Indefinite trans-Sasakian manifold of quasi-constant curvature with lightlike hypersurfaces

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Abstract. In this paper, we study indefinite trans-Sasakian manifold $\bar{M}$ of quasi-constant curvature with lightlike hypersurfaces M. We provide several new results on such a manifold $\bar{M}$, in which the characteristic 1-form θ and the characteristic vector field ζ, defined by (1.1), are identical with the structure 1-form θ and the structure vector field ζ of the indefinite trans-Sasakian structure (J, ζ, θ) of $\bar{M}$, respectively.

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1 Introduction

B.Y. Chen-K. Yano [2] introduced the notion of a *semi-Riemannian manifold of quasi-constant curvature* as a semi-Riemannian manifold $(\bar{M}, \bar{g})$ endowed with the curvature tensor $\bar{R}$ satisfying the following form:

$$\begin{aligned} \bar{R}(X,Y)Z = {} & \lambda\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} \\ & + \mu\{\bar{g}(Y,Z)\theta(X)\zeta - \bar{g}(X,Z)\theta(Y)\zeta \\ & \qquad + \theta(Y)\theta(Z)X - \theta(X)\theta(Z)Y\}, \end{aligned} \tag{1.1}$$

for any vector fields X, Y and Z of $\bar{M}$, where λ and μ are smooth functions, ζ is a smooth vector field and θ is a 1-form associated with ζ by $\theta(X) = \bar{g}(X, \zeta)$. In this case, ζ and θ are called the *characteristic vector field* and the *characteristic* 1*-form* of $\bar{M}$, respectively. It is well known that if the curvature tensor $\bar{R}$ is of the form (1.1), then $\bar{M}$ is conformally flat. If $\mu = 0$, then $\bar{M}$ is a space of constant curvature λ.

J.A. Oubina [9] introduced the notion of a trans-Sasakian manifold of type (α, β). Sasakian manifold is an important kind of trans-Sasakian manifold such that $\alpha = 1$ and $\beta = 0$. Cosymplectic manifold is another kind of trans-Sasakian manifold such that $\alpha = \beta = 0$. Kenmotsu manifold is also an example with $\alpha = 0$ and $\beta = 1$.

The theory of lightlike hypersurfaces is an important topic of research in differential geometry due to its application in mathematical physics. The study of such

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notion was initiated by K.L. Duggal-A. Bejancu [3] and later studied by many authors [4, 5]. In this paper, we study indefinite trans-Sasakian manifold $\bar{M}$ of quasi-constant curvature with lightlike hypersurfaces M such that the 1-form θ and its associated vector field ζ, defined by (1.1), are identical with the structure 1-form θ and the structure vector field ζ of the indefinite trans-Sasakian structure (J, ζ, θ) of $\bar{M}$.

2 Lightlike hypersurfaces

Let (M, g) be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the normal bundle $TM^\perp$ of M is a vector subbundle of the tangent bundle TM of M, of rank 1. Therefore there exists a non-degenerate complementary vector bundle $S(TM)$ of $TM^\perp$ in TM, which is called a *screen distribution* of M, such that

$$TM = TM^\perp \oplus_{orth} S(TM),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E. Also denote by $(2.5)_i$ the i-th equation of the two equations in (2.5). We use same notations for any others. It is known [3] that, for any null section ξ of $TM^\perp$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique vector bundle $tr(TM)$ of rank 1 in $S(TM)^\perp$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall\, X \in \Gamma(S(TM)).$$

In this case, the tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as

$$T\bar{M} = TM \oplus tr(TM) = \{TM^\perp \oplus tr(TM)\} \oplus_{orth} S(TM).$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to the screen distribtion $S(TM)$ respectively.

