Existence of free boundaries using the mean curvature

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Abstract. This paper deals with a free boundary problem for both Laplacian and p-Laplacian operators. We begin by proving the existence of solution (which is of class C^2) for the associated shape optimization problem. Then, after performing the shape derivative we will present two approaches in order to get sufficient conditions of existence of the free boundaries. The first one needs the use of some maximum principle. The second one uses the monotonicity of the mean curvature and can be applied for general divergence operators.

M.S.C. 2010: 35A15, 35J65, 49J20.
Key words: Dirichlet problem; free boundaries; Laplacian; p-Laplacian; mean curvature; minimal surface; shape derivative; shape optimization.

1 Introduction

Let μ be a positive measure with compact support K_μ (with a nonempty interior) and let $k > 0$ be a parameter. We look for an open and bounded set $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ such that

1. Ω strictly contains K_μ and
2. there exists a function u_Ω, satisfying the following overdetermined problem

$$(FB)\begin{cases} -\operatorname{div}(\mathcal{A}(|\nabla u_\Omega|)\nabla u_\Omega) = \mu \quad \text{in } \Omega, \\ u_\Omega = 0 \quad \text{on } \partial\Omega, \\ |\nabla u_\Omega| = k \quad \text{on } \partial\Omega \quad (\text{overdetermined condition}). \end{cases}$$

Imposing boundary conditions for both u_Ω and $|\nabla u_\Omega|$ on $\partial\Omega$ makes problem (FB) *overdetermined*, so that in general without any assumptions on data this problem has no solution. Notice that since $u_\Omega = 0$ on $\partial\Omega$ then $|\nabla u_\Omega| = -\dfrac{\partial u_\Omega}{\partial \nu}$, where ν is the outward normal vector to $\partial\Omega$.

Balkan Journal of Geometry and Its Applications, Vol.21, No.1, 2016, pp. 15-26.

In the linear case, when $\mathcal{A} = 1$ and the equation becomes $-\Delta u = \mu$, (FB) is called the quadrature surfaces free boundary problem and arises in many areas of physics (free streamlines, jets, Hele-show flows, electromagnetic shaping, gravitational problems etc.) It has been intensively studied from different points of view, by several authors. For more details about the methods used for solving this problem see the introduction in [12]. In [2] using the maximum principle together with the compatibility condition of the Neumann problem, the authors gave sufficient condition of existence for problem (FB). When $\mathcal{A}(t) = t^{p-2}$the equation becomes $-\Delta_p u = \mu$. As far as the authors know this problem still open. In [4] using essentially the Hopf's comparison principle (see Lemma 2.5 below), the author gave a sufficient condition of existence for this problem. The purpose of the present paper is to put conditions on μ and k in order to satisfy 1 and 2. Our approach here consists on solving the shape optimization problem associated to (FB). Then performing the shape derivative, we will get the overdetermined condition but not in the entire boundary of Ω. To conclude, we will give two theorems. The proof of the first one needs the use of some maximum principle. For the second theorem, we will use the monotonicity of the mean curvature for the domains which are of class C^2. The outline of the paper is as follows. Section 2 contains some preliminary results. Section 4 is devoted to the shape optimization problems while some auxiliary results are stated and proved in Section 4. In Section 5, we state and prove the main theorems. Section 6 contains some concluding remarks.

2 Preliminaries

Let D be an open ball of $\mathbb{R}^N$ $(N \geq 2)$which will contain all the sets we use in this paper.

Definition 2.1. Let K_1 and K_2 be two compact subsets of D. We call a Hausdorff distance of K_1 and K_2 (or briefly $d_H(K_1, K_2)$), the following positive number:

$$d_H(K_1, K_2) = \max\left[\rho(K_1, K_2), \rho(K_2, K_1)\right],$$

where $\rho(K_i, K_j) = \max\limits_{x \in K_i} d(x, K_j)$ $\ i,\ j = 1,\ 2$ and $d(x, K_j) = \min\limits_{y \in K_j} |x - y|$.

