Ricci solitons and concurrent vector fields

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Abstract. A Ricci soliton (M, g, v, λ) on a Riemannian manifold (M, g) is said to have concurrent potential field if its potential field v is a concurrent vector field. In the first part of this paper we classify Ricci solitons with concurrent potential fields. In the second part we derive a necessary and sufficient condition for a submanifold to be a Ricci soliton in a Riemannian manifold equipped with a concurrent vector field. In the last part, we completely classify shrinking Ricci solitons with $\lambda = 1$ on Euclidean hypersurfaces. Several applications of our results are also presented.

1 Introduction

Through this article we only consider connected manifolds. A smooth vector field ξ on a Riemannian manifold (M, g) is said to define a *Ricci soliton* if it satisfies

$$(1.1) \qquad \frac{1}{2}\mathcal{L}_\xi g + Ric = \lambda g,$$

where $\mathcal{L}_\xi g$ is the Lie-derivative of the metric tensor g with respect to ξ, Ric is the Ricci tensor of (M, g) and λ is a constant. Ricci solitons model the formation of singularities in the Ricci flow and they correspond to self-similar solutions. We shall denote a Ricci soliton by (M, g, ξ, λ). We call the vector field ξ the *potential field* of the Ricci soliton. A Ricci soliton (M, g, ξ, λ) is called *shrinking, steady* or *expanding* according to $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. A trivial Ricci soliton is one for which ξ is zero or Killing, in which case the metric is Einstein.

A Ricci soliton (M, g, ξ, λ) is called *gradient* if its potential field ξ is the gradient of some smooth function f on M. We shall denote a gradient Ricci soliton by (M, g, f, λ) and call the smooth function f the *potential function*. A gradient Ricci soliton (M, g, f, λ) is called *trivial* if its potential function f is a constant. It follows from (1.1) that trivial gradient Ricci solitons are trivial Ricci solitons since $\xi = \nabla f$. It was proved in [19] that if (M, g, ξ, λ) is a compact Ricci soliton, the potential field

ξ is a gradient of some smooth function f up to the addition of a Killing field. Thus compact Ricci solitons are gradient Ricci solitons.

During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Grigory Perelman applied Ricci solitons to solve the long standing Poincaré conjecture posed in 1904. G. Perelman observed in [19] that the Ricci solitons on compact simply connected Riemannian manifolds are gradient Ricci solitons as solutions of Ricci flow.

There are two aspects of the study of Ricci solitons, one looking at the influence on the topology by the Ricci soliton structure of the Riemannian manifold (see e.g. [11, 15]) and the other looking at its influence on its geometry (see e.g. [7, 12, 13]). In this paper we are interested in the geometry of Ricci solitons arisen from concurrent vector fields on Riemannian manifolds.

If the holonomy group of M leaves a point invariant, then it was proved in [24] that there exists a vector field v on M which satisfies

$$(1.2) \qquad\qquad \nabla_Z v = Z$$

for any vector Z tangent to M, where ∇ denotes the Levi-Civita connection of M. Such a vector field is called a *concurrent vector field*. Riemannian manifolds equipped with concurrent vector fields have been studied by many mathematician (see, e.g. [3, 8, 17, 18, 20, 21, 24, 25]). Concurrent vector fields have also been studied in Finsler geometry since the beginning of 1950s (see, e.g. [16, 23]).

In the first part of this paper we completely classify Ricci solitons with concurrent potential fields. In the second part we derive a necessary and sufficient condition for a submanifold to be a Ricci soliton in a Riemannian manifold equipped with a concurrent vector field. In the last part, we classify shrinking Ricci solitons with $\lambda = 1$ on Euclidean hypersurfaces. Several applications of our results are also presented.

2 Preliminaries

2.1 Basic formulas and definitions for submanifolds

For general references on Riemannian submanifolds, we refer to [4, 5, 6].

Let $(N^m, \tilde{g})$ denote an m-dimensional Riemannian manifold and $\phi : M^n \to N^m$ an isometric immersion from an n-dimensional Riemannian manifold (M^n, g) into $(N^m, \tilde{g})$. Denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections on (M^n, g) and $(N^m, \tilde{g})$, respectively.

