Certain QR-submanifolds of maximal QR-dimension in a quaternionic space form

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Abstract. The purpose of this paper is to study n-dimensional QR-submanifolds of maximal QR-dimension isometrically immersed in a quaternionic space form and to classify such submanifolds under certain conditions concerning the second fundamental form and the induced almost contact 3-structure.

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1 Introduction

Let M be a connected real n-dimensional submanifold of real codimension p immersed in a real $(n+p)$-dimensional quaternionic Kähler manifold $\overline{M}$ with quaternionic Kähler structure $\{F, G, H\}$. If there exists an r-dimensional normal distribution ν of the normal bundle $TM^{\perp}$ such that

$$F\nu_x \subset \nu_x,\ G\nu_x \subset \nu_x,\ H\nu_x \subset \nu_x,$$
$$F\nu_x^{\perp} \subset T_xM,\ G\nu_x^{\perp} \subset T_xM,\ H\nu_x^{\perp} \subset T_xM$$

at each point $x \in M$, then M is called a *QR-submanifold of r QR-dimension*, where $\nu^{\perp}$ denotes the complementary orthogonal distribution to ν in $TM^{\perp}$ (cf. [1], [5], [9], [13], [14] etc.). Real hypersurfaces, which are typical examples of QR-submanifold with $r = 0$, have been investigated in many papers (cf. [15], [16] and [17] etc.) in connection with the shape operator and the induced almost contact 3-structure (for definition, see [11]).

On the other hand, for a QR-submanifold M of maximal QR-dimension(that is, $(p-1)$ QR-dimension), we can take a distinguished normal vector field ξ to M so that $\nu^{\perp} = \mathrm{Span}\{\xi\}$. Many authors (cf. [5], [8], [9], [13] and [14]) studied QR-submanifolds M of maximal QR-dimension under the following additional condition:

The distinguished normal vector field ξ is parallel with respect to the normal connection induced on the normal bundle of M.

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In this paper we shall determine QR-submanifolds of maximal QR-dimension isometrically immersed in a quaternionic space form under the conditions given in (3.1) without the additional condition mentioned above. In particular we have Theorems 3.3 and 5.3 which are improvements of theorems provided in [9, Theorem 1.1, p.656] and [5, Theorem 2, p.588], respectively.

All manifolds, submanifolds and geometric objects will be assumed to be connected, differentiable and of class C^∞, and all maps also be of class C^∞ if not stated otherwise.

2 Preliminaries

Let $\overline{M}$ be a real $(n+p)$-dimensional quaternionic Kähler manifold. Then, by definition, there is a 3-dimensional vector bundle V consisting of tensor fields of type (1,1) over $\overline{M}$ satisfying the following conditions (a), (b) and (c):

(a) In any coordinate neighborhood $\overline{\mathcal{U}}$, there is a local basis $\{F,\ G,\ H\}$ of V such that

$$\begin{aligned} &F^2=-I,\ G^2=-I,\ H^2=-I,\\ &FG=-GF=H,\ GH=-HG=F,\ HF=-FH=G. \end{aligned} \tag{2.1}$$

(b) There is a Riemannian metric g which is Hermitian with respect to all of F, G and H.

(c) For the Riemannian connection $\overline{\nabla}$ with respect to g, we have

$$\begin{pmatrix}\overline{\nabla}F\\ \overline{\nabla}G\\ \overline{\nabla}H\end{pmatrix}=\begin{pmatrix}0 & r & -q\\ -r & 0 & p\\ q & -p & 0\end{pmatrix}\begin{pmatrix}F\\ G\\ H\end{pmatrix}, \tag{2.2}$$

where p, q and r are local 1-forms defined in $\overline{\mathcal{U}}$. Such a local basis $\{F,G,H\}$ is called a *canonical local basis* of the bundle V in $\overline{\mathcal{U}}$ (cf. [6] and [7]).

For canonical local bases $\{F,G,H\}$ and $\{'F,'G,'H\}$ of V in coordinate neighborhoods $\overline{\mathcal{U}}$ and $'\overline{\mathcal{U}}$ respectively, it follows that in $\overline{\mathcal{U}}\cap{}'\overline{\mathcal{U}}$

$$\begin{pmatrix}'F\\ 'G\\ 'H\end{pmatrix}=(s_{xy})\begin{pmatrix}F\\ G\\ H\end{pmatrix},\qquad (x,\ y=1,\ 2,\ 3),$$

where s_{xy} are local differentiable functions with $(s_{xy})\in SO(3)$ as a consequence of (2.1). It is well known that every quaternionic Kähler manifold is orientable (cf. [6] and [7]).

Now let M be an n-dimensional QR-submanifold of maximal QR-dimension, namely, $(p-1)$ QR-dimension isometrically immersed in $\overline{M}$. Then by definition there is a unit normal vector field ξ such that $\nu_x^\perp=\mathrm{Span}\{\xi\}$ at each point $x\in M$. We set

$$U=-F\xi,\quad V=-G\xi,\quad W=-H\xi. \tag{2.3}$$

Denoting by $\mathcal{D}_x$ the maximal quaternionic invariant subspace

$$T_xM \cap FT_xM \cap GT_xM \cap HT_xM$$

of T_xM, we have $\mathcal{D}_x^\perp \supset \mathrm{Span}\{U, V, W\}$, where $\mathcal{D}_x^\perp$ means the complementary orthogonal subspace to $\mathcal{D}_x$ in T_xM. But, using (2.1) and (2.3), we can prove that $\mathcal{D}_x^\perp = \mathrm{Span}\{U, V, W\}$ (cf. [1] and [14]). Thus we have

$$T_xM = \mathcal{D}_x \oplus \mathrm{Span}\{U, V, W\}, \quad \forall x \in M,$$

which together with (2.1) and (2.3) implies

$$FT_xM,\ GT_xM,\ HT_xM \subset T_xM \oplus \mathrm{Span}\{\xi\}.$$

Therefore, for any tangent vector field X and for a local orthonormal basis $\{\xi_\alpha\}_{\alpha=1,\dots,p}$ ($\xi_1 := \xi$) of normal vectors to M, we have the following decomposition in tangential and normal components:

$$FX = \phi X + u(X)\xi, \quad GX = \psi X + v(X)\xi, \quad HX = \theta X + w(X)\xi, \tag{2.4}$$

$$F\xi_\alpha = \sum_{\beta=2}^{p} P_{1\alpha\beta}\xi_\beta, \quad G\xi_\alpha = \sum_{\beta=2}^{p} P_{2\alpha\beta}\xi_\beta, \quad H\xi_\alpha = \sum_{\beta=2}^{p} P_{3\alpha\beta}\xi_\beta, \quad \alpha = 2, \dots, p. \tag{2.5}$$