Definition 2.2. Let ω_n be a sequence of open subsets of D and ω be an open subset of D. Let K_n and K be their complements in $\overline{D}$. We say that the sequence ω_n converges in the Hausdorff sense, to ω (or briefly $\omega_n \xrightarrow{H} \omega$) if

$$\lim_{n \to +\infty} d_H(K_n, K) = 0.$$

Definition 2.3. Let ω_n be a sequence of open subsets of D and ω be an open subset of D. We say that the sequence ω_n converges in the compact sense, to ω (or briefly $\omega_n \xrightarrow{K} \omega$) if

- every compact subset of ω is included in ω_n, for n *large enough*, and
- every compact subset of $\overline{\omega}^c$ is included in $\overline{\omega}_n^c$,for n *large enough.*

Definition 2.4. Let ω_n be a sequence of open subsets of D and ω be an open subset of D. We say that the sequence ω_n converges in the sense of characteristic functions, to ω (or briefly $\omega_n \xrightarrow{L} \omega$) if χ_{ω_n} converges to χ_ω in $L^p_{loc}(\mathbb{R}^N)$, $p \neq \infty$, (χ_ω is the characteristic function of ω).

Lemma 2.1. *([8], [18]) If ω_n is a sequence of open subsets of D, there exists a subsequence (still denoted by ω_n) which converges, in the Hausdorff sense, to some open subset of D.*

Definition 2.5. [3] Let C be a compact convex set, the bounded domain ω satisfies C-GNP if

1. $\omega \supset int(C)$,
2. $\partial\omega \setminus C$ is locally Lipschitz,
3. for any $c \in \partial C$ there is an outward normal ray Δ_c such that $\Delta_c \cap \omega$ is connected, and
4. for every $x \in \partial\omega \setminus C$ the inward normal ray to ω (if exists) meets C.

Remark 2.6. If Ω satisfies the C-GNP and C has a nonempty interior, then Ω is connected.

Theorem 2.2. *If $\omega_n \in \mathcal{O}_C$, then there exists an open subset $\omega \subset D$ and a subsequence (again labeled ω_n) such that (i) $\omega_n \xrightarrow{H} \omega$, (ii) $\omega_n \xrightarrow{K} \omega$, (iii) χ_{ω_n} converges to χ_ω in $L^1(D)$ and (iv) $\omega \in \mathcal{O}_C$.*

For the proof of this theorem, see Theorem 3.1 in [3].

Proposition 2.3. *Let $\{\omega_n, \omega\} \subset \mathcal{O}_C$ such that $\omega_n \xrightarrow{H} \omega$. Let u_n and u_ω be respectively the solutions of $P(\omega_n, \mu)$ and $P(\omega, \mu)$. Then u_n converges strongly in $H^1_0(D)$ to u_ω (u_n and u_ω are extended by zero in D).*

This proposition is proven for $N = 2$ or 3 (see Theorem 4.3 in [3]).

Definition 2.7. Let C be a convex set. We say that an open subset ω has the C-SP, if

1. $\omega \supset int(C)$,
2. $\partial\omega \setminus C$ is locally Lipschitz,
3. for any $c \in \partial C$ there is an outward normal ray Δ_c such that $\Delta_c \cap \omega$ is connected, and
4. $\forall\, x \in \partial\omega \setminus C \;\; K_x \cap \omega = \varnothing$,where K_x is the closed cone defined by
$$\left\{ y \in \mathbb{R}^N \;:\; (y - x).(z - x) \leq 0,\ \forall\, z \in C \right\}.$$

Remark 2.8. K_x is the normal cone to the convex hull of C and $\{x\}$.

Proposition 2.4. *ω has the C-GNP if and only if ω satisfies the C-SP.*

For the proof of this proposition see Proposition 2.3 in [3].

