) - D^l(Y, h^s(X,Z)) + (\nabla_X h^s)(Y,Z) \\ &\quad - (\nabla_Y h^s)(X,Z) + D^s(X, h^l(Y,Z)) - D^s(Y, h^l(X,Z)).\end{aligned} \tag{4.14}$$

An indefinite contact space form is a connected indefinite Sasakian manifold of constant ϕ-holomorphic sectional curvature c, denoted by $\bar{M}(c)$, whose curvature tensor $\bar{R}$, for X, Y, Z vector fields on $\bar{M}$, is given by (see [5])

$$\begin{aligned}\bar{R}(X,Y)Z &= \frac{c+3}{4}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} + \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &\quad + \bar{g}(X,Z)\eta(Y)V - \bar{g}(Y,Z)\eta(X)V + \bar{g}(\phi Y,Z)\phi X - \bar{g}(\phi X,Z)\phi Y \\ &\quad - 2\bar{g}(\phi X,Y)\phi Z\}.\end{aligned} \tag{4.15}$$

Theorem 4.4. *There do not exist totally contact umbilical proper hemi-slant lightlike submanifolds of an indefinite contact space form $\bar{M}(c)$ such that $c \neq 1$.*

Proof. Let M be a totally contact umbilical hemi-slant lightlike submanifold of $\bar{M}(c)$ such that $c \neq 1$. Then using (4.15), for $X \in \Gamma(D^\theta)$ and $\xi, \xi' \in \Gamma(Rad(TM))$, we get

$$\bar{g}(\bar{R}(X,\phi X)\xi', \xi) = -\frac{c-1}{2} g(X,X) g(\phi\xi', \xi). \tag{4.16}$$

On the other hand using (4.14), we get

$$\bar{g}(\bar{R}(X,\phi X)\xi', \xi) = \bar{g}((\nabla_X h^l)(\phi X, \xi'), \xi) - \bar{g}((\nabla_{\phi X} h^l)(X, \xi'), \xi). \tag{4.17}$$

On using (4.2), we get $(\nabla_X h^l)(\phi X, \xi') = -g(\nabla_X \phi X, \xi')H^l - g(\phi X, \nabla_X \xi')H^l = \bar{g}(h^l(X,TX), \xi')H^l = g(X,\phi X)\bar{g}(H^l, \xi') = 0$ and similarly $(\nabla_{\phi X} h^l)(X, \xi') = 0$. Thus from (4.16) and (4.17), we obtain $\frac{c-1}{2} g(X,X) g(\phi\xi', \xi) = 0$. Since g is a Riemannian metric on D^θ and (2.3) implies that $g(\phi\xi', \xi) \neq 0$, therefore $c = 1$. This contradiction completes the proof. □

In [4], a minimal lightlike submanifold M is defined when M is a hypersurface of a 4-dimensional Minkowski space. Then in [1], a general notion of minimal lightlike submanifold of a semi-Riemannian manifold $\bar{M}$ is introduced, which is as below.

Definition 4.2. A lightlike submanifold $(M, g, S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is minimal if $h^s = 0$ on $Rad(TM)$ and *trace* $h = 0$, where trace is written with respect to g restricted to $S(TM)$.

Theorem 4.5. *Let M be a totally contact umbilical proper hemi-slant lightlike submanifold of $\bar{M}$. Then M is minimal.*

Proof. From Theorem 4.2, we know $H^l = 0 = H^s$ and $\eta(\xi) = 0$, for any $\xi \in \Gamma(Rad(TM))$ then using (4.3), we have $h^s(\xi, \xi) = 0$, that is, $h^s = 0$ on $Rad(TM)$. From (2.14), we have $\bar{\nabla}_V V = 0$, implies that $h(V,V) = 0$. Let $\{e_1, ..., e_k\}$ be an orthonormal basis of D^θ then using the fact that $\eta(e_i) = 0$, $i \in \{1, 2, ..., k\}$ with (4.2) and (4.3), we have $h(e_i, e_i) = 0$, hence *trace* $h|_{S(TM)} = 0$. □

A lightlike submanifold is called an irrotational submanifold, if and only if, $\bar{\nabla}_X\xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$. This implies that if M is an irrotational lightlike submanifold then $\bar{\nabla}_X\xi = \nabla_X\xi$, $h^l(X,\xi) = 0$ and $h^s(X,\xi) = 0$, for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$.