Then it is easily seen that $\{\phi, \psi, \theta\}$ are skew-symmetric endomorphisms acting on T_xM. Moreover, from (2.3), (2.4), (2.5) and the Hermitian property of $\{F, G, H\}$, it follows that

$$\begin{aligned} &g(U, X) = u(X), \quad g(V, X) = v(X), \quad g(W, X) = w(X), \\ &u(U) = 1, \quad v(V) = 1, \quad w(W) = 1, \\ &\phi U = 0, \quad \psi V = 0, \quad \theta W = 0. \end{aligned} \tag{2.6}$$

Next, applying F to the first equation of (2.4) and using (2.1), (2.3), (2.4) and (2.6), we have

$$\phi^2 X = -X + u(X)U, \quad u(\phi X) = 0.$$

Similarly taking account of the second and the third equations of (2.4), consequently we get

$$\phi^2 X = -X + u(X)U, \quad \psi^2 X = -X + v(X)V, \quad \theta^2 X = -X + w(X)W, \tag{2.7}$$

$$u(\phi X) = g(\phi X, U) = 0, \quad v(\psi X) = g(\psi X, V) = 0, \quad w(\theta X) = g(\theta X, W) = 0. \tag{2.8}$$

Applying G and H respectively to the first equation of (2.4) and using (2.1), (2.3) and (2.4), we have

$$\begin{aligned} \theta X + w(X)\xi &= -\psi(\phi X) - v(\phi X)\xi + u(X)V, \\ \psi X + v(X)\xi &= \theta(\phi X) + w(\phi X)\xi - u(X)W, \end{aligned}$$

respectively. Thus we can see that

$$\tag{2.9} \begin{aligned} &\psi(\phi X) = -\theta X + u(X)V, \quad v(\phi X) = -w(X), \\ &\theta(\phi X) = \psi X + u(X)W, \quad w(\phi X) = v(X). \end{aligned}$$

Therefore, according to similar method as the above, the second and the third equations of (2.4) also yield respectively

$$\tag{2.10} \begin{aligned} &\phi(\psi X) = \theta X + v(X)U, \quad u(\psi X) = w(X), \\ &\theta(\psi X) = -\phi X + v(X)W, \quad w(\psi X) = -u(X), \end{aligned}$$

$$\tag{2.11} \begin{aligned} &\phi(\theta X) = -\psi X + w(X)U, \quad u(\theta X) = -v(X), \\ &\psi(\theta X) = \phi X + w(X)V, \quad v(\theta X) = u(X). \end{aligned}$$

Moreover, from (2.8) joined with the skew-symmetry of ϕ, ψ and θ, it follows that

$$\tag{2.12} \begin{aligned} &\psi U = -W, \quad v(U) = 0, \quad \theta U = V, \quad w(U) = 0, \\ &\phi V = W, \quad u(V) = 0, \quad \theta V = -U, \quad w(V) = 0, \\ &\phi W = -V, \quad u(W) = 0, \quad \psi W = U, \quad v(W) = 0, \end{aligned}$$

where we have used (2.9), (2.10) and (2.11).

The equations (2.6)-(2.12) tell us that M admits the so-called almost contact 3-structure (for definition, see [11]) and consequently it is seen that the dimension n of M satisfies the equality $n = 4m + 3$ for some integer m.

On the other hand, since the normal distribution ν is quaternionic invariant, we can take a local orthonormal basis $\{\xi, \xi_a, \xi_{a^*}, \xi_{a^{**}}, \xi_{a^{***}}\}_{a=1,\dots,q:=\frac{p-1}{4}}$ of normal vectors to M such that

$$\tag{2.13} \xi_{a^*} := F\xi_a, \quad \xi_{a^{**}} := G\xi_a, \quad \xi_{a^{***}} := H\xi_a.$$

Now let ∇ be the Levi-Civita connection on M and let $\nabla^\perp$ the normal connection of $TM^\perp$ induced from $\overline{\nabla}$. Then Gauss and Weingarten formulae are given by

$$\tag{2.14} \overline{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$\tag{2.15$_1$} \begin{aligned} \overline{\nabla}_X \xi = -AX + \nabla^\perp_X \xi = -AX + \sum_{a=1}^{q} \{ &s_a(X)\xi_a + s_{a^*}(X)\xi_{a^*} \\ &+ s_{a^{**}}(X)\xi_{a^{**}} + s_{a^{***}}(X)\xi_{a^{***}}\}, \end{aligned}$$

$$\tag{2.15$_2$} \begin{aligned} \overline{\nabla}_X \xi_a = -A_a X - s_a(X)\xi + \sum_{b=1}^{q} \{ &s_{ab}(X)\xi_b + s_{ab^*}(X)\xi_{b^*} \\ &+ s_{ab^{**}}(X)\xi_{b^{**}} + s_{ab^{***}}(X)\xi_{b^{***}}\}, \end{aligned}$$

$$\tag{2.15$_3$} \begin{aligned} \overline{\nabla}_X \xi_{a^*} = -A_{a^*} X - s_{a^*}(X)\xi + \sum_{b=1}^{q} \{ &s_{a^*b}(X)\xi_b + s_{a^*b^*}(X)\xi_{b^*} \\ &+ s_{a^*b^{**}}(X)\xi_{b^{**}} + s_{a^*b^{***}}(X)\xi_{b^{***}}\}, \end{aligned}$$