Lemma 2.5. (**Hopf's Comparison principle**). *Let $U \subset \mathbb{R}^N$ be open and bounded, and $v_1,\ v_2 \in C^1\left(\overline{U}\right)$, with $\Delta_p v_1 \leq \Delta_p v_2$. Then the following hold.*

1. *If $v_1 \geq v_2$ on ∂U, then $v_1 \geq v_2$ in U.*
2. *Suppose $v_1 > v_2$ in U, $v_1(x) = v_2(x)$ for some $x \in \partial U$, $|\nabla v_2| \geq \gamma$ in U (for some $\gamma > 0$), and U satisfies the interior sphere condition. Then $\frac{\partial v_2}{\partial \nu}(x) > \frac{\partial v_1}{\partial \nu}(x)$, where ν is the unit outward normal vector on ∂U, at x.*
3. *If $v_1 \geq v_2$ and $v_1 \neq v_2$ in U, $|\nabla v_2| \geq \gamma$ in U (for some $\gamma > 0$), then $v_1 > v_2$ in U.*

This lemma is proven in ([23], Lemma 3.2, Proposition 3.4.1, 3.4.2)

As in the linear case, to obtain a continuity result for the Dirichlet problem in the non linear case, we can use the compact convergence and the p-stability of the limit domain (we say that an open set Ω is p-stable if for any $u \in H^{1,p}\left(\mathbb{R}^N\right)$ such that $u = 0$ a.e. in $int(\Omega^c)$, we get $u_{|\Omega} \in H_0^{1,p}(\Omega)$). Here, we will use the theorem (see below) obtained by Bucur and Trebeschi where they generalize the Sverak's result [21].

In [7], the authors gave a compactness-continuity result for the solution of a non linear Dirichlet problems (in particular with the p-Laplacian operator) when the domain varies.

Definition 2.9. (**γ_p-convergence**) We say that a sequence Ω_n of open subsets of D γ_p-converges to Ω if and only if for any $\mu \in H^{-1,q}(D)$ ($\frac{1}{p} + \frac{1}{q} = 1$) the solutions u_n of the Dirichlet problems $P(\Omega_n, \mu)$ converges strongly in $H_0^{1,p}(D)$, as $n \to +\infty$, to the solution u_Ω of $P(\Omega, \mu)$, (u_n and u_Ω are extended by zero to D).

Set
$$\mathcal{O}_l(D) = \{\omega \subseteq D \ \mid \ \sharp\omega^c \leq l\}$$
where $\sharp\omega^c$ denotes the number of connected components of the complement of ω.

Theorem 2.6. **[7]** *Let $N \geq p > N-1$. Consider $\Omega_n \in \mathcal{O}_l(D)$ and assume $\Omega_n \xrightarrow{H} \Omega$, then $\Omega \in \mathcal{O}_l(D)$ and Ω_n γ_p-converges to Ω.*

Remark 2.10. If $p > N$, any sequence of open sets which converge in the Hausdorff sense is γ_p-convergent.

Corollary 2.7. *Assume that the convex C has a nonempty interior. If $\Omega_n \in \mathcal{O}_C$ and $\Omega_n \xrightarrow{H} \Omega$, then Ω_n γ_p-converges to Ω.*

Proof. If the interior of C is nonempty and $\Omega_n \in \mathcal{O}_C$, according to Remark 2.6, Ω_n is connected. Therefore $\Omega_n \in \mathcal{O}_l(D)$. Now, if $\Omega_n \xrightarrow{H} \Omega$, by the previous theorem Ω_n γ_p-converges to Ω. $\square$

Theorem 2.8. *Let L be a compact subset of $\mathbb{R}^N$. Let f_n be a sequence a functions defined on L. We assume that the f_n are of class C^3 and*

$$\left|\frac{\partial f_n}{\partial x_i}\right| \leq M, \ \left|\frac{\partial^2 f_n}{\partial x_i \partial x_j}\right| \leq M, \ \left|\frac{\partial^3 f_n}{\partial x_i \partial x_j \partial x_k}\right| \leq M,$$

where M is a strictly positive constant and is independent of n.

