On a class of pseudo calibrated generalized complex structures related to Norden, para-Norden and statistical manifolds

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Abstract. We consider pseudo calibrated generalized complex structures, $\widehat{J}$, defined by a pseudo Riemannian metric g and a $g-$symmetric operator H such that $H^2 = \mu I$, $\mu \in \mathbb{R}$, on a smooth manifold M. These structures include the case of complex Norden manifolds for $\mu = -1$, studied in [20], the case of almost tangent structures for $\mu = 0$, $ImH = KerH$, and the case of para Norden manifolds for $\mu = 1$. The special case $H = O$ is described in [19]. We study integrability conditions of $\widehat{J}$, with respect to a linear connection ∇, and we describe examples of geometric structures that naturally give rise to integrable pseudo calibrated generalized complex structures. We prove that for $\mu \neq -1$ integrability implies that the $\pm i-$eigenbundles of $\widehat{J}$, $E^{1,0}_{\widehat{J}}$, $E^{0,1}_{\widehat{J}}$, are complex Lie algebroids. We define the concept of generalized $\overline{\partial}_{\widehat{J}}$-operator of (M, H, g, ∇) and we study holomorphic sections.

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1 Introduction

Generalized complex structures were introduced by Hitchin in [9], and further investigated by Gualtieri in [10], in order to unify symplectic and complex geometry. In this paper we consider the concept of generalized complex structure introduced in [16], [17] and also studied in [5], [18], [19], [20].

Let (M, g) be a smooth pseudo Riemannian manifold, let $T(M)$ be the tangent bundle, let $T^*(M)$ be the cotangent bundle and let $E = T(M) \oplus T^*(M)$ be the generalized tangent bundle of M. A pseudo calibrated generalized complex structure of M is a complex structure on E which is pseudo calibrated with respect to the canonical symplectic structure of E. A linear connection, ∇, on M defines a bracket, $[\ ,\ ]_\nabla$,

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on sections of E and we can define the concept of $\nabla-$integrability for generalized complex structures. We consider pseudo calibrated generalized complex structures $\widehat{J}$ defined by a pseudo Riemannian metric g and a $g-$ symmetric operator H such that $H^2 = \mu I$, $\mu \in \mathbb{R}$, on M. These structures include the case of complex Norden manifolds for $\mu = -1$, studied in [20], the case of almost tangent structures for $\mu = 0$, $ImH = KerH$, and the case of para Norden manifolds for $\mu = 1$. The special case $H = O$ is described in [19].

We study the integrability conditions of $\widehat{J}$, with respect to a linear connection ∇ with torsion T^{∇}, and we describe examples of geometric structures that naturally give rise to integrable pseudo calibrated generalized complex structures. Then we prove that for $\mu \neq -1$ integrability implies that the $\pm i-$eigenbundles of $\widehat{J}$, $E^{1,0}_{\widehat{J}}$, $E^{0,1}_{\widehat{J}}$, are complex Lie algebroids. We define the concept of generalized $\overline{\partial}_{\widehat{J}}-$ operator of (M, H, g, ∇); from the Jacobi identity on $E^{1,0}_{\widehat{J}}$ it follows $(\overline{\partial}_{\widehat{J}})^2 = 0$ and, as $\overline{\partial}_{\widehat{J}}$ is the exterior derivative of the Lie algebroid $E^{1,0}_{\widehat{J}}$, we get that $(C^{\infty}(\wedge^{\bullet}(E^{1,0}_{\widehat{J}})), \wedge, \overline{\partial}_{\widehat{J}}, [\ ,\ ]_{\nabla})$ is a differential Gerstenhaber algebra, where $\wedge$ denotes the Schouten bracket, [13], [28]. Finally we study certain holomorphic sections.

The paper is organized as in the following. In section 2 we introduce preliminary material of the generalized tangent bundle and of generalized complex structures; in section 3 we compute integrability conditions and in section 4 we give examples of integrable structures; section 5 is devoted to the study of complex Lie algebroids naturally associated to integrable pseudo calibrated generalized complex structures; in section 6 we define the concept of generalized $\overline{\partial}_{\widehat{J}}-$operator on M and in section 7 we study some generalized holomorphic sections, in particular, in this context, Hessian manifolds occur as interesting examples.

This paper is a generalization of our previous papers [19], [20] and allows us to unify complex Norden and para Norden manifolds through almost tangent structures and statistical manifolds. The theory reveals that the case of complex Norden manifolds is special.

2 Preliminaries

Let M be a smooth manifold of real dimension n and let $E = T(M) \oplus T^*(M)$ be the generalized tangent bundle of M. Smooth sections of E are elements $X + \xi \in C^{\infty}(E)$ where $X \in C^{\infty}(T(M))$ is a vector field and $\xi \in C^{\infty}(T^*(M))$ is a $1-$ form.

E is equipped with a natural symplectic structure, $(\ ,\ )$, defined by:

$$(X + \xi, Y + \eta) = -\frac{1}{2}(\xi(Y) - \eta(X)) \tag{2.1}$$

and a natural indefinite metric, $\langle\ ,\ \rangle$, defined by:

$$\langle X + \xi, Y + \eta\rangle = -\frac{1}{2}(\xi(Y) + \eta(X)). \tag{2.2}$$

$\langle\ ,\ \rangle$ is non degenerate and of signature (n, n).

A linear connection on M, ∇, defines, in a canonical way, a bracket $[\ ,\ ]_\nabla$ on $C^\infty(E)$, as follows:

$$[X+\xi, Y+\eta]_\nabla = [X,Y] + \nabla_X \eta - \nabla_Y \xi. \tag{2.3}$$

Like in [16], a direct computation gives the following:

Lemma 2.1. *For all $X, Y \in C^\infty(T(M))$, for all $\xi, \eta \in C^\infty(T^*(M))$ and for all $f \in C^\infty(M)$ we have*:
1. $[X+\xi, Y+\eta]_\nabla = -[Y+\eta, X+\xi]_\nabla$,
2. $[f(X+\xi), Y+\eta]_\nabla = f[X+\xi, Y+\eta]_\nabla - Y(f)(X+\xi)$,
3. *Jacobi's identity holds for $[\ ,\ ]_\nabla$ if and only if ∇ has zero curvature.*

We consider the following concept of generalized complex structure:

Definition 2.2. A *generalized complex structure* on M is an endomorphism $\widehat{J} : E \to E$ such that $\widehat{J}^2 = -I$.

Definition 2.3. *A* generalized complex structure $\widehat{J}$ is called *pseudo calibrated* if it is $(\ ,\ )-$invariant and if the bilinear symmetric form defined by $(\ ,\widehat{J})$ on $T(M)$ is non degenerate, moreover $\widehat{J}$ is called *calibrated* if it is pseudo calibrated and $(\ ,\widehat{J}\ )$ is positive definite.

From the definition we get that a pseudo calibrated complex structure $\widehat{J}$ can be written in the following block matrix form:

$$\widehat{J} = \begin{pmatrix} H & -(I+H^2)g^{-1} \\ g & -H^* \end{pmatrix} \tag{2.4}$$

where $g : T(M) \to T^*(M)$ is identified to the bemolle musical isomorphism of the pseudo Riemannian metric g on M, $H : T(M) \to T(M)$ is a g-symmetric operator and $H^* : T^*(M) \to T^*(M)$ is the dual operator of H defined by:
$H^*(\xi)(X) = \xi(H(X))$.

We have:

$$(g(X))(Y) = g(X,Y) = 2\left(X, \widehat{J}Y\right) \tag{2.5}$$

for all $X, Y \in T(M)$.