For vector fields X, Y tangent to M^n and η normal to M^n, the formula of Gauss and the formula of Weingarten are given respectively by

$$(2.1) \qquad\qquad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$(2.2) \qquad\qquad \tilde{\nabla}_X \eta = -A_\eta X + D_X \eta,$$

where $\nabla_X Y$ and $h(X, Y)$ are the tangential and the normal components of $\tilde{\nabla}_X Y$. Similarly, $-A_\eta X$ and $D_X \eta$ are the tangential and normal components of $\tilde{\nabla}_X \eta$. These two formulas define the second fundamental form h, the shape operator A, and the normal connection D of M^n in the ambient space N^m.

For a normal vector $\eta \in T_p^\perp M$, $p \in M$, A_η is a self-adjoint endomorphism. The shape operator and the second fundamental form are related by

$$(2.3) \qquad \tilde{g}(h(X,Y),\eta) = g(A_\eta X, Y).$$

The *mean curvature vector* H of M^n in N^m is defined by

$$(2.4) \qquad H = \left(\frac{1}{n}\right) \operatorname{trace} h.$$

The equations of Gauss and Codazzi are given respectively by

$$(2.5) \qquad g(R(X,Y)Z,W) = \tilde{g}(\tilde{R}(X,Y)Z,W) + \tilde{g}(h(X,W),h(Y,Z))$$
$$- \tilde{g}(h(X,Z),h(Y,W)),$$

$$(2.6) \qquad (\tilde{R}(X,Y)Z)^\perp = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z),$$

for vectors X, Y, Z, W tangent to M and ζ, η normal to M, where $(\tilde{R}(X,Y)Z)^\perp$ is the normal component of $\tilde{R}(X,Y)Z$ and $\bar{\nabla} h$ is defined by

$$(2.7) \qquad (\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

2.2 Examples of Riemannian manifolds endowed with concurrent fields

The best known example of Riemannian manifolds endowed with concurrent vector fields is the Euclidean space with the concurrent vector field given by its position vector field $\mathbf{x}$ (with respect to the origin).

For more general examples of Riemannian manifolds with concurrent vector fields, let us consider warped product manifolds of the form: $I \times_s F$, where I is an open interval of the real line $\mathbf{R}$ with s as its arclength and F is an arbitrary Riemannian manifold. The metric tensor g of $I \times_s F$ is given by $g = ds^2 + s^2 g_F$, where g_F is the metric tensor of the second factor F. Let us put $v = s\frac{\partial}{\partial s}$. It follows easily from Proposition 4.1 of [5, page 79] that the vector field v satisfies (1.2) for any vector Z tangent to $I \times_s F$. Therefore $I \times_s F$ admits a concurrent vector field: $v = s\partial/\partial s$.

3 Ricci solitons with concurrent potential fields

The following theorem classifies Ricci solitons on Riemannian manifolds endowed with a concurrent potential field.

Theorem 3.1. *A Ricci soliton (M^n, g, v, λ) on a Riemannian n-manifold (M^n, g) has concurrent potential field v if and only if the following two conditions hold:*

(a) *The Ricci soliton is a shrinking Ricci soliton with $\lambda = 1$.*

(b) *M^n is an open part of a warped product manifold $I \times_s F$, where I is an open interval with arclength s and F is an Einstein $(n-1)$-manifold whose Ricci tensor satisfies $Ric_F = (n-2)g_F$, g_F is the metric tensor of F.*

Proof. Assume that (M^n, g, v, λ) is a Ricci soliton on a Riemannian n-manifold equipped with a concurrent potential field v. Then we have

(3.1) $$\nabla_X v = X, \ \ \forall X \in TM^n.$$

It follows from (3.1) that the concurrent vector field v vanishes on a measure zero subset of M^n at most. By applying (3.1) and the definition of sectional curvature, it is easy to verify that the sectional curvature of M^n satisfies

(3.2) $$K(X, v) = 0.$$

for each unit vector X orthogonal to v. Hence the Ricci tensor of M^n satisfies

(3.3) $$Ric(v, v) = 0.$$

Let us put $v = \mu e_1$, where e_1 is a unit vector field tangent to M^n. Also let us extend e_1 to a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n. Denote by $\{\omega^1, \ldots, \omega^n\}$ the dual frame of 1-forms of $\{e_1, \ldots, e_n\}$.