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingartan formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.1}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N; \tag{2.2}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.3}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi,$$

for all $X,\, Y \in \Gamma(TM)$, where ∇ and ∇^* are the liner connections on M and $S(TM)$ respectively, B and C are the local second fundamental forms on M and $S(TM)$, respectively, A_N and A^*_ξ are the shape operators on M and $S(TM)$ respectively and τ is a 1-form on TM. Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free and B is symmetric. The induced connection ∇ of M is not metric and satisfies

$$(\nabla_X g)(Y, Z) = B(X, Y)\,\eta(Z) + B(X, Z)\,\eta(Y),$$

for any $X,\, Y,\, Z \in \Gamma(TM)$, where η is a 1-form such that

$$\eta(X) = \bar{g}(X, N), \quad \forall\, X \in \Gamma(TM).$$

From the fact that $B(X,Y) = \bar{g}(\bar{\nabla}_X Y,\, \xi)$, we show that B is independent of the choice of a screen distribution $S(TM)$ and satisfies

$$B(X,\xi) = 0, \quad \forall\, X \in \Gamma(TM). \tag{2.4}$$

Two local second fundamental forms B and C are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{2.5}$$

$$C(X,PY) = g(A_{_N} X, PY), \quad \bar{g}(A_{_N} X, N) = 0. \tag{2.6}$$

From (2.5), the operator A^*_ξ is $S(TM)$-valued self-adjoint on TM such that

$$A^*_\xi \xi = 0.$$

From now and in the sequel, let X, Y, Z and W be the vector fields on M, unless otherwise specified. Denote by $\bar{R}$, R and R^* the curvature tensors of the Levi-Civita connection $\bar{\nabla}$ of $\bar{M}$, the induced connection ∇ on M and the induced connection ∇^* on $S(TM)$ respectively. Using the Gauss-Weingarten formulas for M and $S(TM)$, we obtain the Gauss equations for M and $S(TM)$ such that

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z &+ B(X,Z)A_{_N}Y - B(Y,Z)A_{_N}X \\ &+ \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &\quad + \tau(X)B(Y,Z) - \tau(Y)B(X,Z)\}N,\end{aligned} \tag{2.7}$$

$$\begin{aligned}R(X,Y)PZ = R^*(X,Y)PZ &+ C(X,PZ)A^*_\xi Y - C(Y,PZ)A^*_\xi X \\ &+ \{(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\ &\quad - \tau(X)C(Y,PZ) + \tau(Y)C(X,PZ)\}\xi.\end{aligned} \tag{2.8}$$

In case $R = 0$, we say that M is *flat*.

3 Indefinite trans-Sasakian manifolds

An odd-dimensional semi-Riemannian manifold $(\bar{M},\, \bar{g})$ is called an *indefinite almost contact metric manifold* if there exist a (1, 1)-type tensor field J, a vector field ζ which is called the *structure vector field*, and a 1-form θ such that

$$J^2X = -X + \theta(X)\zeta, \quad \bar{g}(JX,\, JY) = \bar{g}(X,\, Y) - \epsilon\,\theta(X)\theta(Y), \quad \theta(\zeta) = 1, \tag{3.1}$$

for any vector fields X and Y on $\bar{M}$, where ϵ denotes the signature of ζ, *i.e.*, $\bar{g}(\zeta,\zeta) = \epsilon$. In this case, the set $\{J,\, \zeta,\, \theta,\, \bar{g}\}$ is called an *indefinite almost contact metric structure* of $\bar{M}$. From (3.1), we see that $J\zeta = 0$, $\theta \circ J = 0$ and $\theta(X) = \epsilon\bar{g}(X,\zeta)$. Also, we see that ζ is a non-null vector field. In the entire discussion of this article, we shall assume that ζ to be unit spacelike, *i.e.*, $\epsilon = 1$, without loss generality.

Definition 3.1. An indefinite almost contact metric manifold $(\bar{M},\bar{g})$ is said to be an *indefinite trans-Sasakian manifold* [7, 8, 9] if, for any vector fields X and Y on $\bar{M}$, there exist two smooth functions α and β such that

$$(\bar{\nabla}_X J)Y = \alpha\{\bar{g}(X,Y)\zeta - \theta(Y)X\} + \beta\{\bar{g}(JX,Y)\zeta - \theta(Y)JX\}. \tag{3.2}$$

We say that $\{J,\zeta,\theta,\bar{g}\}$ is an *indefinite trans-Sasakian structure of type* (α,β).