In the following we will consider g-symmetric operators $H : T(M) \to T(M)$ such that $H^2 = \mu I$ where $\mu \in \mathbb{R}$ and I denotes identity. In this case we have:

$$\widehat{J} = \begin{pmatrix} H & \lambda g^{-1} \\ g & -H^* \end{pmatrix} \tag{2.6}$$

where $\lambda = -1 - \mu$.

We remark that for $\mu = -1$ (M, H, g) is a Norden manifold, [20], for $\mu = 0$ and $ImH = KerH$, (M, H) is an almost tangent manifold, [2], and for $\mu = 1$ (M, H, g) is a para Norden manifold. The special case $H = O$ is described in [19].

3 Integrability

Let ∇ be a linear connection on M and let $[\ ,\ ]_{\nabla}$ be the bracket on $C^{\infty}(E)$ defined by ∇, the following holds:

Lemma 3.1. ([17]) *Let $\widehat{J} : E \to E$ be a generalized complex structure on M and let*

$$N^{\nabla}(\widehat{J}) : C^{\infty}(E) \times C^{\infty}(E) \to C^{\infty}(E) \tag{3.1}$$

defined by:

$$N^{\nabla}(\widehat{J})(\sigma,\tau) = \left[\widehat{J}\sigma, \widehat{J}\tau\right]_{\nabla} - \widehat{J}\left[\widehat{J}\sigma, \tau\right]_{\nabla} - \widehat{J}\left[\sigma, \widehat{J}\tau\right]_{\nabla} - [\sigma,\tau]_{\nabla}, \tag{3.2}$$

for all $\sigma, \tau \in C^{\infty}(E)$; $N^{\nabla}(\widehat{J})$ is a skew symmetric tensor.

Definition 3.2. $N^{\nabla}(\widehat{J})$ is called the *Nijenhuis tensor of $\widehat{J}$ with respect to* ∇.

Definition 3.3. Let $\widehat{J} : E \to E$ be a generalized complex structure on M, $\widehat{J}$ is called $\nabla-$*integrable* if $N^{\nabla}(\widehat{J}) = 0$.

Let T^{∇} be the torsion of ∇:

$$T^{\nabla}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{3.3}$$

and let d^{∇} be the exterior differential associated to ∇:

$$\left(d^{\nabla} g\right)(X,Y) = \left(\nabla_X g\right)(Y) - \left(\nabla_Y g\right)(X) + g\left(T^{\nabla}(X,Y)\right) \tag{3.4}$$

for all $X, Y \in C^{\infty}(TM)$.

We have the following:

Proposition 3.4. *Let $\widehat{J} = \begin{pmatrix} H & \lambda g^{-1} \\ g & -H^* \end{pmatrix}$ be the pseudo calibrated generalized complex structure on M defined by a pseudo Riemannian metric g and a $g-$symmetric operator H of $T(M)$ such that $H^2 = (-1-\lambda)I$. Let $N^{\nabla}(\hat{J})$ be the generalized Nijenhuis tensor of $\widehat{J}$ defined by (3.2), then for all $X, Y \in C^{\infty}(T(M))$ we have*:

$$\begin{gathered} N^{\nabla}(\widehat{J})(X,Y) = N(H)(X,Y) - \lambda g^{-1}\left((d^{\nabla}g)(X,Y)\right) + \\ +(d^{\nabla}g)(HX,Y) + (d^{\nabla}g)(X,HY) + g((\nabla_Y H)(X) - (\nabla_X H)(Y)) \end{gathered} \tag{3.5}$$

$$\begin{gathered} N^{\nabla}(\widehat{J})(X,g(Y)) = \lambda\left((\nabla_X H)(Y) - (\nabla_Y H)(X)\right) + \\ -\lambda g^{-1}((\nabla_{HX} g)(Y) - (\nabla_X g)(HY)) + \\ +\lambda g^{-1}(T^{\nabla}(HX,Y) - HT^{\nabla}(X,Y)) + \\ +\lambda((d^{\nabla}g)(X,Y)) - g((\nabla_{HX} H)(Y) - H(\nabla_X H)(Y)) \end{gathered} \tag{3.6}$$

$$N^{\nabla}(\widehat{J})(g(X),g(Y)) = -\lambda^2 g^{-1}\left((d^{\nabla}g)(X,Y)\right) + \\ +\lambda g((\nabla_Y H)(X) - (\nabla_X H)(Y)) \tag{3.7}$$

where $N(H)$ is the Nijenhuis tensor of H defined by:

$$N(H)(X,Y) = [HX,HY] - H[HX,Y] - H[X,HY] + H^2[X,Y]. \tag{3.8}$$

Proof. Direct computations give:

$$\begin{aligned}
&N^{\nabla}(\widehat{J})(X,Y) = [HX+g(X),HY+g(Y)]_{\nabla} + \\
&-\widehat{J}[HX+g(X),Y)]_{\nabla} - \widehat{J}[X,HY+g(Y)]_{\nabla} - [X,Y]_{\nabla} \\
&= [HX,HY] + \nabla_{HX}g(Y) - \nabla_{HY}g(X) + \\
&-\widehat{J}([HX,Y] - \nabla_Y g(X) + [X,HY] + \nabla_X g(Y)) - [X,Y] \\
&= [HX,HY] - H[HX,Y] - H[X,HY] + \lambda g^{-1}(\nabla_Y g(X) - \nabla_X g(Y)) + \\
&-[X,Y] + \nabla_{HX}g(Y) - \nabla_{HY}g(X) - g([HX,Y]) + \\
&-H^*(\nabla_Y g(X)) - g([X,HY]) + H^*(\nabla_X g(Y)) \\
&= N(H)(X,Y) + \lambda[X,Y] + \lambda g^{-1}((\nabla_Y g)(X) - g(\nabla_X Y)) + \\
&+(\nabla_{HX}g)(Y) - (\nabla_{HY}g)(X) + g((\nabla_Y H)(X) - (\nabla_X H)(Y)) + \\
&-H^*((\nabla_Y g)(X)) + H^*((\nabla_X g)(Y)) + g\left(T^{\nabla}(HX,Y\right) + T^{\nabla}(X,HY)) \\
&= N(H)(X,Y) + \lambda g^{-1}((\nabla_Y g)(X) - (\nabla_X g)(Y) - \lambda T^{\nabla}(X,Y)) + \\
&+(\nabla_{HX}g)(Y) - (\nabla_{HY}g)(X) + g((\nabla_Y H)(X) - (\nabla_X H)(Y)) + \\
&-((\nabla_Y g)(HX)) + ((\nabla_X g)(HY)) + g\left(T^{\nabla}(HX,Y\right) + T^{\nabla}(X,HY)) \\
&= N(H)(X,Y) - \lambda g^{-1}\left((d^{\nabla}g)(X,Y)\right) + \\
&+(d^{\nabla}g)(HX,Y) + (d^{\nabla}g)(X,HY) + g((\nabla_Y H)(X) - (\nabla_X H)(Y))
\end{aligned}$$