Define the connection forms ω_i^j $(i, j = 1, \ldots, n)$ on M^n by

(3.4) $$\nabla_X e_i = \sum_{j=1}^{n} \omega_i^j(X) e_j, \ \ i = 1, \ldots, n.$$

From (3.1) with $X = e_1$, (3.4) and the continuity we find

(3.5) $$e_1 \mu = 1,$$
(3.6) $$\nabla_{e_1} e_1 = 0.$$

Put $\mathcal{D}_1 = \text{Span}\{e_1\}$ and $\mathcal{D}_2 = \text{Span}\{e_2, \ldots, e_n\}$. It follows from (3.6) that $\mathcal{D}_1$ is a totally geodesic distribution so that the leaves of $\mathcal{D}_1$ are geodesics of M^n. Also, we may derive from (3.1) with $X = e_i$ $(i = 2, \ldots, n)$ that

(3.7) $$e_2 \mu = \cdots = e_n \mu = 0,$$
(3.8) $$\mu \omega_i^1(e_i) = -1,$$
(3.9) $$\omega_j^1(e_i) = 0, \ \ i \neq j.$$

From Cartan's structure equations, we have

(3.10) $$d\omega^i = -\sum_{j=1}^{n} \omega_j^i \wedge \omega^j, \ \ i = 1, \ldots, n.$$

Thus, after applying (3.9) and (3.10), we obtain $d\omega^1 = 0$. Hence we have locally $\omega^1 = ds$ for some function s on M^n. It follows from (3.9) that

(3.11) $$g([e_i, e_j], e_1) = \omega_j^1(e_i) - \omega_i^1(e_j) = 0, \ \ 2 \leq i \neq j \leq n.$$

Therefore $\mathcal{D}_2$ is an integrable distribution. Moreover, from (3.8) we know that the second fundamental form $\hat{h}$ of each leaf L of $\mathcal{D}_2$ in M^n satisfies

(3.12) $$\hat{h}(e_i, e_j) = -\frac{\delta_{ij}}{\mu} e_1, \ \ 2 \leq i, j \leq n,$$

which shows that the mean curvature of each leaf L is given by $-\mu^{-1}$.

Equation (3.12) implies that each leaf of $\mathcal{D}_2$ is a totally umbilical hypersurface of M^n whose mean curvature vector is $\hat{H} = -e_1/\mu$. Furthermore, by applying (3.7) we conclude that $\mathcal{D}_2$ is a spherical distribution, i.e., the mean curvature vector of each totally umbilical leaf is parallel in the normal bundle. Consequently, a result of S. Hiepko and Ponge-Reckziegel (see, e.g., [22] or [5, page 90]) implies that M^n is locally a warped product manifold $I \times_{f(s)} F$ whose warped metric is given by

$$(3.13) \qquad\qquad g = ds^2 + f^2(s) g_F$$

such that $e_1 = \partial/\partial s$.

It follows from (3.13) that the sectional curvature of M^n satisfies

$$(3.14) \qquad\qquad K(X, v) = -\frac{f''(s)}{f(s)}$$

for each unit vector X orthogonal to v. Now, after comparing (3.2) with (3.14) we obtain $f''(s) = 0$. Therefore we obtain $f(s) = as + b$ for some constants a and b.

If $a = 0$ holds, then the warped product manifold $I \times_{f(s)} F$ is a Riemannian product, which implies that every leaf of $\mathcal{D}_2$ is totally geodesic in M^n. Hence μ must be zero, which contradicts to (3.12). Therefore we must have $a \neq 0$. Hence, after applying a suitable translation and dilation in s we get $f(s) = s$. Consequently, M^n is locally a warped product manifold $I \times_s F$.