$$\overline{\nabla}_X\xi_{a^{**}} = -A_{a^{**}}X - s_{a^{**}}(X)\xi + \sum_{b=1}^{q}\{s_{a^{**}b}(X)\xi_b + s_{a^{**}b^*}(X)\xi_{b^*} + s_{a^{**}b^{**}}(X)\xi_{b^{**}} + s_{a^{**}b^{***}}(X)\xi_{b^{***}}\}, \tag{2.15$_4$}$$

$$\overline{\nabla}_X\xi_{a^{***}} = -A_{a^{***}}X - s_{a^{***}}(X)\xi + \sum_{b=1}^{q}\{s_{a^{***}b}(X)\xi_b + s_{a^{***}b^*}(X)\xi_{b^*} + s_{a^{***}b^{**}}(X)\xi_{b^{**}} + s_{a^{***}b^{***}}(X)\xi_{b^{***}}\} \tag{2.15$_5$}$$

for vector fields X and Y tangent to M, where $s's$ are the coefficients of the normal connection $\nabla^{\perp}$. Here and in the sequel h denotes the second fundamental form and A, A_a, A_{a^*}, $A_{a^{**}}$, $A_{a^{***}}$ the shape operators corresponding to the normals ξ, ξ_a, ξ_{a^*}, $\xi_{a^{**}}$, $\xi_{a^{***}}$, respectively. They are related by

$$h(X,Y) = g(AX,Y)\xi + \sum_{a=1}^{q}\{g(A_aX,Y)\xi_a + g(A_{a^*}X,Y)\xi_{a^*} + g(A_{a^{**}}X,Y)\xi_{a^{**}} + g(A_{a^{***}}X,Y)\xi_{a^{***}}\}. \tag{2.16}$$

By means of (2.1)-(2.4), (2.13) and (2.15$_{1-5}$), it can be easily verified that

$$\begin{aligned} A_aX &= -\phi A_{a^*}X + s_{a^*}(X)U \\ &= -\psi A_{a^{**}}X + s_{a^{**}}(X)V = -\theta A_{a^{***}}X + s_{a^{***}}(X)W, \end{aligned} \tag{2.17$_1$}$$

$$\begin{aligned} A_{a^*}X &= \phi A_aX - s_a(X)U \\ &= \psi A_{a^{***}}X - s_{a^{***}}(X)V = -\theta A_{a^{**}}X + s_{a^{**}}(X)W, \end{aligned} \tag{2.17$_2$}$$

$$\begin{aligned} A_{a^{**}}X &= -\phi A_{a^{***}}X + s_{a^{***}}(X)U \\ &= \psi A_aX - s_a(X)V = \theta A_{a^*}X - s_{a^*}(X)W, \end{aligned} \tag{2.17$_3$}$$

$$\begin{aligned} A_{a^{***}}X &= \phi A_{a^{**}}X - s_{a^{**}}(X)U \\ &= -\psi A_{a^*}X + s_{a^*}(X)V = \theta A_aX - s_a(X)W, \end{aligned} \tag{2.17$_4$}$$

$$s_a(X) = -u(A_{a^*}X) = -v(A_{a^{**}}X) = -w(A_{a^{***}}X), \tag{2.18$_1$}$$

$$s_{a^*}(X) = u(A_aX) = v(A_{a^{***}}X) = -w(A_{a^{**}}X), \tag{2.18$_2$}$$

$$s_{a^{**}}(X) = -u(A_{a^{***}}X) = v(A_aX) = w(A_{a^*}X), \tag{2.18$_3$}$$

$$s_{a^{***}}(X) = u(A_{a^{**}}X) = -v(A_{a^*}X) = w(A_aX). \tag{2.18$_4$}$$

Moreover, since ϕ, ψ, θ are skew-symmetric and A_a, A_{a^*}, $A_{a^{**}}$, $A_{a^{***}}$ are symmetric, (2.17_{1-4}) together with (2.6) yield

$$\begin{aligned} g((A_a\phi + \phi A_a)X, Y) &= s_a(X)u(Y) - s_a(Y)u(X), \\ g((A_a\psi + \psi A_a)X, Y) &= s_a(X)v(Y) - s_a(Y)v(X), \\ g((A_a\theta + \theta A_a)X, Y) &= s_a(X)w(Y) - s_a(Y)w(X), \end{aligned} \tag{2.19_1}$$

$$\begin{aligned} g((A_{a^*}\phi + \phi A_{a^*})X, Y) &= s_{a^*}(X)u(Y) - s_{a^*}(Y)u(X), \\ g((A_{a^*}\psi + \psi A_{a^*})X, Y) &= s_{a^*}(X)v(Y) - s_{a^*}(Y)v(X), \\ g((A_{a^*}\theta + \theta A_{a^*})X, Y) &= s_{a^*}(X)w(Y) - s_{a^*}(Y)w(X), \end{aligned} \tag{2.19_2}$$

$$\begin{aligned} g((A_{a^{**}}\phi + \phi A_{a^{**}})X, Y) &= s_{a^{**}}(X)u(Y) - s_{a^{**}}(Y)u(X), \\ g((A_{a^{**}}\psi + \psi A_{a^{**}})X, Y) &= s_{a^{**}}(X)v(Y) - s_{a^{**}}(Y)v(X), \\ g((A_{a^{**}}\theta + \theta A_{a^{**}})X, Y) &= s_{a^{**}}(X)w(Y) - s_{a^{**}}(Y)w(X), \end{aligned} \tag{2.19_3}$$

$$\begin{aligned} g((A_{a^{***}}\phi + \phi A_{a^{***}})X, Y) &= s_{a^{***}}(X)u(Y) - s_{a^{***}}(Y)u(X), \\ g((A_{a^{***}}\psi + \psi A_{a^{***}})X, Y) &= s_{a^{***}}(X)v(Y) - s_{a^{***}}(Y)v(X), \\ g((A_{a^{***}}\theta + \theta A_{a^{**}})X, Y) &= s_{a^{***}}(X)w(Y) - s_{a^{***}}(Y)w(X). \end{aligned} \tag{2.19_4}$$