$$\begin{aligned}
&N^{\nabla}(\widehat{J})(X,g(Y)) = [HX+g(X),\lambda Y - g(HY)]_{\nabla} - \widehat{J}[HX+g(X),g(Y)]_{\nabla} + \\
&-\widehat{J}[X,\lambda Y - g(HY)]_{\nabla} - \nabla_X g(Y) \\
&= \lambda[HX,Y] - \nabla_{HX}g(HY) - \lambda\nabla_Y g(X) - \widehat{J}\lambda[X,Y] + \\
&+\widehat{J}\nabla_X g(HY) - \widehat{J}\nabla_{HX}g(Y) - \nabla_X g(Y) \\
&= \lambda[HX,Y] - \nabla_{HX}g(HY) - \lambda\nabla_Y g(X) - \widehat{J}\lambda[X,Y] + \\
&-\lambda H[X,Y] - \lambda g([X,Y]) + \lambda g^{-1}(\nabla_X g(HY)) - \lambda g^{-1}((\nabla_{HX}g(Y) - \nabla_Y g(X)) \\
&+H^*(\nabla_{HX}g(Y)) - H^*(\nabla_X g(HY)) - \nabla_X g(Y) \\
&= \lambda((\nabla_X H)(Y) - (\nabla_Y H)(X) - g^{-1}((\nabla_{HX}g)(Y) - (\nabla_X g)(HY) + \\
&+g(T^{\nabla}(HX,Y) - HT^{\nabla}(X,Y))) + \lambda(d^{\nabla}g)(X,Y) - g((\nabla_{HX}H)(Y) - (\nabla_X H)(Y))
\end{aligned}$$

$$N^{\nabla}(\widehat{J})(g(X),g(Y)) = [\lambda X - g(HX),\lambda Y - g(HY)]_{\nabla} +$$

$$
\begin{aligned}
&-\widehat{J}\,[\lambda X-g(HX),Y]_{\nabla}-\widehat{J}\,[g(X),\lambda Y-g(HY)]_{\nabla}\\
&=\lambda^{2}\,[X,Y]-\lambda\left\{\lambda g^{-1}((\nabla_X g(Y)-\nabla_Y g(X))\right\}+\\
&-\lambda\left\{-H^{*}\left(\nabla_X g(Y)\right)+H^{*}\left(\nabla_Y g(X)\right)\right\}\\
&=\lambda^{2}\left(-T^{\nabla}(X,Y)-g^{-1}((\nabla_X g)\,(Y)-(\nabla_Y g)\,(X)\right)+\\
&-\lambda\left(g((\nabla_Y H)(X)-(\nabla_X H)\,(Y))\right)\\
&=-\lambda^{2}g^{-1}\left((d^{\nabla}g)(X,Y)\right)+\lambda g((\nabla_Y H)X-(\nabla_X H)\,Y).
\end{aligned}
$$

□

In particular we get:

Theorem 3.5. *For* $\lambda(\lambda+1)\neq 0$ $\widehat{J}=\begin{pmatrix} H & \lambda g^{-1}\\ g & -H^{*}\end{pmatrix}$ *is* $\nabla-$*integrable if and only if the following conditions hold:*

$$
\left\{\begin{array}{l} d^{\nabla}g=0\\ N(H)=0\\ \nabla H=0. \end{array}\right. \tag{3.9}
$$

For $\lambda=0$ $\widehat{J}=\begin{pmatrix} H & O\\ g & -H^{*}\end{pmatrix}$ is $\nabla-$*integrable if and only if the following conditions hold:*

$$
\left\{\begin{array}{l} N(H)=0\\ (\nabla_{HX}H)=H\,(\nabla_X H)\\ \left(d^{\nabla}g\right)(HX,Y)+(d^{\nabla}g)(X,HY)-g((\nabla_X H)\,(Y)-(\nabla_Y H)\,(X))=0. \end{array}\right. \tag{3.10}
$$

For $\lambda=-1$ $\widehat{J}=\begin{pmatrix} H & -g^{-1}\\ g & -H^{*}\end{pmatrix}$ is $\nabla-$*integrable if and only if the following conditions hold:*

$$
\left\{\begin{array}{l} d^{\nabla}g=0\\ N(H)=0\\ (\nabla_X H)\,(Y)=(\nabla_Y H)\,(X)\\ (\nabla_{HX}H)=H\,(\nabla_X H)\,. \end{array}\right. \tag{3.11}
$$

Proof. If $\lambda\neq 0$ then from (3.7) and (3.5) we get immediately the first and second condition in (3.9) and (3.11) and the third in (3.11) . Moreover:

$$
\begin{aligned}
&(\nabla_{HX}g)\,(Y)-(\nabla_X g)\,(HY)+g(T^{\nabla}(HX,Y)-HT^{\nabla}(X,Y))\\
=\;&\left(d^{\nabla}g\right)(HX,Y)+(\nabla_Y g)\,(HX)-(\nabla_X g)\,(HY)-g(HT^{\nabla}(X,Y))\\
=\;&\left(d^{\nabla}g\right)(HX,Y)+H^{*}((\nabla_Y g)\,(X)-(\nabla_X g)\,(Y)-g(T^{\nabla}(X,Y))\\
=\;&\left(d^{\nabla}g\right)(HX,Y)-H^{*}\left(\left(d^{\nabla}g\right)(X,Y)\right);
\end{aligned}
$$

then we get (3.11) . In order to obtain (3.9) remark that if $\lambda \neq -1$, we have

$$\begin{aligned}
(\nabla_X H)(Y) &= H^{-1}((\nabla_{HX} H)(Y)) = \frac{1}{-1-\lambda} H((\nabla_{HX} H)(Y)) \\
&= \frac{1}{-1-\lambda} H((\nabla_Y H)(HX)) \\
&= \frac{1}{-1-\lambda} H\left(\nabla_Y H^2 X - H \nabla_Y HX)\right) \\
&= \frac{1}{-1-\lambda} H^2 (H \nabla_Y X - \nabla_Y HX) \\
&= -(\nabla_Y H)(X) \\
&= -(\nabla_X H)(Y);
\end{aligned}$$

thus the third condition in (3.9) is obtained. Finally, (3.10) and (3.12) immediately follow. On the other hand if (3.9), respectively (3.10), (3.11) hold, then $N^{\nabla}(\widehat{J}) = 0$, and the proof is complete. □

Corollary 3.6. *If* $H = O$ $\widehat{J} = \begin{pmatrix} O & -g^{-1} \\ g & O \end{pmatrix}$ *is* $\nabla-$ *integrable if and only if*

$$d^{\nabla} g = 0. \tag{3.12}$$

4 Examples

Examples of integrable structures with $H = 0$ can be found in the context of quasi statistical manifolds.

Definition 4.1. ([1]), ([21]) Let (M, g, ∇) be a pseudo Riemannian manifold with a torsion free linear connection, if ∇g is symmetric then (M, g, ∇) is called a *statistical manifold.*

The concept of statistical manifold can be generalized to *statistical manifolds admitting torsion* or *quasi statistical manifolds* [12]:

Definition 4.2. Let (M, g) be a pseudo Riemannian manifold and let ∇ be a linear connection on M with torsion T^{∇} then (M, g, ∇) is called a *quasi statistical manifold* or *statistical manifold admitting torsion* if, for all $X, Y \in C^{\infty}(T(M))$, the following formula holds:

$$(\nabla_X g)Y - (\nabla_Y g)X + g(T^{\nabla}(X, Y)) = 0. \tag{4.1}$$

As a direct consequence of (3.4) and (3.12) we get the following:

Corollary 4.3. *Let* (M, g) *be a pseudo Riemannian manifold and let* ∇ *be a linear connection on* M *with torsion* T^{∇}*, let*

$$\widehat{J} = \begin{pmatrix} O & -g^{-1} \\ g & O \end{pmatrix} \tag{4.2}$$

be the generalized complex structure on M defined by g, $\widehat{J}$ is $\nabla-$ integrable if and only if (M, g, ∇) is a quasi statistical manifold.