On the other hand, it follows from the definition of Lie-derivative and condition (3.1) that the Lie-derivative satisfies

$$(3.15) \qquad (\mathcal{L}_v g)(X, Y) = g(\nabla_X v, Y) + g(\nabla_Y v, X) = 2g(X, Y)$$

for any X, Y tangent to M^n. Combining (3.15) with (1.1) gives

$$(3.16) \qquad\qquad Ric(X, Y) = (\lambda - 1)g(X, Y),$$

which shows that M^n is an Einstein $(n-1)$-manifold. After comparing (3.3) and (3.16) we conclude that M^n is a Ricci flat space. Hence we get $\lambda = 1$. Consequently, the Ricci soliton (M^n, g, v, λ) is a shrinking one.

Since M^n is a Ricci flat space, it follows from Corollary 4.1(3) of [5, page 82] or formula (9.109) of [2, page 267] that the second factor F of the warped product manifold $I \times_s F$ is an Einstein manifold satisfying $Ric_F = (n-2)g_F$.

The converse can be verified by direct computation. $\qquad\qquad\square$

Theorem 3.1 implies the following

Corollary 3.2. *There do not exist steady or expanding Ricci solitons with concurrent potential fields.*

Corollary 3.3. *Every Ricci soliton (M^n, g, v, λ) with concurrent potential field v is gradient.*

Proof. Since v is concurrent, we have $\frac{1}{2}\mathcal{L}_v g = g$. So, it follows from the Ricci soliton equation and $\lambda = 1$ that $Ric = 0$. But $f := \frac{1}{2}g(v, v)$ satisfies $Hess(f) = g$. Thus we get $Ric + Hess(f) = \lambda g$, which implies that the Ricci soliton is gradient. $\qquad\square$

Remark 3.1. If M^n is a complete Riemannian manifold which admits a concurrent vector field v, then M^n is isometric to the Euclidean n-space. Moreover, we have $v = r\nabla r$, where r is the distance function from the origin (see [3, Theorem 4]).

4   Riemannian submanifolds as Ricci solitons

From now on, we make the following

Assumption. $(N^m, \tilde{g})$ *is a Riemannian m-manifold endowed with a concurrent vector field v.*

For an isometric immersion $\phi : M^n \to N^m$ of a Riemannian n-manifold (M^n, g) into $(N^m, \tilde{g})$, we denote by v^T and $v^\perp$ the tangential and normal components of v on M^n, respectively. As before, we denote by h, A and D the second fundamental form, the shape operator and the normal connection of the submanifold M^n in N^m, respectively.

Theorem 4.1. *A submanifold M^n in N^m admits a Ricci soliton (M^n, g, v^T, λ) if and only if the Ricci tensor of (M^n, g) satisfies*

$$(4.1) \qquad Ric(X, Y) = (\lambda - 1)g(X, Y) - \tilde{g}(h(X, Y), v^\perp)$$

for any X, Y tangent to M^n.

Proof. Let $\phi : M^n \to N^m$ denote the isometric immersion. We have

$$(4.2) \qquad v = v^T + v^\perp.$$

Since v is a concurrent vector field on the ambient space N^m, it follows from (1.2), (4.2) and formulas of Gauss and Weingarten that

$$(4.3) \qquad X = \tilde{\nabla}_X v^T + \tilde{\nabla}_X v^\perp \nabla_X v^T + h(X, v^T) - A_{v^\perp} X + D_X v^\perp$$

for any X tangent to M^n. By comparing the tangential and normal components from (4.3) we obtain

$$(4.4) \qquad \nabla_X v^T = A_{v^\perp} X + X,$$

$$(4.5) \qquad h(X, v^T) = -D_X v^\perp.$$

From the definition of Lie derivative and (4.4) we obtain

$$(4.6) \qquad \begin{aligned} (\mathcal{L}_{v^T} g)(X, Y) &= g(\nabla_X v^T, Y) + g(\nabla_Y v^T, X) \\ &= 2g(X, Y) + 2g(A_{v^\perp} X, Y) \\ &= 2g(X, Y) + 2\tilde{g}(h(X, Y), v^\perp) \end{aligned}$$

for X, Y tangent to M^n. Consequently, by applying (1.1) and (4.5), we conclude that (M^n, g, v^T, λ) is a Ricci soliton if and only if we have

$$(4.7) \qquad Ric(X, Y) + g(X, Y) + \tilde{g}(h(X, Y), v^\perp) = \lambda g(X, Y),$$

which is nothing but (4.1). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Recall that the position vector field $\mathbf{x}$ of a Euclidean m-space $\mathbb{E}^m$ is a concurrent vector field. The simplest examples of Ricci solitons (M^n, g, v^T, λ) on submanifolds in a Riemannian manifold with concurrent field are the following ones.