On the other side, since the ambient manifold is a quaternionic Kählerian manifold, differentiating the first equation of (2.4) covariantly and using (2.2), (2.4), (2.14), (2.15_1) and (2.16), we have

$$\begin{aligned} (\nabla_Y\phi)X &= r(Y)\psi X - q(Y)\theta X + u(X)AY - g(AY, X)U, \\ (\nabla_Y u)X &= r(Y)v(X) - q(Y)w(X) + g(\phi AY, X). \end{aligned} \tag{2.20}$$

Similarly, from the second and the third equations of (2.4), we also get respectively

$$\begin{aligned} (\nabla_Y\psi)X &= -r(Y)\phi X + p(Y)\theta X + v(X)AY - g(AY, X)V, \\ (\nabla_Y v)X &= -r(Y)u(X) + p(Y)w(X) + g(\psi AY, X), \end{aligned} \tag{2.21}$$

$$\begin{aligned} (\nabla_Y\theta)X &= q(Y)\phi X - p(Y)\psi X + w(X)AY - g(AY, X)W, \\ (\nabla_Y w)X &= q(Y)u(X) - p(Y)v(X) + g(\theta AY, X). \end{aligned} \tag{2.22}$$

Next, differentiating the first equation of (2.3) covariantly and making use of (2.2), (2.3), (2.4), (2.14) and (2.15_1), we obtain

$$\nabla_Y U = r(Y)V - q(Y)W + \phi AY, \tag{2.23}$$

From the other equations of (2.3), similarly we obtain

$$\nabla_Y V = -r(Y)U + p(Y)W + \psi A_1 Y, \tag{2.24}$$

$$\nabla_Y W = q(Y)U - p(Y)V + \theta A_1 Y. \tag{2.25}$$

Finally if the ambient manifold is a quaternionic space form $\overline{M}(c)$, namely, a quaternionic Kählerian manifold of constant Q-sectional curvature c, its curvature tensor $\overline{R}$ satisfies

$$\begin{aligned}\overline{R}(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y\\ &+ g(FY,Z)FX - g(FX,Z)FY - 2g(FX,Y)FZ\\ &+ g(GY,Z)GX - g(GX,Z)GY - 2g(GX,Y)GZ\\ &+ g(HY,Z)HX - g(HX,Z)HY - 2g(HX,Y)HZ\},\end{aligned}$$

for vector fields X,Y,Z tangent to $\overline{M}$ (cf. [6] and [7]). Hence equations of Gauss, Codazzi and Ricci imply

$$\begin{aligned}R(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y\\ &+ g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\\ &+ g(\psi Y,Z)\psi X - g(\psi X,Z)\psi Y - 2g(\psi X,Y)\psi Z\\ &+ g(\theta Y,Z)\theta X - g(\theta X,Z)\theta Y - 2g(\theta X,Y)\theta Z\}\\ &+ g(AY,Z)AX - g(AX,Z)AY\\ + \sum_{a=1}^{q}\{&g(A_aY,Z)A_aX - g(A_aX,Z)A_aY\\ &+ g(A_{a^*}Y,Z)A_{a^*}X - g(A_{a^*}X,Z)A_{a^*}Y\\ &+ g(A_{a^{**}}Y,Z)A_{a^{**}}X - g(A_{a^{**}}X,Z)A_{a^{**}}Y\\ &+ g(A_{a^{***}}Y,Z)A_{a^{***}}X - g(A_{a^{***}}X,Z)A_{a^{***}}Y\},\end{aligned} \tag{2.26}$$

$$\begin{aligned}&g((\nabla_X A)Y - (\nabla_Y A)X, Z)\\ &= \frac{c}{4}\{g(\phi Y,Z)u(X) - g(\phi X,Z)u(Y) - 2g(\phi X,Y)u(Z)\\ &\quad + g(\psi Y,Z)v(X) - g(\psi X,Z)v(Y) - 2g(\psi X,Y)v(Z)\\ &\quad + g(\theta Y,Z)w(X) - g(\theta X,Z)w(Y) - 2g(\theta X,Y)w(Z)\}\\ &\quad + \sum_{a=1}^{q}\{g(A_aX,Z)s_a(Y) - g(A_aY,Z)s_a(X)\\ &\quad + g(A_{a^*}X,Z)s_{a^*}(Y) - g(A_{a^*}Y,Z)s_{a^*}(X)\\ &\quad + g(A_{a^{**}}X,Z)s_{a^{**}}(Y) - g(A_{a^{**}}Y,Z)s_{a^{**}}(X)\\ &\quad + g(A_{a^{***}}X,Z)s_{a^{***}}(Y) - g(A_{a^{***}}Y,Z)s_{a^{***}}(X)\},\end{aligned} \tag{2.27}$$

$$g(\overline{R}(X,Y)\xi_\alpha,\xi_\beta) = g(R^\perp(X,Y)\xi_\alpha,\xi_\beta) + g([A_\beta,A_\alpha]X,Y) \tag{2.28}$$

for any vector fields X,Y,Z tangent to M, where R and $R^\perp$ denote the curvature tensor of ∇ and $\nabla^\perp$, respectively (cf. [1] and [3]).

3 Fundamental aspects concerning the conditions

Let M be an n-dimensional QR-submanifold of maximal QR dimension in a quaternionic Kähler manifold. From now on we assume that the equalities

$$h(X,\phi Y)+h(\phi X,Y)=0,\quad h(X,\psi Y)+h(\psi X,Y)=0,\quad h(X,\theta Y)+h(\theta X,Y)=0 \tag{3.1}$$

hold on M. Then it follows from (2.16) and (3.1) that

$$A\phi=\phi A,\quad A\psi=\psi A,\quad A\theta=\theta A, \tag{3.2}$$

$$A_a\phi=\phi A_a,\quad A_a\psi=\psi A_a,\quad A_a\theta=\theta A_a, \tag{3.3_1}$$

$$A_{a^*}\phi=\phi A_{a^*},\quad A_{a^*}\psi=\psi A_{a^*},\quad A_{a^*}\theta=\theta A_{a^*}, \tag{3.3_2}$$

$$A_{a^{**}}\phi=\phi A_{a^{**}},\quad A_{a^{**}}\psi=\psi A_{a^{**}},\quad A_{a^{**}}\theta=\theta A_{a^{**}}, \tag{3.3_3}$$

$$A_{a^{***}}\phi=\phi A_{a^{***}},\quad A_{a^{***}}\psi=\psi A_{a^{***}},\quad A_{a^{***}}\theta=\theta A_{a^{***}}. \tag{3.3_4}$$

Furthermore, taking account of (2.6), (2.10), (2.12) and (3.2), we can easily obtain that

$$AU=\lambda U,\quad AV=\lambda V,\quad AW=\lambda W, \tag{3.4}$$

where $\lambda:=u(AU)=v(AV)=w(AW)$.