Examples of integrable structures with $H^2 = -I$ can be found in the context of Norden manifolds.

Norden manifolds were introduced by A. P. Norden in [22] and then studied also under the names of almost complex manifolds with B-metric and anti Kählerian manifolds, [3], [11]. They have applications in mathematics and in theoretical physics.

Definition 4.4. Let (M, H) be an almost complex manifold of real dimension $2n$ and let g be a pseudo Riemannian metric on M, if H is a g-symmetric operator then g is called *Norden metric* and (M, H, g) is called *Norden manifold.* If (M, H, g) is a Norden manifold with H integrable then it is called *complex Norden manifold.*

Let (M, H, g) be a complex Norden manifold, the following holds:

Theorem 4.5. ([11]) *On a complex manifold with Norden metric (M, H, g) there exists a unique linear connection D with torsion T such that*:

$$(D_X g)(Y, Z) = 0 \tag{4.3}$$

$$T(HX, Y) = -T(X, HY) \tag{4.4}$$

$$g(T(X, Y), Z) + g(T(Y, Z), X) + g(T(Z, X), Y) = 0 \tag{4.5}$$

for all vector fields X, Y, Z on M.

D is called the *natural canonical connection* of the Norden manifold or *$B-$connection* and it is defined by:

$$D_X Y = \nabla_X Y - \frac{1}{2} H(\nabla_X H)Y \tag{4.6}$$

where ∇ is the Levi-Civita connection of g.

In particular, if D is the natural canonical connection of the complex Norden manifold (M, H, g), then

$$DH = 0. \tag{4.7}$$

Corollary 4.6. *Let (M, H, g) be a complex Norden manifold and let D be the natural canonical connection, let*

$$\widehat{J} = \begin{pmatrix} H & O \\ g & -H^* \end{pmatrix} \tag{4.8}$$

be the generalized complex structure defined by H and g, $\widehat{J}$ is $D-$ integrable.

Definition 4.7. Let (M, H, g) be a Norden manifold and let ∇ be the Levi-Civita connection of g, if $\nabla H = 0$ then (M, H, g) is called *Kähler Norden manifold.*

We remark that for a Kähler Norden manifold (M, H, g) the structure H is integrable and the natural canonical connection is the Levi Civita connection.

Examples of integrable structures with $H^2 = I$ are given by para Norden manifolds, [4], [25].

Definition 4.8. An *almost product structure* on a differentiable manifold M is a $(1,1)$ tensor field H on M such that $H^2 = I$. The pair (M, H) is called an *almost product manifold.*

Definition 4.9. An *almost paracomplex manifold* is an almost product manifold (M, H) such that the two eigenbundles, $T^+(M)$, $T^-(M)$, associated to the two eigenvalues, $+1$ and -1 of H respectively, have the same rank.

Definition 4.10. An *almost paracomplex Norden manifold* (M, H, g) is a real smooth manifold of dimension $2n$ with an almost paracomplex structure H and a pseudo Riemannian metric g such that H is a $g-$symmetric operator.

Definition 4.11. A *paraholomorphic Norden manifold,* or *para Kähler Norden manifold,* is an almost paracomplex Norden manifold (M, H, g) such that $\nabla H = 0$, where ∇ is the Levi Civita connection of g.

We remark that for an almost paracomplex structure H the vanishing of the Nijenhuis tensor $N(H)$ is equivalent to the existence of a torsion free linear connection ∇ such that $\nabla H = 0$, [25]. In particular from (3.9) we get immediately the following:

Corollary 4.12. *Let (M, H, g) be a paraholomorphic Norden manifold and let ∇ be the Levi Civita connection of g, let*

$$\widehat{J} = \begin{pmatrix} H & -2g^{-1} \\ g & -H^* \end{pmatrix} \tag{4.9}$$

be the generalized complex structure on M defined by H and *g, $\widehat{J}$ is $\nabla-$ integrable.*

5 Complex Lie algebroids

Lie algebroids were introduced by J. Pradines in [23]; we recall here the definition and the main properties.

Definition 5.1. A *complex Lie algebroid* is a complex vector bundle L over a smooth real manifold M such that: a Lie bracket $[\ ,\ ]$ is defined on $C^\infty(L)$, a smooth bundle map $\rho : L \to T(M) \otimes \mathbb{C}$, called *anchor*, is defined and, for all $\sigma, \tau \in C^\infty(L)$, for all $f \in C^\infty(M)$, the following conditions hold:
1. $\rho([\sigma, \tau]) = [\rho(\sigma), \rho(\tau)]$
2. $[f\sigma, \tau] = f([\sigma, \tau]) - (\rho(\tau)(f))\sigma$.

Let L and its dual vector bundle L^* be Lie algebroids; on sections of $\wedge L$, respectively $\wedge L^*$, the *Schouten bracket* is defined by:

$$[\ ,\ ]_L : C^\infty(\wedge^p L) \times C^\infty(\wedge^q L) \longrightarrow C^\infty\left(\wedge^{p+q-1} L\right) \tag{5.1}$$

$$\begin{gathered}[X_1 \wedge ... \wedge X_p, Y_1 \wedge ... \wedge Y_q]_L = \\ = \sum_{i=1}^{p} \sum_{j=1}^{q} (-1)^{i+j} [X_i, Y_j]_L \wedge X_1 \wedge ..\widehat{i}.. \wedge X_p \wedge Y_1 \wedge ..\widehat{j}.. \wedge Y_q\end{gathered} \tag{5.2}$$

and, for $f \in C^\infty(M)$, $X \in C^\infty(L)$

$$[X, f]_L = -[f, X]_L = \rho(X)(f); \tag{5.3}$$

respectively, by:

$$[\ ,\ ]_{L^*} : C^\infty(\wedge^p L^*) \times C^\infty(\wedge^q L^*) \longrightarrow C^\infty\left(\wedge^{p+q-1} L^*\right) \tag{5.4}$$

$$\begin{gathered}\left[X_1^* \wedge ... \wedge X_p^*, Y_1^* \wedge ... \wedge Y_q^*\right]_{L^*} = \\ = \sum_{i=1}^{p} \sum_{j=1}^{q} (-1)^{i+j} \left[X_i^*, Y_j^*\right]_{L^*} \wedge X_1^* \wedge ..\widehat{i}.. \wedge X_p^* \wedge Y_1^* \wedge ..\widehat{j}.. \wedge Y_q^*\end{gathered} \tag{5.5}$$

and, for $f \in C^\infty(M)$, $X \in C^\infty(L^*)$

$$[X, f]_{L^*} = -[f, X]_{L^*} = \rho(X)(f). \tag{5.6}$$

Moreover the *exterior derivatives* d and d_* associated with the Lie algebroid structure of L and L^* are defined respectively by:

$$d : C^\infty(\wedge^p L^*) \longrightarrow C^\infty\left(\wedge^{p+1} L^*\right) \tag{5.7}$$

$$\begin{gathered}(d\alpha)(\sigma_0, ..., \sigma_p) = \\ = \sum_{i=0}^{p} (-1)^i \rho(\sigma_i)\, \alpha\left(\sigma_0, ..\widehat{i}.., \sigma_p\right) + \sum_{i<j} (-1)^{i+j} \alpha\left([\sigma_i, \sigma_j]_L, \sigma_0, ..\widehat{i}..\widehat{j}.., \sigma_p\right)\end{gathered} \tag{5.8}$$

for $\alpha \in C^\infty(\wedge^p L^*)$, $\sigma_0, ..., \sigma_p \in C^\infty(L)$,
and:

$$d_* : C^\infty(\wedge^p L) \longrightarrow C^\infty\left(\wedge^{p+1} L\right) \tag{5.9}$$

$$\begin{gathered}(d_*\alpha)(\sigma_0, ..., \sigma_p) = \\ = \sum_{i=0}^{p} (-1)^i \rho(\sigma_i)\, \alpha\left(\sigma_0, ..\widehat{i}.., \sigma_p\right) + \sum_{i<j} (-1)^{i+j} \alpha\left([\sigma_i, \sigma_j]_{L^*}, \sigma_0, ..\widehat{i}..\widehat{j}.., \sigma_p\right)\end{gathered} \tag{5.10}$$

for $\alpha \in C^\infty(\wedge^p L)$, $\sigma_0, ..., \sigma_p \in C^\infty(L^*)$.