Theorem 4.6. *Let M be an irrotational hemi-slant lightlike submanifold of $\bar{M}$. Then M is minimal, if and only if, trace $A_{W_q}|_{S(TM)} = 0$, trace $A^*_{\xi_j}|_{S(TM)} = 0$, where $\{W_q\}_{q=1}^l$ is a basis of $S(TM^\perp)$ and $\{\xi_j\}_{j=1}^r$ is a basis of $Rad(TM)$.*

Proof. Let M be an irrotational lightlike submanifold then $h^s(X,\xi) = 0$ for $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$ implies that h^s vanishes on $Rad(TM)$ and $\bar{\nabla}_V V = 0$, implies that $h(V,V) = 0$. Hence M is minimal, if and only if, *trace* $h = 0$ on D^θ, that is, M is minimal if and only $\sum_{i=1}^k h(e_i,e_i) = 0$, where $\{e_i\}_{i=1}^k$ be an orthonormal basis of D^θ. Using (2.8) and (2.11) we obtain $\sum_{i=1}^k h(e_i,e_i) = \sum_{i=1}^k\{\frac{1}{r}\sum_{j=1}^r g(A^*_{\xi_j}e_i,e_i)N_j + \frac{1}{l}\sum_{q=1}^l g(A_{W_q}e_i,e_i)W_q\}$, and the assertion follows. □

Theorem 4.7. *Let M be a proper hemi-slant lightlike submanifold of $\bar{M}$. Then M is minimal, if and only if, trace $A_{W_q}|_{S(TM)} = 0$, trace $A^*_{\xi_j}|_{S(TM)} = 0$, and $\bar{g}(D^l(X,W),Y) = 0$, for any $X,Y \in \Gamma(Rad(TM))$, where $\{W_q\}_{q=1}^l$ is a basis of $S(TM^\perp)$ and $\{\xi_j\}_{j=1}^r$ is a basis of $Rad(TM)$.*

Proof. Let $X,Y \in \Gamma(Rad(TM))$ then using (2.8), it is clear that $h^s = 0$ on $Rad(TM)$, if and only if, $\bar{g}(D^l(X,W),Y) = 0$ and $\bar{\nabla}_V V = 0$, implies that $h(V,V) = 0$. Therefore M is minimal, if and only if, $\sum_{i=1}^k h(e_i,e_i) = 0$, where $\{e_i\}_{i=1}^k$ be an orthonormal basis of D^θ. Then following the proof of the Theorem 4.6, the assertion follows. □

Lemma 4.8. *Let M be a proper hemi-slant lightlike submanifold of $\bar{M}$ such that $dim(D^\theta) = dim(S(TM^\perp))$. If $\{e_i\}_{i=1}^k$ is a local orthonormal basis of $\Gamma(D^\theta)$ then $\{csc\theta Fe_i\}_{i=1}^k$ is a orthonormal basis of $S(TM^\perp)$.*

Proof. Let $\{e_1,.....,e_k\}$ be a local orthonormal basis of $\Gamma(D^\theta)$ then using the Lemma 3.2, $Fe_i \in \Gamma(S(TM^\perp))$, for $i \in \{1,....,k\}$. Since $S(TM)$ and $S(TM^\perp)$ are Riemannian therefore using (3.7), we obtain $\bar{g}(csc\theta Fe_i, csc\theta Fe_j) = csc^2\theta sin^2\theta g(e_i,e_j) = \delta_{ij}$, this proves the assertion. □

Theorem 4.9. *Let M be a proper hemi-slant lightlike submanifold of $\bar{M}$ such that $dim(D^\theta) = dim(S(TM^\perp))$. Then M is minimal, if and only if, $traceA_{csc\theta Fe_i}|_{S(TM)} = 0$, $traceA^*_{\xi_j}|_{S(TM)} = 0$, and $\bar{g}(D^l(X,Fe_i),Y) = 0$, for any $X,Y \in \Gamma(Rad(TM))$, where $\{e_i\}_{i=1}^k$ is a basis of D^θ.*