From (3.1) and (3.2), we get

$$\bar{\nabla}_X \zeta = -\alpha JX + \beta(X - \theta(X)\zeta), \qquad d\theta(X, Y) = g(X, JY). \tag{3.3}$$

In the sequel, let M be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\bar{M}$ such that the structure vector field ζ of $\bar{M}$ is tangent to M. Călin [1] proved that if ζ is tangent to M, then it belongs to $S(TM)$ which we assume in this work. It is known [7, 8] that, for any lightlike hypersurface M of an indefinite almost contact metric manifold $\bar{M}$, the distributions $J(TM^\perp)$ and $J(tr(TM))$ are vector subbundles of $S(TM)$, of rank 1, and $J(TM^\perp) \cap J(tr(TM)) = \{0\}$. Thus we see that $J(TM^\perp) \oplus J(tr(TM))$ is a vector subbundle of $S(TM)$ of rank 2. Therefore there exists two non-degenerate almost complex distributions D_o and D with respect to the structure tensor J, *i.e.*, $J(D_o) = D_o$ and $J(D) = D$, such that

$$S(TM) = J(TM^\perp) \oplus J(tr(TM)) \oplus_{orth} D_o,$$
$$D = TM^\perp \oplus_{orth} J(TM^\perp) \oplus_{orth} D_o.$$

Using these distributions, TM is decomposed as follow:

$$TM = D \oplus J(tr(TM)).$$

Consider the local lightlike vector fields U and V such that

$$U = -JN, \qquad V = -J\xi. \tag{3.4}$$

Denote by S the projection morphism of TM on D. Using this operator,

$$X = SX + u(X)U, \qquad \forall\, X \in \Gamma(TM),$$

where u and v are 1-forms locally defined on M by

$$u(X) = g(X, V), \qquad v(X) = g(X, U). \tag{3.5}$$

Using (3.4), the action JX of any vector field X on M by J is expressed as

$$JX = FX + u(X)N, \tag{3.6}$$

where F is a tensor field of type $(1, 1)$ globally defined on M by $F = J \circ S$.

Applying $\bar{\nabla}_X$ to (3.4) ∼ (3.6) and using (2.1) ∼ (2.4) and (3.4) ∼ (3.6), we have

$$B(X, U) = C(X, V), \tag{3.7}$$
$$\nabla_X U = F(A_N X) + \tau(X)U - \{\alpha\eta(X) + \beta v(X)\}\zeta, \tag{3.8}$$
$$\nabla_X V = F(A^*_\xi X) - \tau(X)V - \beta u(X)\zeta, \tag{3.9}$$
$$\begin{aligned}(\nabla_X F)(Y) = {} & u(Y)A_N X - B(X, Y)U \\ & + \alpha\{g(X, Y)\zeta - \theta(Y)X\} + \beta\{\bar{g}(JX, Y)\zeta - \theta(Y)FX\},\end{aligned} \tag{3.10}$$
$$(\nabla_X u)(Y) = -u(Y)\tau(X) - \beta\theta(Y)u(X) - B(X, FY), \tag{3.11}$$
$$(\nabla_X v)(Y) = v(Y)\tau(X) - \theta(Y)\{\alpha\eta(X) + \beta v(X)\} - g(A_N X, FY). \tag{3.12}$$

Applying $\bar{\nabla}_X$ to $g(\zeta, \xi) = 0$ and $\bar{g}(\zeta, N) = 0$ and using (3.3), we have

$$B(X, \zeta) = -\,\alpha u(X), \qquad C(X, \zeta) = -\,\alpha v(X) + \beta\eta(X). \tag{3.13}$$

Substituting (3.6) into (3.3), we see that

$$\nabla_X \zeta = -\alpha F X + \beta(X - \theta(X)\zeta). \tag{3.14}$$

Applying J to (3.6) and using (3.1) and (3.4), we have

$$F^2 X = -X + u(X)U + \theta(X)\zeta, \quad FU = 0, \quad F\zeta = 0, \quad u(U) = 1. \tag{3.15}$$

We say that U is the *canonical structure vector field* of M.