Define a sequence Ω_n, by $\Omega_n = \{x \in L \,:\, f_n(x) > 0\}$ and suppose there exists $\alpha > 0$ such that $|f_n(x)| + |\nabla f_n(x)| \geq \alpha$ for all x in L. If the Ω_n have the C-GNP, then there exists Ω of class C^2 and a subsequence (still denoted by Ω_n) such that Ω_n converges in the compact sense, to Ω.

3 Shape optimization problems

Up to now, $\mu = f$ where $f \in L^2(D)$ for $p = 2$ or $f \in L^\infty(D)$ for $p \neq 2$.

In [12],[20] (for $p = 2$) or in [14] (for $p \neq 2$) by using the moving plane method [11], the authors showed that if the problem (FB) admits a solution (Ω, u_Ω) such that Ω is of class C^2 and $u_\Omega \in C^2(\overline{\Omega} \setminus K_\mu) \cap C^1\left(\overline{\Omega}\right)$, then all the inward normals at the boundary $\partial\Omega$ of Ω meet C (the convex hull of K_μ). Since we relate the existence of a solution for Problem (FB) to the existence of a minimum of some shape optimization problem, it is natural to solve this one in a class of domains with this geometric normal property.

Using the shape derivative, the problem (FB) can be seen as the Euler equation of the following problem of minimization, e.g. [22] and [17]:

$$(OP) \ \text{ Find } \Omega \in \mathcal{O}_C \text{ such that } J(\Omega) = \min_{\omega \in \mathcal{O}_C} J(\omega),$$

where

$$\mathcal{O}_C = \{\omega \subset D \,:\, \omega \text{ satisfies } C\text{-GNP}\}$$

and

$$J(\omega) = \int_\omega \left(\frac{1}{p}|\nabla u_\omega|^p - f u_\omega + \frac{k^p}{p}\right) dx$$

with u_ω the solution of the Dirichlet problem.

$$P(\omega, f) \begin{cases} -\Delta_p u_\omega = f & \text{in } \ \omega, \\ u_\omega = 0 & \text{on } \ \partial\omega. \end{cases}$$

3.1 Existence of the minima

Theorem 3.1. *There exists $\Omega \in \mathcal{O}_C$ which minimizes the functional J on $\mathcal{O}_C$. Ω is of class C^2.*

We will give the proof in the case where $p \neq 2$. For $p = 2$, just replace in the proof the Hopf's comparison principle by the maximum principle. The continuity result thanks to Proposition 2.3 from above.

Proof. Using the variational formulation of the Dirichlet problem $P(\omega, f)$, we get

$$\int_\omega |\nabla u_\omega(x)|^p dx = \int_\omega f u_\omega.$$

If u_D denotes the solution of the Dirichlet problem $P(D, f)$, by the Hopf's comparison principle (see Lemma 2.5 part 1.), $0 \leq u_\omega \leq u_D$ so

$$J(\omega) = -\frac{p-1}{p}\int_\omega f u_\omega + \frac{k^p}{p}\int_\omega dx \geq -\frac{p-1}{p}\int_D f u_D$$

and $\inf J$ exists. Let Ω_n be a minimizing sequence in $\mathcal{O}_C$. (one can choose it as in Theorem 2.8 from above). Since $int(C) \subset \Omega_n \subset D$, according to (i) of the Theorem and the continuity of the inclusion for the Hausdorff topology, there exist an open set Ω and a subsequence of Ω_n (still denoted by Ω_n) such that $\Omega_n \xrightarrow{H} \Omega$ and $int(C) \subset \Omega \subset D$. (ii) of Theorem 2.8 together with Theorem 2.8 imply that Ω is of class C^2. Now by (iii) of Theorem 2.2 $\int_{\Omega_n} dx$ converges to $\int_\Omega dx$, and by Corollary 2.7, $\int_D f u_n \chi_{\Omega_n}$ converges to $\int_D f u_\Omega \chi_\Omega = \int_\Omega |\nabla u_\Omega(x)|^p dx$. Hence $J(\Omega) \leq \liminf_{n\to+\infty} J(\Omega_n)$. According to (iv) of Theorem 2.2, $\Omega \in \mathcal{O}_C$, therefore $J(\Omega) = \min_{\omega\in\mathcal{O}_C} J(\omega)$. The regularity C^2 of Ω thanks to Theorem 2.8. □