Let M be a smooth manifold and let $\widehat{J} = \begin{pmatrix} H & \lambda g^{-1} \\ g & -H^* \end{pmatrix}$ be the generalized pseudo calibrated complex structure on M defined by a pseudo Riemannian metric g and a $g-$symmetric operator H of $T(M)$ such that $H^2 = (-1-\lambda)I$.

Let

$$E^{\mathbb{C}} = (T(M) \oplus T^*(M)) \otimes \mathbb{C} \tag{5.11}$$

be the complexified generalized tangent bundle. The splitting in $\pm i$ eigenspaces of $\widehat{J}$ is denoted by:

$$E^{\mathbb{C}} = E^{1,0}_{\widehat{J}} \oplus E^{0,1}_{\widehat{J}} \tag{5.12}$$

with

$$E^{0,1}_{\widehat{J}} = \overline{E^{1,0}_{\widehat{J}}}. \tag{5.13}$$

A direct computation gives:

(5.14)
$$E^{1,0}_{\widehat{J}} = \{ Z - iHZ + g(W + iHW - iZ) + i(-\lambda)W \mid Z, W \in C^\infty(T(M) \otimes \mathbb{C}) \},$$

equivalently $E^{1,0}_{\widehat{J}}$ is generated by elements of the following type:

$$Z - iHZ - ig(Z) \tag{5.15}$$

with $Z \in C^\infty(T(M))$,

$$-\lambda i W + g(W + iHW) \tag{5.16}$$

with $W \in C^\infty(T(M))$.

Analogously we have:

(5.17)
$$E^{0,1}_{\widehat{J}} = \{ Z + iHZ + g(W - iHW + iZ) - i(-\lambda)W \mid Z, W \in C^\infty(T(M) \otimes \mathbb{C}) \}$$

and $E^{0,1}_{\widehat{J}}$ is generated by elements of the following type:

$$Z + iHZ + ig(Z) \text{ with } Z \in C^\infty(T(M)), \tag{5.18}$$

$$\lambda i W + g(W - iHW) \text{ with } W \in C^\infty(T(M)). \tag{5.19}$$

Lemma 5.2. *For $\lambda \neq 0$ the map*

$$\psi : T(M) \otimes \mathbb{C} \to T(M) \otimes \mathbb{C}$$

defined by:

$$\psi(Z) = Z + iHZ \tag{5.20}$$

is an isomorphism and

$$\psi(Z - iHZ) - ig(\psi(Z)) = -\lambda Z - ig(Z + iHZ) = -i(-\lambda i Z + g(Z + iHZ)). \tag{5.21}$$

Proof. We have that ψ injective if and only if $(I+iH)$ is invertible, or i is not an eigenvalue of H. Moreover a direct computation, by using the condition $H^2 = (-1-\lambda)I$, gives (5.21). □

Corollary 5.3. *If* $\lambda \neq 0$ *then:*

$$E^{1,0}_{\widehat{J}} = \{-\lambda Z - ig(Z+iHZ) \mid Z \in C^\infty(T(M)\otimes\mathbb{C})\}, \tag{5.22}$$

$$E^{0,1}_{\widehat{J}} = \{-\lambda Z + ig(Z-iHZ) \mid Z \in C^\infty(T(M)\otimes\mathbb{C})\}. \tag{5.23}$$

Moreover, for any linear connection ∇, the following holds:

Lemma 5.4. $E^{1,0}_{\widehat{J}}$ *and* $E^{0,1}_{\widehat{J}}$ *are* $[\ ,\ ]_\nabla$ *−involutive if and only if* $N^\nabla(\widehat{J}) = 0$.

Proof. Let $P_+ : E^{\mathbb{C}} \to E^{1,0}_{\widehat{J}}$ and $P_- : E^{\mathbb{C}} \to E^{0,1}_{\widehat{J}}$ be the projection operators:

$$P_\pm = \frac{1}{2}(I \mp i\widehat{J}), \tag{5.24}$$

for all $\sigma, \tau \in C^\infty(E^{\mathbb{C}})$ we have:

$$\begin{aligned} &P_\mp \left[P_\pm(\sigma), P_\pm(\tau)\right]_\nabla = P_\mp \left[\tfrac{1}{2}\left(\sigma \mp i\widehat{J}\sigma\right), \tfrac{1}{2}\left(\tau \mp i\widehat{J}\tau\right)\right]_\nabla \\ &= -\tfrac{1}{8}(N^\nabla(\widehat{J})(\sigma,\tau) \pm i\widehat{J}N^\nabla(\widehat{J})(\sigma,\tau)) = -\tfrac{1}{4}P_\mp\left(N^\nabla(\widehat{J})(\sigma,\tau)\right), \end{aligned} \tag{5.25}$$

and the proof is complete. □

Theorem 5.4. *Let* $\widehat{J} = \begin{pmatrix} H & \lambda g^{-1} \\ g & -H^* \end{pmatrix}$ be *the pseudo calibrated generalized complex structure on* M *defined by a pseudo Riemannian metric* g and a $g-$*symmetric operator* H *of* $T(M)$ *such that* $H^2 = (-1-\lambda)I$ with $\lambda \neq 0$, let ∇ *be a linear connection on* M*, if* $\widehat{J}$ *is* $\nabla-$*integrable then* $E^{1,0}_{\widehat{J}}$ *and* $E^{0,1}_{\widehat{J}}$ *are complex Lie algebroids.*

Proof. The anchor

$$\rho : E^{1,0}_{\widehat{J}} \to T(M)\otimes\mathbb{C} \tag{5.26}$$

is defined by

$$\rho(-\lambda Z - ig(Z+iHZ)) = -\lambda Z. \tag{5.27}$$

Conditions 1. and 2. of the definition are trivially satisfied then we need to prove the the Jacobi identity holds.
Let $\sigma, \tau, \upsilon \in E^{1,0}_{\widehat{J}}$, we denote *Jacobiator* with Jac, that is:

$$Jac\,[[\sigma,\tau]_\nabla, \upsilon]_\nabla = [[\sigma,\tau]_\nabla, \upsilon]_\nabla + [[\tau,\upsilon]_\nabla, \sigma]_\nabla + [[\upsilon,\sigma]_\nabla, \tau]_\nabla. \tag{5.28}$$

We may assume, without loss of generality, that:

$$
\begin{aligned}
\sigma &= -\lambda Z - ig(Z + iHZ) \\
\tau &= -\lambda W - ig(W + iHW) \\
\upsilon &= -\lambda U - ig(U + iHU),
\end{aligned}
\tag{5.29}
$$

with $Z, W, U \in C^\infty(T(M) \otimes \mathbb{C})$ and use integrability conditions.