Next, applying ϕ to the first equation of (3.3_1) and using (2.6) and (2.7), we have

$$A_aU=u(A_aU)U.$$

On the other hand, since (2.18_2) gives $u(A_aU)=s_{a^*}(U)$, consequently we get

$$A_aU=s_{a^*}(U)U.$$

Similarly, we also have

$$A_aU=s_{a^*}(U)U,\quad A_aV=s_{a^{**}}(V)V,\quad A_aW=s_{a^{***}}(W)W. \tag{3.5}$$

By the same method as the above, from $(3.3)_{2-4}$, we can easily verify that

$$A_{a^*}U=-s_a(U)U,\quad A_{a^*}V=-s_{a^{***}}(V)V,\quad A_{a^*}W=s_{a^{**}}(W)W, \tag{3.6_1}$$

$$A_{a^{**}}U=s_{a^{***}}(U)U,\quad A_{a^{**}}V=-s_a(V)V,\quad A_{a^{**}}W=-s_{a^*}(W)W, \tag{3.6_2}$$

$$A_{a^{***}}U=-s_{a^{**}}(U)U,\quad A_{a^{***}}V=s_{a^*}(V)V,\quad A_{a^{***}}W=-s_a(W)W. \tag{3.6_3}$$

Hence (2.18_1) and (3.6_{1-3}) reduce to

$$s_a(X) = s_a(U)u(X) = s_a(V)v(X) = s_a(W)w(X),$$

from which together with (2.12), it follows that $s_a = 0$. Likewise, taking account of (2.18_{2-4}) and (3.6_{1-3}), we also obtain

$$\begin{aligned} s_{a^*}(X) &= s_{a^*}(U)u(X) = s_{a^*}(V)v(X) = s_{a^*}(W)w(X), \\ s_{a^{**}}(X) &= s_{a^{**}}(U)u(X) = s_{a^{**}}(V)v(X) = s_{a^{**}}(W)w(X), \\ s_{a^{***}}(X) &= s_{a^{***}}(U)u(X) = s_{a^{***}}(V)v(X) = s_{a^{***}}(W)w(X), \end{aligned}$$

which also yield $s_{a^*} = s_{a^{**}} = s_{a^{***}} = 0$. Summing up, we have

$$s_a = s_{a^*} = s_{a^{**}} = s_{a^{***}} = 0, \tag{3.7}$$

or equivalently $\nabla^\perp \xi = 0$. Thus we get

Lemma 3.1. *Let M be an n-dimensional QR-submanifold of maximal QR-dimension in a quaternionic Kähler manifold. If the equalities appeared in* (3.1) *hold on M, then the distinguished normal vector field ξ is parallel with respect to the normal connection.*

By means of Lemma 3.1, we can see that the distinguished normal vector field ξ is parallel with respect to $\nabla^\perp$, namely, that (3.7) establishes on M. Hence it is clear from (2.17_{2-4}) and (3.5) that

$$\phi A_a = A_{a^*}, \quad \psi A_a = A_{a^{**}}, \quad \theta A_a = A_{a^{***}}, \quad a = 1, \cdots, q, \tag{3.8}$$

$$A_a U = 0, \quad A_a V = 0, \quad A_a W = 0, \quad a = 1, \cdots, q. \tag{3.9}$$

On the other hand, it follows from (2.19_1), (3.3_1) and (3.7) that

$$\phi A_a = 0, \quad \psi A_a = 0, \quad \theta A_a = 0,$$

which together with (2.7) and (3.9) gives $A_a = 0$. Then this equation combined with (3.8) yields

$$A_a = A_{a^*} = A_{a^{**}} = A_{a^{***}} = 0, \quad a = 1, \cdots, q. \tag{3.10}$$

Owing to Lemma 3.1 and (3.10), we can use the codimension reduction theorems provided in [4, Theorem, p.339], [10, Theorem 4.3. p.32] and [12 Theorem 3.4, p.115] and therefore prove

Theorem 3.2. *Let M be an n-dimensional QR-submanifold of maximal QR-dimension in a quaternionic space form $\overline{M}^{(n+p)/4}(c)$ of constant Q-sectional curvature c. If the equalities appeared in* (3.1) *hold on M, then there exists a real $(n+1)$-dimensional totally geodesic quaternionic space form $\overline{M}^{(n+1)/4}(c)$ such that $M \subset \overline{M}^{(n+1)/4}(c)$.*

Proof. Lemma 3.1 and (3.10) imply that the first normal space of M is contained in Span$\{\xi\}$ which is invariant under parallel translation with respect to the normal connection $\nabla^\perp$. Thus we can apply to M the codimension reduction theorems provided in [12, Theorem 3.4, p.115] (in the case of $c = 4$), [4 Theorem, p.339] (in the case of $c = 0$) and [10, Theorem 4.3, p.32] (in the case of $c = -4$) and verify that there exists a real $(n+1)$-dimensional totally geodesic submanifold $\overline{M}^{n+1}$ such that $M \subset \overline{M}^{n+1}$.

Tentatively we denote $\overline{M}^{n+1}$ by M' and by i_1 the immersion of M into M' and by i_2 the totally geodesic immersion of M' into $\overline{M}^{(n+p)/4}(c)$. Then it is clear from (2.16) that

$$\nabla'_{i_1 X} i_1 Y = i_1 \nabla_X Y + h'(X,Y) = i_1 \nabla_X Y + g(A'X,Y)\xi', \tag{3.11}$$

where ∇' is the induced connection on M' from that of $\overline{M}^{(n+p)/4}(c)$, h' the second fundamental form of M in M' and A' the corresponding shape operator to a unit normal vector field ξ' to M in M'.