Proof. Let $\{e_i\}_{i=1}^k$ be a basis of D^θ then using the Lemma 4.8, $\{csc\theta Fe_i\}_{i=1}^k$ is a basis of $S(TM^\perp)$. Therefore $h^s(X,X)$ can be written as linear combination of $\{csc\theta Fe_i\}_{i=1}^k$, that is, we can write $h^s(X,X) = \sum_{i=1}^k \lambda_i csc\theta Fe_i$, for any $X \in \Gamma(D^\theta)$ and for some functions $\lambda_i, i \in \{1,...,k\}$. Using (2.8), we have $\bar{g}(h^s(X,X), csc\theta Fe_i) = \bar{g}(A_{csc\theta Fe_i}X,X)$, for any $X \in \Gamma(D^\theta)$, therefore using (3.7), it further leads to $\lambda_i = \bar{g}(A_{csc\theta Fe_i}X,X)$ and hence we get $h^s(X,X) = \sum_{i=1}^k csc\theta\bar{g}(A_{csc\theta Fe_i}X,X)Fe_i$, for any $X \in \Gamma(D^\theta)$. Then the assertion comes from the Theorem 4.7. □

References

[1] C. L. Bejan and K. L. Duggal, *Global lightlike manifolds and harmonicity*, Kodai Math. J. 28 (2005), 131-145.

[2] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Birkhauser, New York 2002.

[3] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez and M. Fernandez, *Slant submanifolds in Sasakian manifolds*, Glasg. Math. J. 42 (2000), 125-138.

[4] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of semi-Riemannian Manifolds and Applications*, Kluwer Academic Publishers, The Netherlands 1996.

[5] Rakesh Kumar, Rachna Rani and R. K. Nagaich, *On sectional curvature of ϵ-Sasakian manifolds*, Int. J. Math. Math. Sci. Article ID 93562 (2007), 1-8.

[6] A. Lotta, *Slant submanifolds in contact geometry*, Bull. Math. Soc. Sci. Math. Roumanie 39 (1996), 183-198.

[7] S. Maclane, *Geometrical Mechanics II*, Lecture Notes, University of Chicago, Chicago 1968.

[8] V. E. Nazaikinskii, V. E. Shatalov and B. Y. Sternin, *Contact Geometry and Linear Differential Equations*, Walter de Gruyter, Berlin 1992.

[9] Rashmi Sachdeva, Rakesh Kumar and S. S. Bhatia, *Totally contact umbilical slant lightlike submanifolds of indefinite Cosymplectic manifolds*, ISRN Geometry Article ID 231869 (2013), 1-8.

[10] Rashmi Sachdeva, Rakesh Kumar and S. S. Bhatia, *Non existence of totally umbilical proper slant lightlike submanifold of indefinite Sasakian manifolds*, Bull. Iranian Math. Soc. 40, 5 (2014), 1135-1151.

[11] Rashmi Sachdeva, Rakesh Kumar and S. S. Bhatia, *Totally umbilical hemi-slant lightlike submanifolds*, New York J. Math. 21 (2015), 191-203.

[12] B. Sahin and C. Yildirim, *Slant lightlike submanifolds of indefinite Sasakian manifolds*, Filomat 26 (2012), 277-287.

Author's address:

Rashmi Sachdeva
Department of Basic and Applied Sciences, Punjabi University, Patiala-147002, India.
E-mail: rashmi.sachdeva86@gmail.com

Rakesh Kumar (Corresponding Author)
Department of Basic and Applied Sciences, Punjabi University, Patiala-147002, India.
E-mail: dr_rk37c@yahoo.co.in

Rachna Rani
Department of Mathematics, University College, Ghanaur-140702, Punjab
E-mail: rachna@pbi.ac.in

Satvinder Singh Bhatia
School of Mathematics, Thapar University, Patiala-147001, India.
E-mail: ssbhatia@thapar.edu