We compute

$$Jac\,[[-\lambda Z - ig(Z + iHZ), -\lambda W - ig(W + iHW)]_\nabla\,, -\lambda U - ig(U + iHU)]_\nabla\,.$$

As well, we have

$$
\begin{aligned}
& [-\lambda Z - ig(Z + iHZ), -\lambda W - ig(W + iHW)]_\nabla \\
= \;& -\lambda^2\,[Z, W] + i\lambda \nabla_Z g(W + iHW) - i\lambda \nabla_W g(Z + iHZ) \\
= \;& -i\lambda\,\{-i\lambda\,[Z, W] - (\nabla_Z g)\,(W + iHW) + (\nabla_W g)\,(Z + iHZ)\} + \\
& + i\lambda\,\{g\,(\nabla_Z (W + iHW)) - g\,(\nabla_W (Z + iHZ))\} \\
= \;& -i\lambda\,\{-i\lambda\,[Z, W] - g\,([Z, W] - iH\,[Z, W])\} + \\
& - \left(d^\nabla g\right)(Z, W) - iH^*(d^\nabla g)(Z, W) \\
= \;& -i\lambda\,\{-i\lambda\,[Z, W] + g\,([Z, W] + iH\,[Z, W])\}\,,
\end{aligned}
$$

and then

$$
\begin{aligned}
& Jac\,[[-\lambda Z - ig(Z + iHZ), -\lambda W - ig(W + iHW)]_\nabla\,, -\lambda U - ig(U + iHU)]_\nabla \\
= \;& -i\lambda\,\{-i\lambda Jac\,[[Z, W]\,, U] + g\,(Jac\,[[Z, W]\,, U] + iHJac\,[[Z, W]\,, U])\} = O,
\end{aligned}
$$

or,

$$Jac\,[[\sigma, \tau]_\nabla\,, \upsilon]_\nabla = O.$$

A similar computation for $E^{0,1}_{\widehat{J}}$ gives the statement, and the proof is complete. □

We remark that in the case $\lambda = 0$, $\nabla-$integrability of $\widehat{J}$ is not sufficient to have the Jacobi identity on $E^{1,0}_{\widehat{J}}$ and $E^{0,1}_{\widehat{J}}$. Namely, in this case, we get the following:

Proposition 5.5. ([20]) *Let (M, H, g) be a complex Norden manifold, the Jacobi identity holds on $E^{1,0}_{\widehat{J}}$ and $E^{0,1}_{\widehat{J}}$ if and only if the following conditions are satisfied:*

$$R^D(HY, HZ) - HR^D(HY, Z) - HR^D(Y, HZ) - R^D(Y, Z) = O \tag{5.30}$$

$$
\begin{aligned}
&\left(R^D(HX, Y) + R^D(X, HY)\right) Z + \left(R^D(HZ, X) + R^D(Z, HX)\right) Y + \\
&\qquad + \left(R^D(Y, HZ) + R^D(HY, Z)\right) X = O
\end{aligned}
\tag{5.31}
$$

for all $X, Y, Z \in C^\infty(T(M))$, where R^D denotes the curvature operator of the natural canonical connection D on M,

$$R^D(X, Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}. \tag{5.32}$$

In particular we have the following, [20]:

Theorem 5.6. *Let (M, H, g) be a Kähler Norden manifold then $E^{1,0}_{\widehat{J}}$ and $E^{0,1}_{\widehat{J}}$ are complex Lie algebroids.*

6 The generalized $\overline{\partial}_{\widehat{J}}$-operator

The following holds:

Proposition 6.1. *Let M be a smooth manifold and let $\widehat{J} = \begin{pmatrix} H & \lambda g^{-1} \\ g & -H^* \end{pmatrix}$ be the generalized pseudo calibrated complex structure on M defined by a pseudo Riemannian metric g and a $g-$symmetric operator H of $T(M)$ such that $H^2 = (-1-\lambda)I$. The natural symplectic structure on E defines a canonical isomorphism between $E^{0,1}_{\widehat{J}}$ and the dual bundle of $E^{1,0}_{\widehat{J}}$, $\left(E^{1,0}_{\widehat{J}}\right)^*$.*

Proof. We define

$$\varphi : E^{0,1}_{\widehat{J}} \to \left(E^{1,0}_{\widehat{J}}\right)^* \tag{6.1}$$

by:

$$\begin{gathered}(\varphi(Z + iHZ + g(W - iHW + iZ) + i\lambda W)) \\ (X - iHX + g(Y + iHY - iX) - i\lambda Y) = \\ = (Z + iHZ + g(W - iHW + iZ) + i\lambda W, X - iHX + g(Y + iHY - iX) - i\lambda Y),\end{gathered} \tag{6.2}$$

for all $X, Y, Z, W \in C^\infty(T(M) \otimes \mathbb{C})$ and we extend by linearity.
A direct computation gives:

$$\begin{gathered}(\varphi(Z + iHZ + g(W - iHW + iZ) + i\lambda W)) \\ (X - iHX + g(Y + iHY - iX) - i\lambda Y) = \\ = g(Y, Z) - g(W, X) + i\left(g(W, HX) + g(Y, HZ) - g(X, Z)\right).\end{gathered} \tag{6.3}$$

We have immediately that φ is injective and furthermore φ is an isomorphism. □

The canonical isomorphism φ between $E^{0,1}_{\widehat{J}}$ and the dual bundle $\left(E^{1,0}_{\widehat{J}}\right)^*$ allows us to define the $\overline{\partial}_{\widehat{J}}-$*operator* associated to the complex structure $\widehat{J}$, and to the fixed linear connection ∇ on M, as in the following:
let $f \in C^\infty(M)$ and let $df \in C^\infty(T^*(M)) \hookrightarrow C^\infty(T(M) \oplus T^*(M))$, we pose

$$\overline{\partial}_{\widehat{J}} f = 2\,(df)^{0,1} = df + i\widehat{J}df, \tag{6.4}$$

or,

$$\begin{aligned}\overline{\partial}_{\widehat{J}} f &= df - iJ^*(df) \\ &= df - i\,(df)\,J;\end{aligned} \tag{6.5}$$

moreover, we define:

$$\overline{\partial}_{\widehat{J}} : C^\infty\left(E^{0,1}_{\widehat{J}}\right) \to C^\infty\left(\wedge^2\left(E^{0,1}_{\widehat{J}}\right)\right) \tag{6.6}$$

via the natural isomorphism

$$E_{\widehat{J}}^{0,1} \overset{\wp}{\simeq} \left(E_{\widehat{J}}^{1,0}\right)^{*}, \tag{6.7}$$

as:

$$\overline{\partial}_{\widehat{J}} : C^{\infty}\left(\left(E_{\widehat{J}}^{1,0}\right)^{*}\right) \to C^{\infty}\left(\wedge^{2}\left(E_{\widehat{J}}^{1,0}\right)^{*}\right) \tag{6.8}$$

$$\left(\overline{\partial}_{\widehat{J}}\alpha\right)(\sigma,\tau) = \rho(\sigma)\,\alpha(\tau) - \rho(\tau)\,\alpha(\sigma) - \alpha\left([\sigma,\tau]_{\nabla}\right), \tag{6.9}$$

for $\alpha \in C^{\infty}\left(\left(E_{\widehat{J}}^{1,0}\right)^{*}\right)$, $\sigma, \tau \in C^{\infty}\left(E_{\widehat{J}}^{1,0}\right)$.