4 Manifold of quasi-constant curvature

Let M be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\bar{M}$ of quasi-constant curvature. Comparing the tangential and transversal components of the two equations (1.1) and (2.7), we obtain

$$\begin{aligned} R(X,Y)Z \;=\; & \lambda\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} \\ & + \mu\{\bar{g}(Y,Z)\theta(X)\zeta - \bar{g}(X,Z)\theta(Y)\zeta + \theta(Y)\theta(Z)X \\ & - \theta(X)\theta(Z)Y\} \;+\; B(Y,Z)A_N X - B(X,Z)A_N Y, \end{aligned} \tag{4.1}$$

$$(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) = 0, \tag{4.2}$$

respectively. Taking the scalar product with N to (2.8), we have

$$\begin{aligned} \bar{g}(R(X,Y)PZ,\, N) = & (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\ & - \tau(X)C(Y,PZ) + \tau(Y)C(X,PZ). \end{aligned}$$

Substituting (4.1) into the last equation, we see that

$$\begin{aligned} & (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) - \tau(X)C(Y,PZ) + \tau(Y)C(X,PZ) \\ & = \lambda\{g(Y,PZ)\eta(X) - g(X,PZ)\eta(Y)\} \\ & \quad + \mu\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(PZ). \end{aligned} \tag{4.3}$$

Theorem 4.1. *Let M be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\bar{M}$ of quasi-constant curvature. Then α is a constant, and*

$$\beta = 0, \qquad \lambda = \alpha^2, \qquad \mu = 0.$$

Proof. Applying ∇_Y to (3.7) and using (3.1), (3.6) $\sim$ (3.9) and (3.13), we have

$$\begin{aligned} (\nabla_X B)(Y,U) \;=\; & (\nabla_X C)(Y,V) - 2\tau(X)C(Y,V) \\ & - \alpha^2 u(Y)\eta(X) - \beta^2 u(X)\eta(Y) + \alpha\beta\{u(X)v(Y) - u(Y)v(X)\} \\ & - g(A^*_\xi X, F(A_N Y)) - g(A^*_\xi Y, F(A_N X)). \end{aligned}$$

Substituting this equation and (3.7) into (4.2) such that $Z = U$, we get

$$\begin{aligned} & (\nabla_X C)(Y,V) - (\nabla_Y C)(X,V) - \tau(X)C(Y,V) + \tau(Y)C(X,V) \\ & = (\alpha^2 - \beta^2)\{u(Y)\eta(X) - u(X)\eta(Y)\} + 2\alpha\beta\{u(Y)v(X) - u(X)v(Y)\}. \end{aligned}$$

Comparing this equation and (4.3) such that $PZ = V$, we obtain

$$(\lambda - \alpha^2 + \beta^2)\{u(Y)\eta(X) - u(X)\eta(Y)\} = 2\alpha\beta\{u(Y)v(X) - u(X)v(Y)\}.$$

Taking $X = \xi$ and $Y = U$, and then, $X = V$ and $Y = U$ to this, we have

$$\lambda = \alpha^2 - \beta^2, \qquad \alpha\beta = 0. \tag{4.4}$$

Applying ∇_X to $B(Y, \zeta) = -\alpha u(Y)$ and using (3.11) and (3.14), we have

$$(\nabla_X B)(Y, \zeta) = -(X\alpha)u(Y) - \beta B(X, Y) + \alpha\{u(Y)\tau(X) + B(X, FY) + B(Y, FX)\},$$

due to $\alpha\beta = 0$. Substituting this equation into (4.2) such that $Z = \zeta$, we have

$$(X\alpha)u(Y) = (Y\alpha)u(X).$$

Replacing Y by U to this equation, we obtain

$$X\alpha = (U\alpha)u(X). \tag{4.5}$$

Applying $\bar{\nabla}_X$ to $\eta(Y) = \bar{g}(Y, N)$ and using (2.1) and (2.2) we have