Since $i = i_2 \circ i_1$ and M' is totally geodesic in $\overline{M}^{(n+p)/4}(c)$, we can easily see that

$$\xi = i_2 \xi', \quad A = A', \tag{3.12}$$

where we have used (2.16) and (3.11). Moreover, since the tangent space of the totally geodesic submanifold M' at $x \in M$ is $T_x M \oplus \mathrm{Span}\{\xi\}$, it is clear from (2.3) and (2.4) that M' is a quaternionic invariant submanifold of $Q^{(n+p)/4}$, namely, a quaternionic space form with constant Q-sectional curvature c. □

Furthermore, owing to Lemma 3.1 and the theorem([9, Theorem 1.1, p.656]) due to the present authors, we immediately have

Theorem 3.3. *Let M be a complete n-dimensional QR-submanifold of maximal QR-dimension in a quaternionic projective space $QP^{(n+p)/4}$. If the equalities appeared in* (3.1) *hold on M, then M is congruent to a tube of some radius $r \in (0, \pi/2)$ around the canonically* (*totally geodesic*) *embeded quaternionic projective space QP^k for some $k \in \{0, \dots, (n+p)/4 - 1\}$.*

Remark. In the proof of Theorem 3.2, M' is a quaternionic invariant submanifold of $\overline{M}^{(n+p)/4}(c)$ and hence, for any vector field X tangent to M,

$$Fi_2 X = i_2 F'X, \; Gi_2 X = i_2 G'X, \; Hi_2 X = i_2 H'X \tag{3.13}$$

are valid, where $\{F', G', H'\}$ is the induced quaternionic Kähler structure on M'. Thus it follows from the first equation of (2.4) and (3.13) that

$$\begin{aligned} FiX = Fi_2 \circ i_1 X = i_2 F' i_1 X &= i_2(i_1 \phi' X + u'(X)\xi') \\ &= i\phi' X + u'(X) i_2 \xi' = i\phi' X + u'(X)\xi, \end{aligned}$$

for any vector field X tangent to M. Comparing this equation with the first equation of (2.4), we have $\phi = \phi'$ and $u = u'$. Similarly, we have

$$\phi = \phi', \; \psi = \psi', \; \theta = \theta', \; u = u', \; v = v', \; w = w'. \tag{3.14}$$

In this sense, by means of (3.2), Theorem 3.2 and the theorem ([17, Theorem 10, p.57]) due to the second author, we can also prove Theorem 3.3.

4 The case of ambient quaternionic hyperbolic space

In this section we specialize to the case of an ambient quaternionic hyperbolic space $QH^{(n+p)/4}$, namely, to the case of a complete simply connected quaternionic Kähler manifold of constant Q-sectional curvature -4, and assume that M is an n-dimensional QR-submanifold of maximal QR-dimension in $QH^{(n+p)/4}$ and the equalities appeared in (3.1) hold on M. As was already shown in Theorem 3.2 and Remark, M can be regarded as a real hypersurface of $QH^{(n+1)/4}$ which is totally geodesic in $QH^{(n+p)/4}$.

In what follows, *we study the QR-submanifold M as a real hypersurface of $QH^{(n+1)/4}$ and use the same notations and related equations as in* §1 *and* §2 *in the sense of* (3.12) *and* (3.13) .

A real hypersurface of a Riemannian manifold $\overline{M}$ is said to be *curvature-adapted* if the shape operator A of M with respect to a unit normal vector field ξ and the normal Jacobi operator $K(\cdot) := \overline{R}(\cdot, \xi)\xi$ are simultaneously diagonalizable (i.e. $K \circ A = A \circ K$), where $\overline{R}$ denotes the curvature tensor of $\overline{M}$.

On the other hand, for a real hypersurface M in a quaternionic Kähler manifold $\overline{M}$, TM can be decomposed into subbundles $\mathcal{D} \oplus \mathcal{D}^{\perp}$ by use of the maximal quaternionic invariant subbundle $\mathcal{D}$. J. Berndt([2]) pointed out that a real hypersurface in a non-flat quaternionic space form is curvature-adapted if and only if one of the following two conditions holds:

(i) the subbundle $\mathcal{D}$ is invariant under the shape operator A,

(ii) the subbundle $\mathcal{D}^{\perp}$ is invariant under the shape operator A.

Moreover, from this fact, in [2] J.Berndt provided the following theorem:

Let M be a connected curvature-adapted real hypersurface in $QH^n (n \geq 2)$ with constant principal curvatures λ_1, λ_2 and α_1.

(B_1) If λ_1 and λ_2 (resp. α_1) belong to $A|\mathcal{D}$ (resp. $A|\mathcal{D}^{\perp}$), then M is congruent to an open part of a tube of some radius $r \in (0, \infty)$ around a canonically embedded totally geodesic quaternionic hyperbolic space QH^k for some $k \in \{0, \ldots, n-1\}$.

(B_2) If $\lambda_1 = \lambda_2$ (resp. α_1) belongs to $A|\mathcal{D}$ (resp. $A|\mathcal{D}^{\perp}$), then M is congruent to a horosphere in QH^n.

Conversely, these model spaces are curvature-adapted in QH^n and their principal curvatures are constant.

In our case, we first notice that

$$A\phi = \phi A, \quad A\psi = \psi A, \quad A\theta = \theta A, \tag{4.1}$$

which implies

$$AU = \lambda U, \quad AV = \lambda V, \quad AW = \lambda W, \tag{4.2}$$

where $\lambda := u(AU) = v(AV) = w(AW)$. Since $\mathcal{D}^{\perp} = \mathrm{Span}\{U, V, W\}$ (see §2) is invariant under the shape operator A because of (4.2), M is curvature-adapted in $QH^{(n+1)/4}$. Hence, owing to this fact and Berndt's theorem([2], Theorem 2, p.10), we can verify

Theorem 4.1. *Let M be a complete n-dimensional QR-submanifold of maximal QR-dimension in a quaternionic hyperbolic space $QH^{(n+p)/4}$. If the equalities appeared* *in* (3.1) *hold on* M*, then* M *is congruent to a tube of some radius* $r \in (0, \infty)$ *around a canonically embedded totally geodesic quaternionic hyperbolic space* QH^k *for some* $k \in \{0, \dots, (n+p-4)/4\}$ *or a horosphere in* $QH^{(n+p)/4}$.