Example 4.1. Let $\gamma(s)$ be a unit speed curve lying in the unit hypersphere $S_o^{m-n}(1)$ of $\mathbb{E}^{m-n+1}$ centered at the origin o. Consider the Riemannian submanifold (M^n, g) of $\mathbb{E}^m$ defined by

$$\phi(s, x_2, \ldots, x_n) = (\gamma(s)x_2, x_2, x_3, \ldots, x_n).$$

Then M^n is a flat space and $(M^n, g, \mathbf{x}^T, \lambda)$ is a shrinking Ricci soliton satisfying (4.1) with $\lambda = 1$. Moreover, $\mathbf{x}^T = \mathbf{x}$ and M^n is generated by lines in $\mathbb{E}^m$ through the origin.

The following provides more examples of Ricci solitons on submanifolds.

Example 4.2. Let k be a natural number such that $2 \leq k \leq n - 1$ and $r = \sqrt{k-1}$. Consider the spherical hypercylinder $\phi : S^k(r) \times \mathbb{E}^{n-k} \to \mathbb{E}^{n+1}$ defined by $\{(\mathbf{y}, x_{k+2}, \ldots, x_{n+1}) \in \mathbb{E}^{n+1} : \mathbf{y} \in \mathbb{E}^{k+1} \ and \ \langle \mathbf{y}, \mathbf{y} \rangle = r^2\}$. It is straightforward to verify that the spherical hypercylinder $S^k(\sqrt{k-1}) \times \mathbb{E}^{n-k}$ in $\mathbb{E}^{n+1}$ satisfies (4.1) with $\lambda = 1$. Hence $(S^k(\sqrt{k-1}) \times \mathbb{E}^{n-k}, g, \mathbf{x}^T, \lambda)$ is a shrinking Ricci soliton with $\lambda = 1$.

Example 4.3. Let $n_1, \ldots, n_p$ be integers ≥ 2 and $r_1, \ldots, r_p$ be positive numbers satisfying $(n_1 - 1)/r_1^2 = \cdots = (n_p - 1)/r_p^2$. Put $n = n_1 + \cdots + n_p$.

Let (M^n, g) denote the Riemannian product $S^{n_1}(r_1) \times \ldots \times S^{n_p}(r_p)$ of p spheres $S^{n_1}(r_1), \ldots, S^{n_p}(r_p)$ of radii $r_1, \ldots, r_p$, respectively, which is isometrically imbedded in $\mathbb{E}^{n+p}$ in the standard way. It is direct to verify that $(M^n, g, \mathbf{x}^T, \lambda)$ is a shrinking Ricci soliton with λ equal to $(n_1 - 1)/r_1^2$.

5   Some applications of Theorem 4.1

A Riemannian submanifold M^n is called η-umbilical (with respect to a normal vector field η) if its shape operated satisfies $A_\eta = \varphi I$, where φ is a function on M^n and I is the identity map.

The following two results are immediate consequences of Theorem 4.1.

Theorem 5.1. A Ricci soliton (M^n, g, v^T, λ) on a submanifold M^n in N^m is trivial if and only if M^n is $v^\perp$-umbilical.

Corollary 5.2. Every Ricci soliton (M^n, g, v^T, λ) on a totally umbilical submanifold M^n of N^m is a trivial Ricci soliton.

Following [5], the scalar curvature τ of (M^n, g) is defined to be

$$(5.1) \qquad\qquad \tau = \sum_{1 \leq i < j \leq n} K(e_i, e_j),$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal frame of M^n.

Another easy application of Theorem 4.1 is the following.

Proposition 5.3. If (M^n, g, v^T, λ) is a Ricci soliton on a minimal submanifold M^n in N^m, then M^n has constant scalar curvature given by $n(\lambda - 1)/2$.