$$(\nabla_X \eta)(Y) = -g(A_N X,\, Y) + \tau(X)\eta(Y).$$

Applying ∇_Y to $(3.13)_2$ and using (3.12), (3.13) and (3.14), we have

$$\begin{aligned}(\nabla_X C)(Y, \zeta) = &-(X\alpha)v(Y) - \alpha v(Y)\tau(X) + \alpha^2\theta(Y)\eta(X)\\ &+ (X\beta)\eta(Y) + \beta\eta(Y)\tau(X) + \beta^2\theta(X)\eta(Y)\\ &+ \alpha\{g(A_N X, FY) + g(A_N Y, FX)\}\\ &- \beta\{g(X, A_N Y) + g(A_N X, Y)\}.\end{aligned}$$

Substituting this equation into (4.3) such that $PZ = \zeta$ and using $(4.4)_1$, we get

$$\{X\beta + \mu\theta(X)\}\eta(Y) - \{Y\beta + \mu\theta(Y)\}\eta(X) = (X\alpha)v(Y) - (Y\alpha)v(X).$$

Taking $X = \xi$ and $Y = \zeta$, and then, $X = U$ and $Y = V$ to this, we get

$$\mu = -\zeta\beta, \tag{4.6}$$

and $U\alpha = 0$. From (4.5) and the result $U\alpha = 0$, we see that α is a constant.

As α is a constant and $\alpha\beta = 0$, if $\alpha \neq 0$, then we have $\beta = 0$.

Assume that $\alpha = 0$. Then the equation (3.14) is reduced to

$$\nabla_Y \zeta = \beta(Y - \theta(Y)\zeta).$$

By straightforward calculations form this equation, we obtain

$$\begin{aligned}R(X, Y)\zeta = &\ (X\beta)Y - (Y\beta)X - \{(X\beta)\theta(Y) - (Y\beta)\theta(X)\}\zeta\\ &+ \beta^2\{\theta(X)Y - \theta(Y)X\} - 2\beta d\theta(X, Y)\zeta.\end{aligned}$$

Comparing this equation and (4.1) such that $Z = \zeta$, we obtain

$$\begin{aligned} &(X\beta)Y - (Y\beta)X - \{(X\beta)\theta(Y) - (Y\beta)\theta(X)\}\zeta \\ &\quad + \beta^2\{\theta(X)Y - \theta(Y)X\} - 2\beta d\theta(X,Y)\zeta \\ &= (\lambda + \mu)\{\theta(Y)X - \theta(X)Y\}. \end{aligned}$$

Taking the scalar product with ζ to this, we get $\beta d\theta(X, Y) = 0$, *i.e.*,

$$\beta g(X, JY) = 0, \quad \forall\, X,\, Y \in \Gamma(TM),$$

due to $(3.3)_2$. Taking $X = U$ and $Y = \xi$ to this, we have $\beta = 0$. As $\beta = 0$, (4.4) and (4.6) are reduced to $\lambda = \alpha^2$ and $\mu = 0$ respectively. □

Corollary 4.2. *Let $\bar{M}$ be an indefinite trans-Sasakian manifold of quasi-constant curvature, of type (α, β), endowed with a lightlike hypersurface. Then $\bar{M}$ is an indefinite α-Sasakian manifold of constant positive curvature α^2.*

Definition 4.1. Let $\nabla^{\perp}_X N = \pi(\bar{\nabla}_X N)$ for any $X \in \Gamma(TM)$, where π is the projection morphism of $T\bar{M}$ on $tr(TM)$. Then $\nabla^{\perp}$ is a linear connection on the transversal vector bundle $tr(TM)$ of M. We say that $\nabla^{\perp}$ is the *transversal connection* of M. We define the curvature tensor $R^{\perp}$ of $tr(TM)$ by

$$R^{\perp}(X,Y)N = \nabla^{\perp}_X \nabla^{\perp}_Y N - \nabla^{\perp}_Y \nabla^{\perp}_X N - \nabla^{\perp}_{[X,Y]} N.$$

The transversal connection $\nabla^{\perp}$ is called *flat* if $R^{\perp}$ vanishes identically [6].