In general:

$$\overline{\partial}_{\widehat{J}} : C^{\infty}\left(\wedge^{p}\left(E_{\widehat{J}}^{1,0}\right)^{*}\right) \to C^{\infty}\left(\wedge^{p+1}\left(E_{\widehat{J}}^{1,0}\right)^{*}\right) \tag{6.10}$$

is defined by:

$$\begin{gathered}\left(\overline{\partial}_{\widehat{J}}\alpha\right)(\sigma_0,...,\sigma_p) = \\ = \sum_{i=0}^{p}(-1)^{i}\rho(\sigma_i)\,\alpha\left(\sigma_0,..\overset{\widehat{i}}{.}..,\sigma_p\right) + \sum_{i<j}(-1)^{i+j}\alpha\left([\sigma_i,\sigma_j]_{\nabla},\sigma_0,..\overset{\widehat{i}}{.}.\overset{\widehat{j}}{.}..,\sigma_p\right),\end{gathered} \tag{6.11}$$

for $\alpha \in C^{\infty}\left(\wedge^{p}\left(E_{\widehat{J}}^{1,0}\right)^{*}\right)$, $\sigma_0, ..., \sigma_p \in C^{\infty}\left(E_{\widehat{J}}^{1,0}\right)$.

Definition 6.2. $\overline{\partial}_{\widehat{J}}$ is called *generalized* $\overline{\partial}-$ *operator* of (M, J, g, ∇) or *generalized* $\overline{\partial}_{\widehat{J}}-$ *operator.*

We get the following:

Proposition 6.3. *If* $E_{\widehat{J}}^{1,0}$ *and* $E_{\widehat{J}}^{0,1}$ *are complex Lie algebroids then* $\left(\overline{\partial}_{\widehat{J}}\right)^{2} = 0$ *and* $\left(\partial_{\widehat{J}}\right)^{2} = 0$.

Proof. It follows from the fact that Jacobi identity holds on $E_{\widehat{J}}^{1,0}$ and $\left(E_{\widehat{J}}^{1,0}\right)^{*}$. □

It turns out that $\overline{\partial}_{\widehat{J}}-$ operator is the exterior derivative, d_L, of the Lie algebroid $L = E_{\widehat{J}}^{1,0}$. Moreover the exterior derivative d_{L^*} of $L^* = \left(E_{\widehat{J}}^{1,0}\right)^{*}$ is given by the operator $\partial_{\widehat{J}}$ defined by:

$$\partial_{\widehat{J}} : C^{\infty}\left(\wedge^{p}\left(E_{\widehat{J}}^{1,0}\right)\right) \to C^{\infty}\left(\wedge^{p+1}\left(E_{\widehat{J}}^{1,0}\right)\right) \tag{6.12}$$

$$\begin{gathered}\left(\partial_{\widehat{J}}\sigma\right)(\alpha_0^*,...,\alpha_p^*) = \\ = \sum_{i=0}^{p}(-1)^{i}\rho(\alpha_i^*)\,\sigma\left(\alpha_0^*,..\overset{\widehat{i}}{.}..,\alpha_p^*\right) + \sum_{i<j}(-1)^{i+j}\sigma\left([\alpha_i^*,\alpha_j^*]_{\nabla},\alpha_0^*,..\overset{\widehat{i}}{.}.\overset{\widehat{j}}{.}..,\alpha_p^*\right)\end{gathered} \tag{6.13}$$

for $\sigma \in C^{\infty}\left(\wedge^{p}\left(E_{\widehat{J}}^{1,0}\right)\right)$, $\alpha_0^*, ..., \alpha_p^* \in C^{\infty}\left(\left(E_{\widehat{J}}^{1,0}\right)^{*}\right)$.

In particular $\left(C^{\infty}\left(\wedge^{\bullet}\left(E_{\widehat{J}}^{1,0}\right)\right), \wedge, \overline{\partial}_{\widehat{J}}, [\ ,\ ]_{\nabla}\right)$ is a differential Gerstenhaber algebra, where $\wedge$ denotes the Schouten bracket, [13], [28].

7 Generalized holomorphic sections

Definition 7.1. $\alpha \in C^\infty\left(\wedge^p\left(E^{1,0}_{\widehat{J}}\right)^*\right)$ is called *generalized holomorphic section* if

$$\overline{\partial}_{\widehat{J}}\alpha = 0. \tag{7.1}$$

We remark that for all $f \in C^\infty(M)$ we have $\overline{\partial}_{\widehat{J}} f = 0$ if and only if $df = 0$, so the generalized holomorphic condition for functions gives only constant functions on connected components of M.

The following holds:

Proposition 7.2. *Let* $\widehat{J} = \begin{pmatrix} H & \lambda g^{-1} \\ g & -H^* \end{pmatrix}$ *be the pseudo calibrated generalized complex structure on* M *defined by a pseudo Riemannian metric* g *and a* $g-$*symmetric operator* H *of* $T(M)$, *assume that* $H^2 = (-1-\lambda)I$ with $\lambda \neq 0$. Let ∇ *be a linear connection on* M *such that* $\widehat{J}$ *is* $\nabla-$*integrable, let* $W \in C^\infty(T(M))$ *and let* $\sigma = -\lambda W + ig(W - iHW) \in E^{0,1}_{\widehat{J}}$, then $\overline{\partial}_{\widehat{J}}\sigma = 0$ if and only if $g(W)$ *is a Lagrangian submanifold of* $T^*(M)$ *with respect to the standard symplectic structure.*

Proof. Let $X, Y \in C^\infty(T(M))$, direct computations give:

$$\begin{aligned} &\left(\overline{\partial}_{\widehat{J}}\sigma\right)(-\lambda X - ig(X + iHX), -\lambda Y - ig(Y + iHY)) = \\ &= -\lambda X(-2i\lambda g(W, Y)) + \lambda Y(-2i\lambda g(X, W)) + \\ &-\sigma([\lambda X, \lambda Y] + ig(X + iHY, \lambda Y) - ig(\lambda X, Y + iHY)) \\ &= 2i\lambda^2\{Xg(W, Y) - Yg(W, X) - g([X, Y], W)\}. \end{aligned} \tag{7.2}$$

In particular we have $\left(\overline{\partial}_{\widehat{J}}\sigma\right) = 0$ if and only if:

$$(d(g(W)))(X, Y) = 0, \tag{7.3}$$

and then, by using a classical result in symplectic geometry, [15], we have that $\sigma = -\lambda W + ig(W - iHW)$ is a generalized holomorphic section of $E^{0,1}_{\widehat{J}}$ if and only if $g(W)$ is a Lagrangian submanifold of $T^*(M)$ with respect to the standard symplectic structure. □

Examples of generalized holomorphic sections can be obtained naturally in the field of Hessian geometry, [24], [26]. The concept of Hessian manifold was inspired by the Bergmann metric on bounded domains in $\mathbb{C}^n$ and now is a very interesting topic, related to many other fields in mathematics and theoretical physics: Kähler and symplectic geometry, affine differential geometry, special manifolds, string theory and mirror symmetry, [6], [7], [8], [14], [26], [27].

Definition 7.3. Let (M, g) be a pseudo Riemannian manifold, g is called of *Hessian type* if there exists $u \in C^\infty(M)$ such that $g = Hess(u) = \nabla^2 u$, where ∇ is the Levi Civita connection of g. (M, g) is called *Hessian pseudo Riemannian manifold* if g is of Hessian type.