Proof. Assume that (M^n, g, v^T, λ) is a Ricci soliton on a submanifold M^n in N^m. Then Theorem 4.1 implies that the Ricci tensor of M^n satisfies

$$(5.2) \qquad Ric(X, Y) = (\lambda - 1)g(X, Y) - \tilde{g}(h(X, Y), v^\perp)$$

for X, Y tangent to M^n. If M^n is minimal in N^m, then the mean curvature vector vanishes identically. In particular, this implies that $\tilde{g}(H, v^\perp) = 0$. Hence, we obtain (5.2) that

$$\sum_{i=1}^{n} Ric(e_i, e_i) = n(\lambda - 1).$$

Therefore M^n has constant scalar curvature $n(\lambda - 1)/2$. $\qquad\square$

Let ∇f denote the gradient of a function f on M^n. By applying (4.4) and (4.5) we have the following.

Lemma 5.4. *Let M^n be a submanifold of N^m. Then we have*

$$(5.3) \qquad\qquad \nabla\psi = -A_{v^\perp} v^T,$$
$$(5.4) \qquad\qquad v^T = \nabla\varphi,$$

where $\psi = \frac{1}{2}\tilde{g}(v^\perp, v^\perp)$ and $\varphi = \frac{1}{2}\tilde{g}(v, v)$.

Proof. Let M^n be a submanifold of N^m. Then we find from (4.5) that

$$X\psi = \tilde{g}(\tilde{\nabla}_X v^\perp, v^\perp) = \tilde{g}(D_X v^\perp, v^\perp) = -g(A_{v^\perp} v^T, X),$$

which implies (5.3). Equation (5.4) follows from

$$X\varphi = \tilde{g}(\tilde{\nabla}_X v, v) = \tilde{g}(X, v) = g(X, v^T)$$

for X tangent to M^n. $\qquad\square$

The next result follows immediately from (5.4) of Lemma 5.4.

Proposition 5.5. *Every Ricci soliton (M^n, g, v^T, λ) on a submanifold M^n of N^m is a gradient Ricci soliton with potential function $\varphi = \frac{1}{2}\tilde{g}(v, v)$.*

This proposition shows that the gradient Ricci soliton $(M^n, g, \varphi, \lambda)$ on M^n is trivial if and only if $\tilde{g}(v, v)$ is constant on M^n.

Corollary 5.6. *A gradient Ricci soliton $(M^n, g, \varphi, \lambda)$ on a submanifold M^n of N^m is trivial if and only if the concurrent vector field v on N^m is normal to M^n.*

Proof. Let M^n be a submanifold of N^m. Suppose that $(M^n, g, \varphi, \lambda)$ is a trivial gradient Ricci soliton. Then $\tilde{g}(v, v)$ is constant on M^n. Thus by taking the derivative of $\tilde{g}(v, v)$ with respect to a tangent vector X, we find $0 = X\tilde{g}(v, v) = 2g(X, v)$ according to (1.2). Because this is true for any arbitrary tangent vector of M^n, the concurrent vector field v must be normal to M^n.

Conversely, if v is normal to M^n, then we have $X\tilde{g}(v, v) = 2g(X, v) = 0$. Thus $\tilde{g}(v, v)$ is constant on M^n. Consequently, the gradient Ricci soliton is a trivial one according to Corollary 5.6. $\qquad\square$

The last result of this section is the following.

Proposition 5.7. *If $(M^n, g, \mathbf{x}^T, \lambda)$ is a Ricci soliton on a hypersurface of M^n of $\mathbb{E}^{n+1}$, then M^n has at most two distinct principal curvatures given by*

$$(5.5) \qquad \kappa_1, \kappa_2 = \frac{n\alpha + \rho \pm \sqrt{(n\alpha + \rho)^2 + 4 - 4\lambda}}{2},$$

where α is the mean curvature and ρ is the support function, i.e., $H = \alpha N$ and $\rho = \langle N, \mathbf{x} \rangle$ with N being a unit normal vector field.