As $\nabla^{\perp}_X N = \tau(X)N$, we show [6] that the transversal connection of M is flat if and only if the 1-form τ is closed, *i.e.*, $d\tau = 0$, on any $\mathcal{U} \subset M$.

In the sequel, we shall denote σ and ρ the 1-forms defined by

$$\sigma(X) = B(X, U) = C(X, V), \qquad \rho(X) = B(X, V).$$

Theorem 4.3. *Let $\bar{M}$ be an indefinite trans-Sasakian manifold of quasi-constant curvature with a lightlike hypersurface M. If one of the following three conditions*
(1) *F is parallel with respect to the induced connection ∇,*
(2) *U is parallel with respect to the induced connection ∇, and*
(3) *V is parallel with respect to the induced connection ∇*
is satisfied, then $\bar{M}$ is a flat manifold with indefinite cosymplectic structure and the transversal connection of M is flat. In case (1), M is also a flat manifold.

Proof. (1) If F is parallel, then, from (3.10) and the fact that $\beta = 0$, we get

$$u(Y)A_N X - B(X,Y)U + \alpha\{g(X,Y)\zeta - \theta(Y)X\} = 0. \tag{4.7}$$

Taking $X = U$ and $Y = V$ to (4.7), we have $\sigma(V)U = \alpha\zeta$. Taking the scalar product with ζ to this result, we get $\alpha = 0$. Therefore, $\lambda = 0$ and $\bar{M}$ is a flat manifold with indefinite cosymplectic structure. Replacing Y by U to (4.7), we obtain

$$A_N X = \sigma(X)U. \tag{4.8}$$

Taking the scalar product with V to (4.7), we get $B(X,Y) = u(Y)\sigma(X)$, *i.e.*,

$$g(A^*_\xi X, Y) = g(\sigma(X)V, Y).$$

As $A^*_\xi X$ and V belong to $S(TM)$, and $S(TM)$ is non-degenerate, we get

$$A^*_\xi X = \sigma(X)V. \tag{4.9}$$

Substituting (4.8) and (4.9) into (4.1) with $\lambda = \mu = 0$, we get

$$R(X,Y)Z = \{\sigma(Y)\sigma(X) - \sigma(X)\sigma(Y)\}u(Z)U = 0,$$

for all X, Y, $Z \in \Gamma(TM)$. Therefore $R = 0$ and M is also flat.

Substituting (4.8) into (3.8) and using the fact that $FU = 0$, we get

$$\nabla_X U = \tau(X)U.$$

Substituting this into $\nabla_X\nabla_Y U - \nabla_Y\nabla_X U - \nabla_{[X,Y]}U = 0$, we get $d\tau = 0$. Thus the transversal connection of M is flat.

(2) If U is parallel, then, from (3.6), (3.8) and the fact that $\beta = 0$, we have

$$J(A_N X) - u(A_N X)N + \tau(X)U - \alpha\eta(X)\zeta = 0.$$

Taking the scalar product with ζ to this equation, we get $\alpha = 0$. Thus $\lambda = 0$ and $\bar{M}$ is a flat manifold with indefinite cosymplectic structure. Taking the scalar product with V to the last equation, we have $\tau = 0$. As $\tau = 0$, we obtain $d\tau = 0$ and the transversal connection of M is flat.