Proposition 7.4. *Let* (M, H, g) be a Hessian pseudo Riemannian manifold such that H *is a* $g-$*symmetric operator and* $H^2 = (-1-\lambda)I$, $\lambda \neq 0$. *Let* ∇ *be the Levi Civita connection of* g, *assume* $\widehat{J} = \begin{pmatrix} H & \lambda g^{-1} \\ g & -H^* \end{pmatrix}$ *is* $\nabla-$*integrable. Let* $\{x_1, ..., x_n\}$ *be local coordinates on* M, let $\left\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right\}$ *be local frames for* $T(M)$, *if the curvature tensor of* ∇, R^{∇}, *vanishes, then for all* $k = 1, ..., n$ *the local section*

$$\sigma_k = -\lambda \frac{\partial}{\partial x_k} + ig\left(\frac{\partial}{\partial x_k} - iH\frac{\partial}{\partial x_k}\right) \in C^{\infty}\left(E^{0,1}_{\widehat{J}}\right)$$

is $\overline{\partial}_{\widehat{J}}-$*closed.*

Proof. Let $g = \nabla^2 u$, then:

$$g_{jk} = \frac{\partial^2 u}{\partial x_j \partial x_k} - \sum_{l=1}^{n} \Gamma^l_{jk} \frac{\partial u}{\partial x_l},$$

where $\left\{\Gamma^l_{jk}\right\}$ are Christoffel's symbols of g. In particular $g\left(\frac{\partial}{\partial x_k}\right)$ is d closed if and only if for all $i, j, k = 1, ..., n$:

$$\begin{aligned} & \frac{\partial^3 u}{\partial x_j \partial x_k \partial x_i} - \sum_{l=1}^{n} \frac{\partial \Gamma^l_{jk}}{\partial x_i}\frac{\partial u}{\partial x_l} - \sum_{l=1}^{n} \Gamma^l_{jk} \frac{\partial^2 u}{\partial x_l \partial x_i} + \\ & - \frac{\partial^3 u}{\partial x_i \partial x_k \partial x_j} + \sum_{l=1}^{n} \frac{\partial \Gamma^l_{ik}}{\partial x_j}\frac{\partial u}{\partial x_l} + \sum_{l=1}^{n} \Gamma^l_{ik} \frac{\partial^2 u}{\partial x_l \partial x_j} \\ = & \sum_{l=1}^{n} \left(\Gamma^l_{ik} \left(\frac{\partial^2 u}{\partial x_l \partial x_j} - \sum_{r=1}^{n} \Gamma^r_{lj} \frac{\partial u}{\partial x_r} \right) - \Gamma^l_{jk} \left(\frac{\partial^2 u}{\partial x_l \partial x_i} - \sum_{r=1}^{n} \Gamma^r_{li} \frac{\partial u}{\partial x_r} \right) \right) \\ = & \ 0 \end{aligned}$$

or:

$$\begin{aligned} & \sum_{l=1}^{n} \left(\frac{\partial \Gamma^l_{ik}}{\partial x_j} - \frac{\partial \Gamma^l_{jk}}{\partial x_i} + \sum_{r=1}^{n} \left(\Gamma^r_{ik}\Gamma^l_{rj} - \Gamma^r_{jk}\Gamma^l_{ri} \right) \right) \frac{\partial u}{\partial x_l} \\ = & \sum_{l=1}^{n} R^i_{ijk} \frac{\partial u}{\partial x_l} = 0 \end{aligned}$$

and thus, by using Proposition 7.2., we have the statement. □

The holomorphic sections in the case $\lambda = 0$ have been studied in [20].

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References

[1] S. Amari, *Differential-Geometrical Methods in Statistics*, Lectures Notes in Statistics 28 (1985).

[2] A.M. Blaga, M. Crasmareanu, *A class of almost tangent structures in generalized geometry*, Balkan J. Geom. Appl. 19 (2014), no. 2, 23-35.

[3] A. Borowiec, M. Francaviglia, I. Volovich, *Anti-Kählerian manifolds*, Diff. Geom. Appl. 12 (2000), 281-289.

[4] V. Cruceanu, P. Fortuny, P. M. Gadea, *A survey on paracomplex geometry*, Rocky Mountain J. Math. 26 (1995), 83-115.

[5] L. David, *On cotangent manifolds, complex structures and generalized geometry*, Annals-Institut Fourier 66 (2016), 1-28.

[6] J. Diustermaat, *On Hessian Riemannian structures*, Asian J. Math. 5 (2001), 79-91.

[7] D. S. Freed, *Special Kähler manifolds*, Comm. in Math. Physics 203(1) (1999), 31-52.

[8] N. Hitchin, *The moduli space of special Lagrangian submanifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25 (1997), 503-515.

[9] N. Hitchin, *Generalized Calabi-Yau manifolds*, Quart. J. Math. Oxford 54, (2003), 281-308.

[10] M. Gualtieri, *Generalized Complex Geometry*, Ann. of Math. (2) 174 (2011), 75-123.

[11] G. Ganchev, V. Mihova, *Canonical connection and the canonical conformal group on an almost complex manifold with B-metric*, Ann. Univ. Sofia Fac. Math. et Inf. 81, (1987), 195-206.

[12] T. Kurose, *Statistical manifolds admitting torsion*, Geometry and Somethong, Fukuoka University, (2007).

[13] Y. Kosmann-Schwarzbach, *Exact Gerstenhaber Algebras and Lie Bialgebroids*, Acta Applicandae Mathematicae 41 (1995), 153-165.

[14] J. Loftin, S.T. Yau, E. Zaslow, *Affine manifolds, SYZ Geometry and the "Y" vertex*, Journal of Differential Geometry 71 (1) (2005), 129-158. .

[15] D. McDuff, D. Salamon, *Introduction to Symplectic Topology*, Oxford Mathematical Monographs, Oxford Science Publications (1995).

[16] A. Nannicini, *Calibrated complex structures on the generalized tangent bundle of a Riemannian manifold*, Journal of Geometry and Physics 56 (2006), 903-916.

[17] A. Nannicini, *Almost complex structures on cotangent bundles and generalized geometry*, Journal of Geometry and Physics 60 (2010), 1781-1791.

[18] A. Nannicini, *Special Kähler manifolds generalized geometry*, Differential Geometry and its Applications 31 (2013), 230-238.

[19] A. Nannicini, *Generalized geometry of pseudo-Riemannian manifolds and generalized $\overline{\partial}-$operator*, Adv. Geom. 16 (2) (2016), 165-173.

[20] A. Nannicini, *Generalized geometry of Norden manifolds*, Journal of Geometry and Physics 99 (2016), 244-255.

[21] M. Nogushi, *Geometry of statistical manifolds*, Differential Geometry and its Applications 2 (1992), 197-222.

[22] A. P. Norden, *On a class of four-dimensional A-spaces*, Russian Math. (Izv VUZ) 17 (4) (1960), 145-157.

[23] J. Pradines, *Théorie de Lie pour les groupoides différentiables. Calcul différentiel dans la catégorie des groupoides infinitésimaux*, C.R.Acad. Sc. Paris, t. 264, Série A, (1967), 245-248.

[24] A. A. Salimov, A. Gezer, *Norden structures of Hessian type*, Turk. J. Math. 38 (2014), 462-469.

[25] A. A. Salimov, M. Iscan, F. Etayo, *Paraholomorphic B-manifolds and its properties*, Topology and its Applications 154 (2007), 925-933.

[26] H. Shima, K.Yagi, *Geometry of Hessian manifolds*, Differential geometry and its applications 7 (1997), 277-290.

[27] B. Totaro, *The curvature of a Hessian manifold*, Int. J. Math. 15 (4) (2004), 369-391.

[28] P. Xu, *Gerstenhaber algebras and BV-algebras in Poisson geometry*, Comm. Math. Phys. 200 (1999), 545-560.

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