Put

$$\mathcal{O}_\Omega = \{\omega \subset \Omega \ : \ \omega \text{ satisfies } C\text{-GNP}\}$$

and

$$j(\omega) = k|\partial\omega| + \int_{\partial\omega} \frac{\partial u_\omega}{\partial \nu} dx$$

where ν is the exterior normal vector to $\partial\omega$, $|\partial\omega|$ denotes the perimeter of ω and u_ω the solution of the Dirichlet problem P(ω ,f). By Green formula, j becomes

$$j(\omega) = k|\partial\omega| - \int_\omega f(x) dx$$

Theorem 3.2. *There exists $\Omega^* \in \mathcal{O}_\Omega$ which minimizes the functional j on $\mathcal{O}_\Omega$. Ω^* is of class C^2.*

For the proof of this theorem, we use (iii) and (iv) of Theorem 2.2. Once again, the C^2 regularity of Ω^* thanks to Theorem 2.8.

3.2 The optimality conditions

In this paragraph, we are going to use the standard tool of the domain derivative to write down the optimality condition. Let us recall the definition of the domain derivative, see for instance [22] and [17]. Since the minimum Ω of the functional J is of class C^2. Let us consider a deformation field $V \in C^2\left(\mathbb{R}^N; \mathbb{R}^N\right)$ and set $\Omega_t = \{x + tV(x),\ x \in \Omega\}$, $t > 0$. The application $Id + tV$ is a perturbation of the identity which is a Lipschitz diffeomorphism for t small enough. By definition, the derivative of J at Ω in the direction V is

$$dJ(\Omega, V) = \lim_{t\to 0} \frac{J(\Omega_t) - J(\Omega)}{t}.$$

As the functional J depends on the domain Ω through the solution of the Dirichlet problem $P(\Omega, f)$, we need to define also the domain derivative of u_Ω. If u'_Ω denotes the domain derivative of u_Ω, then

$$u'_\Omega = \lim_{t\to 0} \frac{u_{\Omega_t} - u_\Omega}{t}.$$

Recall that the shape derivative of the volume is $\int_{\partial\Omega} V.\nu \, d\sigma$
Now for $F(\Omega) = \int_\Omega h(u_\Omega)dx$, the Hadamard formula gives

$$dF(\Omega, V) = \int_\Omega h'(u_\Omega)\, u'_\Omega dx + \int_{\partial\Omega} h(u_\Omega) V \cdot \nu \, d\sigma.$$

Furthermore, we can prove ([22], [17]) that u'_Ω is a solution of some linear Dirichlet problem with

$$u'_\Omega = -\frac{\partial u_\Omega}{\partial \nu} V \cdot \nu \text{ on } \partial\Omega.$$

This, together with $u_\Omega = 0$ on $\partial\Omega$ implies

$$dF(\Omega, V) = \int_{\partial\Omega} h(u_\Omega) V \cdot \nu \, d\sigma.$$

Now by Green formula

$$J(\Omega) = -\frac{1}{p}\int_\Omega |\nabla u_\Omega|^p + \frac{1}{p} k^p \int_\Omega dx.$$

So if we put $h(u_\Omega) = |\nabla u_\Omega|^p$, according to what precedes we obtain

$$dJ(\Omega; V) = \frac{1}{p}\int_{\partial\Omega} \left(k^p - |\nabla u_\Omega(x)|^p\right) V.\nu \, d\sigma. \tag{3.1}$$

where ν is the outward normal vector to $\partial\Omega$.