(3) If V is parallel, then, from (3.6), (3.9) and the fact that $\beta = 0$, we have

$$J(A^*_\xi X) - u(A^*_\xi X)N - \tau(X)V = 0. \tag{4.10}$$

Taking the scalar product with U to (4.10), we have $\tau = 0$. Thus $d\tau = 0$ and the transversal connection of M is flat. Applying J to (4.10) and using (3.13), we have

$$A^*_\xi X = -\alpha u(X)\zeta + \rho(X)U.$$

Taking the scalar product with U to this equation, we obtain $B(X,U) = 0$ for all $X \in \Gamma(TM)$. Replacing X by ζ to this result and using $(3.13)_1$, we get

$$\alpha = \alpha u(U) = -B(U,\zeta) = 0.$$

Thus $\lambda = 0$ and $\bar{M}$ is a flat manifold with indefinite cosymplectic structure. □

5 Two types lightlike hypersurfaces

Definition 5.1. The canonical structure vector field U is called *principal* [7], with respect to the shape operator A^*_ξ, if there exists a smooth function f such that

$$A^*_\xi U = fU. \tag{5.1}$$

A lightlike hypersurface M of an indefinite almost complex manifold $\bar{M}$ is said to be a *Hopf lightlike hypersurface* [7] if it admits a principal canonical structure vector field U, with respect to the shape operator A^*_ξ.

Taking the scalar product with X to (5.1) and using (3.7), we get

$$B(X,U) = fv(X), \qquad C(X,V) = fv(X), \qquad \sigma(X) = fv(X). \tag{5.2}$$

Theorem 5.1. *Let $\bar{M}$ be an indefinite trans-Sasakian manifold of quasi-constant curvature with a Hopf lightlike hypersurface M. Then $\bar{M}$ is a flat manifold with indefinite cosymplectic structure.*

Proof. Replacing X by U to $(3.13)_1$, we have $B(U,\zeta) = -\alpha$. Also, replacing X by ζ to $(5.2)_1$, we have $B(U,\zeta) = fv(\zeta) = -f\theta(JN) = 0$. Therefore $\alpha = 0$. By Theorem 4.1, $\lambda = 0$ and $\bar{M}$ is a flat manifold with indefinite cosymplectic structure. □

Theorem 5.2. *Let M be a Hoph lightlike hypersurfaces of an indefinite trans-Sasakian manifold $\bar{M}$ of quasi-constant curvature. If F is parallel with respect to the induced connection ∇ of M, then $f = 0$ and $S(TM)$ is totally geodesic in M.*

Proof. As M is Hopf lightlike hypersurface, from (4.8) and $(5.2)_3$, we have

$$A_N X = fv(X)U. \tag{5.3}$$

Taking the scalar product with Y to (4.9) and using $(5.2)_3$, we have

$$B(X,Y) = fv(X)u(Y).$$

Taking $X = V,\ Y = U$ and $X = U,\ Y = V$ to this equation by turns, we obtain

$$B(V,U) = f, \qquad B(U,V) = 0.$$

Thus $f = 0$. From (5.3), we get $A_N = 0$ and $S(TM)$ is totally geodesic in M. □

Theorem 5.3. *Let M be a Hopf lightlike hypersurface of an indefinite trans-Sasakian manifold $\bar{M}$ of quasi-constant curvature. If U is parallel with respect to the induced connection ∇ of M, then $S(TM)$ is an integrable distribution.*

Proof. As M is Hopf lightlike hypersurface, from (4.8) and $(5.2)_3$, we obtain

$$A_N X = fv(X)U.$$

Taking the scalar product with Y to this equation, we see that

$$g(A_N X,\, Y) = fv(X)v(Y).$$

It follow that A_N is self-adjoint linear operator with respect to g. Consequently, C is symmetric on $S(TM)$ due to (2.6). By using (2.3) we obtain

$$\eta([X,Y]) = C(X,Y) - C(Y,X) = 0, \quad \forall\, X,\, Y \in \Gamma(S(TM)),$$

which implies that $[X,Y] \in \Gamma(S(TM))$ for any $X,\, Y \in \Gamma(S(TM))$. Therefore, $S(TM)$ is an integrable distribution. □