Proof. It suffices to show that M has two or three constant principal curvatures. We first notice that, in our case, the Codazzi equation (2.27) reduces to

$$\begin{aligned} &g((\nabla_X A)Y - (\nabla_Y A)X, Z) \\ &\quad = \frac{c}{4}\{g(\phi Y, Z)u(X) - g(\phi X, Z)u(Y) - 2g(\phi X, Y)u(Z) \\ &\qquad + g(\psi Y, Z)v(X) - g(\psi X, Z)v(Y) - 2g(\psi X, Y)v(Z) \\ &\qquad + g(\theta Y, Z)w(X) - g(\theta X, Z)w(Y) - 2g(\theta X, Y)w(Z)\}, \end{aligned} \tag{4.3}$$

with $c = -4$ because of (3.7) or (3.10).

Differentiating the first equation of (4.2) covariantly and taking account of (2.23), (4.1) and (4.2) itself, we have

$$g((\nabla_X A)Y, U) + g(\phi A^2 X, Y) = u(Y)X\lambda + \lambda g(\phi AX, Y),$$

from which, taking the skew-symmetric part and using (4.3) with $c = -4$ and (4.1), it follows that

$$\begin{aligned} &-2\{g(\phi X, Y) - w(Y)v(X) + w(X)v(Y)\} - 2g(\phi A^2 X, Y) \\ &= u(X)Y\lambda - u(Y)X\lambda - 2\lambda g(\phi AX, Y). \end{aligned} \tag{4.4}$$

Now we put $Y = U$ in (4.4). Then the skew-symmetry of ϕ, (2.6), (2.12) and (4.2) imply $X\lambda = (U\lambda)u(X)$. Similarly, we also have

$$X\lambda = (U\lambda)u(X) = (V\lambda)v(X) = (W\lambda)w(X)$$

and consequently it is seen that $U\lambda = V\lambda = W\lambda = 0$. Therefore we can see that λ is constant. This fact combined with (4.4) gives

$$\phi A^2 X = -\{\phi X + w(X)V - v(X)W\} + \lambda\phi AX,$$

from which, applying ϕ and taking account of (2.7), (2.12) and (4.2), it turns out to be

$$A^2 X = \lambda AX - \{X - u(X)U - v(X)V - w(X)W\}. \tag{4.5}$$

If X is a non-zero vector field with $X \in \mathcal{D}$ and $AX = \rho X$, then it follows from (4.5) that

$$\rho^2 - \lambda\rho + 1 = 0,$$

and thus we get $\rho \neq \lambda$. Consequently we have the following:

(1) If $\lambda^2 \neq 4$, then M has three constant principal curvatures $(\lambda + \sqrt{\lambda^2 - 4})/2$, $(\lambda - \sqrt{\lambda^2 - 4})/2$ and λ with multiplicities $4k$, $4(m-k)$ and 3, respectively.

(2) If $\lambda^2 = 4$, then M has two constant principal curvatures $\lambda/2$ and λ with multiplicities $4m$ and 3, respectively.

Hence owing to (1) and (2), the table([2], p.11) provided by J. Berndt implies our assertion. □

5 The case of ambient quaternionic number space

In this section we specialize to the case of an ambient quaternionic number space $Q^{(n+p)/4}$, namely, to the case of a quaternionic Kähler manifold of constant Q-sectional curvature $c = 0$, and suppose that M is an n-dimensional QR-submanifold of maximal QR-dimension in $Q^{(n+p)/4}$ and the conditions (3.1) hold on M.

In this case, by means of Theorem 3.2 the submanifold M can be regarded as a real hypersurface of $Q^{(n+1)/4}$ which is totally geodesic in $Q^{(n+p)/4}$.

In what follows, *we study the QR-submanifold M as a real hypersurface of $Q^{(n+1)/4}$ and use the same notations and related equations as in* § 1 *and* § 2.

We first notice that in this case (4.1) and (4.2) are also established on M. Differentiating the first equation of (4.2) covariantly and using (2.23), (4.1) and (4.2) itself, we have

$$g((\nabla_X A)Y, U) + g(\phi A^2 X, Y) = (X\lambda)u(Y) + \lambda g(\phi AX, Y),$$

thus taking the skew-symmetric part of the last equation and making use of (4.3) with $c = 0$ and (4.1), it turns out to be

$$2g(\phi A^2 X, Y) = (X\lambda)u(Y) - (Y\lambda)u(X) + 2\lambda g(\phi AX, Y). \tag{5.1}$$

Now we put $Y = U$ in (5.1). Then the skew-symmetry of ϕ and (2.12) imply $X\lambda = (U\lambda)u(X)$. Similarly, we also have

$$X\lambda = (U\lambda)u(X) = (V\lambda)v(X) = (W\lambda)w(X)$$

and consequently we get

$$U\lambda = V\lambda = W\lambda = 0,$$

which yields that λ is constant. This fact combined with (5.1) gives $\phi(A^2X - \lambda AX) = 0$, and thus applying ϕ and using (2.7) and (4.2), the last equation implies $A^2 = \lambda A$. Therefore we have

Lemma 5.1. *Let M be a real hypersurface of a quaternionic number space $Q^{(n+1)/4}$ on which the equalities appeared in* (3.1) *are valid. Then*

$$A^2 = \lambda A \tag{5.2}$$

and λ is locally constant.