Now since Ω is the minimum for the functional J, $dJ(\Omega; V) \geq 0$ for every admissible direction V. Therefore

$$\int_{\partial\Omega} \left(k^p - |\nabla u_\Omega(x)|^p\right) V.\nu \, d\sigma \geq 0 \text{ for every admissible direction } V.$$

We mean by admissible displacement the one which allows us to keep the C-GNP or the C-SP (according to Proposition 2.4 from above). Since Ω has the C-GNP, it satisfies the C-SP. Then

$$\forall\, x \in \partial\Omega \setminus C \;\; K_x \cap \Omega = \varnothing\,.$$

For t sufficiently small, let $\Omega_t = \Omega + tV(\Omega)$ be the deformation of Ω in the direction V. Let $x_t \in \partial\Omega_t$. There exists $x \in \partial\Omega$ s.t $x_t = x + tV(x)$. Using the definition of K_{x_t} and the equality above, it is obvious to get (for t small enough and for every displacement V) :

$$\forall\, x_t \in \partial\Omega_t \setminus C \;\; K_{x_t} \cap \Omega_t = \varnothing\,,$$

which means that Ω_t satisfies the C-SP (and so the C-GNP) for every displacement V when t is sufficiently small. Then, using V and $-V$, and the fact that the set of the functions $V \cdot \nu$ is dense in $L^2(\partial\Omega)$, we deduce

$$|\nabla u_\Omega(x)| = k \text{ on } \partial\Omega \setminus \partial C. \tag{3.2}$$

On the other hand, the admissible directions V on $\partial\Omega \cap \partial C$ must satisfy $V(x) \cdot \nu(x) \geq 0$, and one gets

$$|\nabla u_\Omega(x)| \leq k \text{ on } \partial\Omega \cap \partial C. \tag{3.3}$$

Now, thanks to Hadamard formula, the shape derivative of j on Ω^* is

$$dj(\Omega^*; V) = \int_{\partial\Omega^*} (NkH_{\partial\Omega^*} - f)\, V.\nu \, d\sigma \geq 0 \text{ for every admissible direction } V.$$

Arguing as above and using the fact that $int(C) \subset \Omega^*$, we get

$$\begin{cases} H_{\partial\Omega^*} = 0 \text{ on } \partial\Omega^* \setminus \partial C \\ H_{\partial\Omega^*} \geq \frac{f}{Nk} \text{ on } \partial\Omega^* \cap \partial C. \end{cases} \tag{3.4}$$

4 Auxiliary results

In this section, we will state and prove some propositions which we will use in the Section 5. Let Ω (resp. Ω^*) be the minimum of J (resp. j). The two first propositions are given for $p \neq 2$. For $p = 2$, the proof is done if we replace The Hopf's comparison principle by the maximum principle.

Proposition 4.1. *Suppose that C is of class C^2 and $|\nabla u_C| \geq \gamma$ in $int(C)$ (for some $\gamma > 0$) and C satisfies the interior sphere condition. Then*

1. *either $\partial\Omega \cap \partial C \neq \varnothing$ and $|\nabla u_C(x)| \leq k$ on $\partial\Omega \cap \partial C$*
2. *or C is strictly contained in Ω.*

Proof. Let $\partial\Omega \cap \partial C \neq \varnothing$ and suppose by contradiction there exists $x \in \partial\Omega \cap \partial C$ such that $|\nabla u_\Omega(x)| > k$. This together with (3) implies that $\partial\Omega \neq \partial C$.
Now, since

$$\Delta_p u_\Omega = -f = \Delta_p u_C \text{ in} int(C) \text{ and } u_\Omega \geq 0 = u_C \text{ on } \partial C,$$

part 1. of Lemma 2.5 implies that

$$u_\Omega \geq u_C \text{ in } int(C).$$

But $u_\Omega \neq u_C$ in $int(C)$, then

$$u_\Omega > u_C \text{ in } int(C).$$

Now, since C satisfies the interior sphere condition, $|\nabla u_C| > \gamma$ on $int(C)$ and

$$u_\Omega = u_C \text{ on } \partial\Omega \cap \partial C,$$

part 2. of Lemma 2.5, gives

$$\frac{\partial u_\Omega}{\partial \nu}(x) < \frac{\partial u_C}{\partial \nu}(x)$$