Proof. Assume that $(M^n, g, \mathbf{x}^T, \lambda)$ is a Ricci soliton on a hypersurface of M^n of $\mathbb{E}^{n+1}$, where $\mathbf{x}^T$ denotes the tangential component of the position vector field $\mathbf{x}$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal frame on M^n such that $e_1, \ldots, e_n$ are eigenvectors of the shape operator A_N. Then we have

$$(5.6) \qquad A_N e_i = \kappa_i e_i, \quad i = 1, \ldots, n.$$

From equation (2.5) of Gauss we obtain

$$(5.7) \qquad Ric(X, Y) = n g_0(h(X, Y), H) - \sum_{i=1}^{n} g_0(h(X, e_i), h(Y, e_i)),$$

where g_0 denotes the Euclidean metric of $\mathbb{E}^{n+1}$. It follows from (5.6), (5.7) and Theorem 4.1 that $(M^n, g, \mathbf{x}^T, \lambda)$ is a Ricci soliton if and only if we have

$$(5.8) \qquad (n\alpha - \kappa_j)\kappa_i \delta_{ij} = (\lambda - 1)\delta_{ij} - \rho \kappa_i \delta_{ij},$$

where δ_{ij} is the Kronecker delta. Equation (5.8) is equivalent to

$$\kappa_i^2 - (n\alpha + \rho)\kappa_i + \lambda - 1 = 0, \quad i = 1, \ldots, 0,$$

which implies the proposition $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

6 Shrinking Ricci solitons on Euclidean hypersurfaces

If the ambient space is complete, then it is isometric to the Euclidean space and, up to isometries, every concurrent vector field is the position vector field (cf. Remark 3.1). Hence, the purpose of this section is to prove the following classification theorem.

Theorem 6.1. *Let $(M^n, g, \mathbf{x}^T, \lambda)$ be a shrinking Ricci soliton on a hypersurface of M^n of $\mathbb{E}^{n+1}$ with $\lambda = 1$. Then M^n is an open portion of one of the following hypersurfaces of $\mathbb{E}^{n+1}$:*

(1) *A totally umbilical hypersurface;*

(2) *A flat hypersurface generated by lines through the origin o of $\mathbb{E}^{n+1}$;*

(3) *A spherical hypercylinder $S^k(\sqrt{k-1}) \times \mathbb{E}^{n-k}$, $2 \le k \le n-1$.*

Proof. Assume that $(M^n, g, \mathbf{x}^T, \lambda)$ is a shrinking Ricci soliton on a hypersurface of M^n of $\mathbb{E}^{n+1}$. Then it follows from Proposition 5.7 that M^n has at most two distinct principal curvatures given by

$$(6.1) \qquad \frac{n\alpha + \rho + \sqrt{(n\alpha + \rho)^2 + 4 - 4\lambda}}{2}, \quad \frac{n\alpha + \rho - \sqrt{(n\alpha + \rho)^2 + 4 - 4\lambda}}{2}.$$

If M^n has only one principal curvature, then M^n is totally umbilical.

Now, let us assume that M^n has two distinct principal curvatures and $\lambda = 1$. Then (6.1) implies that the two distinct principal curvatures are given respectively by 0 and $n\alpha + \rho$. Let κ denote the nonzero principal curvature, i.e., $\kappa = n\alpha + \rho$. Let us assume that the multiplicities of κ and 0 are k and $n - k$, respectively, for some k with $1 \le k < n$. Then we have $n\alpha = k\kappa$. Hence the mean curvature α and the support function ρ are related by

$$(6.2) \qquad n(1 - k)\alpha = k\rho.$$

Case (a): $k = 1$. In this case, (6.2) gives $\rho = \tilde{g}(\mathbf{x}, N) = 0$. Thus the concurrent vector field $\mathbf{x}$ is tangent to M^n. So, it follows from (1.2) that $\tilde{\nabla}_X \mathbf{x} = X$. Hence integral curves of $\mathbf{x}$ are part of lines through the origin in $\mathbb{E}^{n+1}$. Therefore we obtain case (2) of the theorem.