Definition 5.2. The structure tensor field F on M is said to be *recurrent* if there exists a 1-form ϑ on M such that

$$(\nabla_X F)Y = \vartheta(X)FY, \quad \forall\, X,\, Y \in \Gamma(TM).$$

Theorem 5.4. *Let M be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\bar{M}$ of quasi-constant curvature. If F is recurrent, then $\bar{M}$ is a flat manifold with indefinite cosymplectic structure, M is flat and the transversal connection is flat.*

Proof. As M is recurrent, from (3.10) and the fact that $\beta = 0$, we get

$$\vartheta(X)FY = u(Y)A_{_N}X - B(X,Y)U + \alpha\{g(X,Y)\zeta - \theta(Y)X\}.$$

Replacing Y by ξ to this, we get $\vartheta(X)V = 0$ for all $X \in \Gamma(TM)$. Taking the scalar product with U to this result, we obtain $\vartheta = 0$. Therefore, F is parallel with respect to ∇. By (1) of Theorem 4.3, we have our assertion. □

Definition 5.3. The structure tensor field F of M is said to be *Lie recurrent* [7] if there exists a 1-form ω on M such that

$$(\mathcal{L}_{_X}F)Y = \omega(X)FY, \quad \forall\, X,\, Y \in \Gamma(TM), \tag{5.4}$$

where $\mathcal{L}_{_X}$ denotes the Lie derivative on M with respect to X, that is,

$$\begin{aligned}(\mathcal{L}_{_X}F)Y &= [X, FY] - F[X,Y] \\ &= (\nabla_X F)Y - \nabla_{FY}X + F\nabla_Y X.\end{aligned} \tag{5.5}$$

The structure tensor field F is called *Lie parallel* [7] if $\mathcal{L}_{_X}F = 0$.

A lightlike hypersurface M of an indefinite almost complex manifold $\bar{M}$ is called *Lie recurrent* [7] if it admits a Lie recurrent structure tensor field F.

Theorem 5.5. *Let M be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\bar{M}$ of quasi-constant curvature. If F is Lie recurrent, then it is Lie parallel, and $\bar{M}$ is a flat manifold with indefinite cosymplectic structure.*

Proof. As F is Lie recurrent, from (3.10), (5.4) and (5.5) we get

$$\begin{aligned}\omega(X)FY \;=\; & u(Y)A_{_N}X - B(X,Y)U - \nabla_{FY}X + F\nabla_Y X \\ & + \alpha\{g(X,Y)\zeta - \theta(Y)X\}.\end{aligned} \tag{5.6}$$

Replacing Y by ξ to (5.6) and using (2.4), (3.5) and $F\xi = -V$, we have

$$-\,\omega(X)V = \nabla_V X + F\nabla_\xi X. \tag{5.7}$$

Taking the scalar product with V and ζ to this equation by turns, we get

$$u(\nabla_V X) = g(\nabla_V X, V) = 0, \qquad \theta(\nabla_V X) = 0. \tag{5.8}$$

On the other hand, taking $Y = V$ to (5.6) and using (3.5), we have

$$\omega(X)\xi = -B(X,V)U - \nabla_\xi X + F\nabla_V X + \alpha u(X)\zeta,$$

due to $FV = \xi$. Applying F to this equation and using $(3.15)_1$, (5.8) and the facts that $FU = 0$ and $F\zeta = 0$, we have

$$\omega(X)V = \nabla_V X + F\nabla_\xi X.$$

Comparing this with (5.7), we obtain $\omega = 0$. Therefore, F is Lie parallel.

Replacing X by U to (5.6) and using (3.4), (3.5), (3.8) and $(3.13)_2$, we get

$$u(Y)A_N U - F(A_N FY) - \tau(FY)U - A_N Y + \alpha\{v(Y)\zeta - \theta(Y)U\} = 0.$$

Taking $Y = V$ to this and using (3.5) and the fact that $FV = \xi$, we get

$$F(A_N \xi) + \tau(\xi)U + A_N V - \alpha\zeta = 0.$$

Taking the scalar product with ζ to this equation, we see that $\alpha = 0$. Thus $\lambda = 0$ and $\bar{M}$ is a flat manifold with indefinite cosymplectic structure. □

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