or again, since $|\nabla u_\Omega(x)| = -\dfrac{\partial u_\Omega(x)}{\partial \nu(x)}$,

$$|\nabla u_C(x)| < |\nabla u_\Omega(x)|\,.$$

So $|\nabla u_\Omega(x)| > k$ which contradicts (3.3).
□

If we replace, in the preceding proposition, $int(C)$ by Ω^*, we can obtain

Proposition 4.2. *Suppose that $|\nabla u_{\Omega^*}| \geq \gamma$ in Ω^* (for some $\gamma > 0$) and Ω^* satisfies the interior sphere condition. Then*

1. *either $\partial\Omega^* \cap \partial\Omega \neq \varnothing$ and $|\nabla u_{\Omega^*}(x)| \leq k$ on $\partial\Omega \cap \partial\Omega^*$*
2. *or Ω^* is strictly contained in Ω.*

Proposition 4.3. *Suppose that C is of class C^2, then*

1. *either $\partial\Omega^* \cap \partial C \neq \varnothing$ and $H_{\partial\Omega^*} \geq \frac{f}{Nk}$ on $\partial\Omega^* \cap \partial C$*
2. *or C is strictly contained in Ω^*.*

Proof. Suppose there exists $x \in \partial\Omega^* \cap \partial C$ such that $H_{\partial C}(x) < \frac{f(x)}{Nk}$. Since $int(C) \subset \Omega^*$, $x \in \partial\Omega^* \cap \partial C$ and C and Ω^* are of class C^2, then

$$\frac{f(x)}{Nk} \leq H_{\partial\Omega^*} \leq H_{\partial C} < \frac{f(x)}{Nk}$$

which is absurd. □

5 Existence of free boundaries

Theorem 5.1. *Suppose $p \neq 2$ and let Ω and Ω^* be as in Theorems 3.1 and 3.2. If $|\nabla u_{\Omega^*}| > k$ on Ω^* then Ω is a solution of (FB) which strictly contains Ω^*.*

Remark 5.1. For $p = 2$, we can obtain the same result if we replace the condition stated above by the following:

$$|\nabla u_{\Omega^*}| > k \text{ on } \partial\Omega^*$$

Proof. This result is an immediate consequence of Proposition 4.1. □

Theorem 5.2. *Let Ω and Ω^* be as in Theorems 3.1 and 3.2.*

1. *If C is of class C^2 and $H_{\partial C} < \dfrac{f}{Nk}$ on ∂C then*
   (a) C is strictly contained in Ω^*

(b) Ω^* is a minimal surface

(c) Ω is a solution of (FB) which contains Ω^*

2. *Furthermore, if $|\nabla u_{\Omega^*}| \leq k$ on $\partial\Omega^*$ or if $k|\partial\Omega^*| \geq \int_C f$ then Ω is a minimum of j and so it is a minimal surface.*

Proof. (1)

- (a) is an immediate consequence of Proposition 4.3.
- (b) The optimality condition (4) gives $H_{\partial\Omega^*} = 0$ which implies that Ω^* is a minimal surface.
- (c) According to (a), C is strictly contained in Ω^* but $\Omega^* \subset \Omega$. So C is strictly contained in Ω and the optimality conditions (2) and (3) imply $|\nabla u_\Omega| = k$ on $\partial\Omega$

(2) If in addition Ω^* verifies one of the two conditions stated above, then $j(\Omega^*) \geq 0$. But $\Omega \in \mathcal{O}_\Omega$ and by (c) $j(\Omega) = 0$ so $j(\Omega) \leq j(\Omega^*)$. Then we can conclude by (b). $\square$

Replace in the expressions of J and j, f by $1+f$ and denote by J_1 and j_1 the corresponding functionals of domains. We obtain

Theorem 5.3. *Let Ω_1 (resp. Ω_1^*) be the minimum of J_1 (resp. of j_1). If C is of class C^2 and $H_{\partial C} < \dfrac{1+f}{Nk}$ on ∂C then*

1. *C is strictly contained in Ω_1^**
2. *Ω_1^* is a ball with radius Nk*
3. *Ω_1 is a solution of (FB) which contains Ω_1^**

Reasoning like in Theorem 5.3, the first and the third items are immediate. For the second item one can replace, in the optimality conditions (3.4), f by $1+f$. Then using the fact that C is strictly contained in Ω_1^*, one can obtain $H_{\partial\Omega_1^*} = \dfrac{1}{Nk}$ which says that Ω_1^* is a ball with radius Nk thanking to the Alexandrov result [1].