In particular, from Lemmma 5.1 we can prove

Lemma 5.2. *Let M be as in Lemma 5.1. Then*

$$\nabla A = 0. \tag{5.3}$$

Proof. Differentiating (5.2) covariantly and making use of the fact that λ is constant, we get

$$(\nabla_Y A)AX + A(\nabla_Y A)X = \lambda(\nabla_Y A)X, \tag{5.4}$$

thus taking the skew-symmetric part of the last equation and using (4.3) with $c = 0$, we find

$$(\nabla_Y A)AX = (\nabla_X A)AY$$

and hence we get

$$g((\nabla_Y A)AX, Z) = g((\nabla_X A)AY, Z) = g(A(\nabla_X A)Z, Y).$$

On the other side, sine we see that

$$g((\nabla_Y A)AX, Z) = g((\nabla_Z A)AX, Y),$$

which together with the last equation gives

$$g((\nabla_Y A)AX, Z) = g(A(\nabla_X A)Y, Z),$$

that is, $(\nabla_Y A)AX = A(\nabla_Y A)X$. Hence (5.4) reduces to

$$2A(\nabla_Y A)X = \lambda(\nabla_Y A)X,$$

thus applying A to the last equation and using (5.2), we have $\lambda A(\nabla_Y A)X = 0$ and therefore we obtain $\lambda(\nabla_Y A)X = 0$, which completes our assertion because of the fact that λ is constant. □

By means of Lemma 5.1, the eigenvalues κ of the shape operator A satisfy

$$\kappa(\kappa - \lambda) = 0.$$

Moreover it is clear from (4.1) and (4.2) that the multiplicity of λ must be $4m + 3$ for some integer m at each point in M. Since λ is constant and $traceA$ is continuous, the multiplicity r of λ is constant. Hence it suffices to consider the following 3-cases

$$(i) \quad r = 0, \qquad (ii) \quad r = n, \qquad (iii) \quad 3 \le r < n.$$

We will start with the first case of (i). In this case $A = 0$ and consequently M is contained in a totally geodesic hyperplane $\mathbb{R}^n$ of $Q^{(n+1)/4}$.

Next, we consider the case of (ii). In this case $A = \lambda I$. Let $\bar{x}$ be the position vector of M and put $\bar{p} := \bar{x} + \lambda^{-1}\xi$. Then, since $\nabla^{\perp}_X \xi = 0$,

$$\overline{\nabla}_X \bar{p} = \overline{\nabla}_X(\bar{x} + \lambda^{-1}\xi) = X - \lambda^{-1}(AX) = 0,$$

which means that $\bar{p}$ is a fixed point in $Q^{(n+1)/4}$. Moreover, it is clear that $\|\bar{x} - \bar{p}\| = |\lambda|^{-1}$ and consequently M is contained in a hypersphere $S^n(|\lambda|^{-1})$ of radius $|\lambda|^{-1}$ and centered at $\bar{p}$.

Finally we consider the case of (iii). Since the multiplicity r of λ is constant, the eigenspaces corresponding to λ and 0 determine distributions of dimension r and $n - r$, which will be denoted by D_λ and D_0, respectively. Furthermore, by means of Lemma 5.2, $\nabla A = 0$ and consequently it is easily verified that D_λ and D_0 are both involutive and that D_λ is parallel along D_0 and vice versa. Denoting by M_λ and M_0 the integral submanifolds of D_λ and D_0, respectively, we can see that M is locally the Riemannian product $M_\lambda \times M_0$.

From now on we shall study M_λ and M_0 more precisely and start with M_λ. Let $Z_1, \cdots, Z_{n-r}$ be orthonormal vector fields belonging to D_0. Since M_λ is totally geodesic in M, the shape operators $A'_1, \cdots, A'_{n-r}$ corresponding to those normal vectors vanish. On the other hand we may consider M_λ as a submanifold of $Q^{(n+1)/4}$.

Then the vector fields $Z_1, \cdots, Z_{n-r}, \xi$ form an orthonormal set of local vector fields normal to M_λ. In this case the shape operators corresponding to $Z_1, \cdots, Z_{n-r}$ also vanish. Hence it is clear from (2.28) that

$$'R^{\perp}(X,Y)Z_i = 0, \quad i = 1, \ldots, n-r\,, \tag{5.5}$$

where $'R^{\perp}$ denotes the curvature tensor of the normal connection $'\nabla^{\perp}$ of M_λ in $Q^{(n+1)/4}$. Thus, by the same method as that used in the proof of Proposition 1.1 in [3, p.99], we may show that the equation (5.5) yields the existence of the normal vector fields $Z_1, \cdots, Z_{n-r}$ such that

$$'\nabla^{\perp}_X Z_i = 0, \quad i = 1, \ldots, n-r \tag{5.6}$$

for any tangent vector X to M_λ.

Now let $\bar{x}$ be the position vector of M_λ in $Q^{(n+1)/4}$ and put $\bar{p} := \bar{x} + \lambda^{-1}\xi$. Then, for $X \in D_\lambda$, it follows that

$$\overline{\nabla}_X \bar{p} = X - \lambda^{-1}AX = 0 \qquad \text{and} \qquad \|\bar{x} - \bar{p}\| = |\lambda|^{-1},$$

which means that M_λ belongs to the hypersphere of radius $|\lambda|^{-1}$ centered at $\bar{p}$. Further, using (5.7) and $A'_i = 0, \quad i = 1, \cdots, n-r$, we have

$$Xg(\overline{x}, Z_i) = g(X, Z_i) = 0, \quad i = 1, \cdots, n-r,$$

that is,

$$g(\bar{x}, Z_i) = c_i, \quad i = 1, \cdots, n-r \tag{5.7}$$

for $X \in D_\lambda$, where $c_i (i = 1, \cdots, n-r)$ are constants. Hence M_λ belongs to the intersection of the hypersphere of radius $|\lambda|^{-1}$ centered at $\bar{p}$ and the $n-r$ hyperplanes defined by (5.7). We notice that $\overline{p}$ is contained in the $n-r$ hyperplanes.

In a similar way it can be shown that M_0 belongs to the intersection of the $r+1$ hyperplanes given by

$$g(\bar{x}, \xi) = c, \quad g(\bar{x}, Z_s) = c_s, \quad s = n-r+1, \cdots, n,$$

where c and $c_s (s = n-r+1, \cdots, n)$ are constants. Summing up, we yield

Theorem 5.3. *Let M be a complete n-dimensional QR-submanifold of maximal QR-dimension in $Q^{(n+p)/4}$ on which the equalities appeared in* (3.1) *are valid. Then M is isometric to* $\mathbb{R}^n$, S^n *or* $S^r \times \mathbb{R}^{n-r}$.

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