Case (b): $2 \le k \le n - 1$. Without loss of generality, we may assume that

$$(6.3) \qquad A_N = \begin{pmatrix} \kappa I_k & 0 \\ 0 & 0_{n-k} \end{pmatrix}$$

with respect to an orthonormal tangent frame $\{e_1, \ldots, e_n\}$ of M^n, where I_k is an $k \times k$ identity matrix and 0_{n-k} is an $(n - k) \times (n - k)$ zero matrix. We put

$$(6.4) \qquad \mathcal{D}_1 = \mathrm{Span}\{e_1, \ldots, e_k\}, \quad \mathcal{D}_2 = \mathrm{Span}\{e_{k+1}, \ldots, e_n\}.$$

By taking the derivative of (6.2) with respect a tangent vector X of M^n, we find

$$(6.5) \qquad X\alpha = -\frac{k}{n(1 - k)} g(\mathbf{x}^T, A_N X) = \frac{k}{n(k - 1)} g(A_N \mathbf{x}^T, X).$$

Thus we have

$$(6.6) \qquad \nabla\alpha = \frac{k}{n(k - 1)} A_N \mathbf{x}^T,$$

which implies that the gradient $\nabla\alpha$ lies in the distribution $\mathcal{D}_1$. Therefore, without loss of generality, we may assume that

$$(6.7) \qquad \nabla\alpha = \zeta e_1$$

for some function ζ. So we have

$$(6.8) \qquad e_2\alpha = \cdots = e_n\alpha = e_2\kappa = \cdots = e_n\kappa = 0.$$

For any vector fields $X, Y \in \mathcal{D}_1$ and $V, W \in \mathcal{D}_2$, we have

$$(6.9) \qquad h(X, Y) = \kappa g(X, Y), \quad h(X, V) = h(V, W) = 0.$$

It follows from (2.7), (6.9) and equation $(\bar{\nabla}_V h)(W, X) = (\bar{\nabla}_X h)(V, W)$ of Codazzi that $h(\nabla_V W, X) = 0$. Since this is true for any vector field X in $\mathcal{D}_1$, we conclude from (6.3) that $\nabla_V W$ lies in $\mathcal{D}_2$. Therefore $\mathcal{D}_2$ is a totally geodesic integrable distribution, i.e., $\mathcal{D}_2$ is an integrable distribution whose leaves are totally geodesic submanifolds of M^n. Moreover, it follows from $h(V, W) = 0$ that each leaf of $\mathcal{D}_2$ is in fact a totally geodesic submanifold of $\mathbb{E}^{n+1}$. Consequently, M^n are foliated by $(n-k)$-dimensional totally geodesic submanifolds of $\mathbb{E}^{n+1}$.

For $1 \le i \ne j \le k$ and $t \in \{k+1, \ldots, n\}$, (2.7), (6.3), (6.8) and (6.9) give

$$(6.10) \qquad (\bar{\nabla}_{e_i} h)(e_j, e_t) = -h(e_j, \nabla_{e_i} e_t), \quad (\bar{\nabla}_{e_t} h)(e_i, e_j) = 0.$$

Thus from $(\bar{\nabla}_{e_i} h)(e_j, e_t) = (\bar{\nabla}_{e_t} h)(e_t, e_j)$, we obtain $\omega_i^t(e_j) = 0$. Therefore $\mathcal{D}_1$ is also a totally geodesic integrable distribution. Consequently, the de Rham decomposition theorem implies that M^n is locally a Riemannian product, say $M_1^k \times \mathbb{E}^{n-k}$, of a Riemannian r-manifold M_1^k and the Euclidean $(n-k)$-space. Furthermore, due to $h(\mathcal{D}_1, \mathcal{D}_2) = \{0\}$ by (6.3), Moore's lemma implies that the immersion is a direct product immersion, i.e., $S^k \times \mathbb{E}^{n-k} \subset \mathbb{E}^{k+1} \times \mathbb{E}^{n-k}$, where $S^k \subset \mathbb{E}^{k+1}$ is the standard imbedding of a k-sphere. Consequently, we obtain case (3) of the theorem. $\qquad \square$

Remark 6.1. If Theorem 6.1(1) occurs, then M^n is either an open portion of a hyperplane contained the origin or of a hypersphere centered at the origin.

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