Remark 5.2. A simple calculation shows that we cannot put conditions on Ω_1^* (as in (2) of Theorem 5.3) and so Ω_1 cannot be a minimum of j_1.

Remark 5.3. In one hand Ω_1^* is a ball, so it satisfies the Geometric Normal Property w.r.t its center. In the other hand Ω_1^* has the C-GNP. Therefore, the center of Ω_1^* belongs to C.

6 Concluding remarks

Remark 6.1. The aim of Theorem 2.8 is to give the C^2 regularity of the minimum Ω (resp. Ω^*) of J (resp. j). This in order to use the shape derivative and so to resolve Problem (FB). The proof of this theorem uses the following Lemma (see [2]):

Lemma 6.1. *Let L be a compact subset of $\mathbb{R}^N$. Let f_n be a sequence of functions defined as Theorem 2.8. Suppose that Ω is an open subset of L such that*

$$\begin{aligned} \Omega &= \{x \in L \,:\, h(x) > 0\} \text{ and} \\ \partial\Omega &= \{x \in L \,:\, h(x) = 0\}, \end{aligned}$$

where h is a continuous function defined in L. If the f_n converge uniformly to h in L, then the Ω_n converge in the compact sense, to Ω.

Remark 6.2. **a)** The hypothesis in Theorem 2.8 about the local regularity is not too restrictive because of, for instance, results due to E. DiBenditto [10], J.L .Lewis [15] and G.M. Lieberman [16].

b) When $p = 2$ Proposition 4.1 and Theorem 5.1 can be extended to the divergence operator $div(a(x)\nabla u)$. For this kind of operator the continuity result is a simple consequence of Mosco convergence (see for instance [7]).

c) According to the results obtained by Bucur and Trebeschi in [7], Proposition 4.3 and Theorem 5.3 can be extended to other divergence operators like $div(a(x, Du))$.

d) Let $f = a\chi_{B_R}$ where $a > 0$, $B_R \subset \mathbb{R}^2$ is some ball of radius R and χ_{B_R} is its characteristic function. The condition stated in Theorem 5.3 becomes $aR > 2k$. Now if Ω is a regular solution of (FB), then Green formula implies $aR > 2k$, i.e this condition is necessary and sufficient for solving (FB) in this case.

e) Consider the case of (FB) where μ is the uniform density $\delta_{[-1,1]\times\{0\}}$. Let C be the ball of radius 1 and of center 0. According to the preceding remark, $a > 2k$ is a necessary and sufficient condition of existence for a free boundary which contains strictly the segment line $[-1, 1] \times \{0\}$. Notice that in [12], the authors gave $a > 24\pi k$ as sufficient condition of existence for this problem while in [5], the author proposes $a > 3.92k$.

f) Let Ω be a solution of (FB) in the case where $\mu \equiv 1$. Using the same arguments as in Theorem 3.2, we can prove the existence of a minimum Ω^* of j on some class of admissible domains (for instance the domains which are contained in Ω and satisfy the ε-cone property). If both Ω and Ω^* are of class C^2 then by the optimality condition, Ω^* is a ball with radius Nk and $j(\Omega^*) = 0 = j(\Omega)$. Therefore Ω is a minimum of j and so $\Omega = B_{Nk}$, i.e it is the solution of Serrin's problem [19]. Now according to Remark e), this result can be extended to other divergence operators like $div(a(x, Du))$ and according to Remark e), it cannot be obtained when μ is nonconstant.

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