Geometrical aspects of the ball-plate problem

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Abstract. The geometry of the mechanical ball-plate problem is discussed and some of its geometrical properties are pointed out.

1 Introduction

The matrix Lie groups arise naturally in engineering, physics or biology usually as the configuration space of some concrete mechanical problems and then their corresponding dynamics leads naturally to a set of differential equations on the ambient matrix Lie group. For instance, this is the case of electrical networks used for power conversion modeled on the orthogonal group $SO(n)$ and extensively studied by Wood [16], the control tower problem from air traffic, modeled on the special euclidian group $SE(3)$ and extensively studied by Montgomery, Sastry and Walsh [14], the underwater vehicle dynamics modeled on the special euclidian group $SE(3)$ and studied by Leonard [8], the molecular motion in the context of coherent control of quantum dynamics modeled on the complex unitary group $U(n)$ and studied by Dahleh, Pierce, Rabitz and Ramakrishna [4], and the spacecraft dynamics modeled on the special othogonal group $SO(3)$ and extensively studied by Leonard [8] and Puta [12].

The goal of our paper is to add at this list a new example, namely the ball-plate problem and to point out some of its geometrical and dynamical properties.

2 The geometrical picture of the problem

Let us consider the following well-known kinematics problem: a ball rolls without slipping between two horizontal plates separated by the distance equal to the diameter of the ball. It is assumed that the lower plate is fixed and that the ball is rolled through the horizontal movement of the upper plate. The problem is to transfer the ball from a given initial position and a given orientation to a prescribed final position and final orientation along a path which minimizes the functional $\int_0^T \|V(t)\|^2 \, dt$, among all

possible pathes which satisfy these boundary condition. $V(t)$ denotes the the velocity of the moving plate and T is the time transfer. For convenience we have chosen the components of velocity of the center of the ball as the control functions, rather than the components of the velocity of the moving plate and we have also assume that the radius of the ball is 1. Then the velocity of the moving plate is twice the velocity of the center of the ball. The problem can be also formulated as an optimal control problem on a Lie group. This way of looking at the ball-plate problem was pointed out by Jurdjevic ([6]). More exactly we have: let $G = \mathbb{R}^2 \times SO(3)$ be regarded as an 5-dimensional Lie group with the group operation given by:

$$(a_1, R_1) \cdot (a_2, R_2) = (a_1 + a_2, R_1 R_2),$$

for any $(a_i, R_i) \in G, i = 1, 2$. Its Lie algebra $\mathcal{G}$ is $\mathbb{R}^2 \times so(3)$ with bracket operation given by

$$[(a, A), (b, B)] = (0, AB - BA), \quad \forall (a, A), (b, B) \in \mathbb{R}^2 \times so(3).$$

Since $so(3)$ can be identified with $\mathbb{R}^3$ via the canonical map:

$$\wedge : \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3 \mapsto \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \in so(3),$$

we can conclude that $\mathbb{R}^2 \times so(3)$ can be identified with $\mathbb{R}^5$. Moreover, let: $e_1 = (1, 0)^t, (0, 1)^t$ and

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

be respectively the canonical bases of $\mathbb{R}^2$ and respectively $so(3)$. Then:

$$A_1 = (0, 0, 1, 0, 0)^t, \ A_2 = (0, 0, 0, 1, 0)^t, \ A_3 = (0, 0, 0, 0, 1)^t,$$

$$A_4 = (1, 0, 0, 0, 0)^t, \ A_5 = (0, 1, 0, 0, 0)^t$$

is the corresponding basis of $\mathbb{R}^5$. Following Jurdjevic ([6]), the dynamics of the ball-plate can be described by the following left invariant system on $\mathbb{R}^2 \times SO(3)$:

(2.1) $$\dot{X} = X [u_1 (A_4 - A_2) + u_2 (A_5 + A_1)].$$

Using the following Table:

[.,.]	A_1	A_2	A_3	A_4	A_5
A_1	0	A_3	$-A_2$	0	0
A_2	$-A_3$	0	A_1	0	0
A_3	A_2	$-A_1$	0	0	0
A_4	0	0	0	0	0
A_5	0	0	0	0	0

we can conclude that the system (2.1) is controllable and is a single bracket one.

3   An optimal control problem for the system (2.1)

In this section we will present an optimal control problem for the controllable system (2.1). Despite the classic optimization problems solved by Pontryaguin maximum principle, for instance like [9] or [10], this kind of problems are solved by Krishnaprasad Theorem, see [7].

Proposition 3.1. *([13]) Let $\mathcal{J}$ be the cost function given by:*

$$\mathcal{J}(u_1, u_2) = \frac{1}{2} \int_0^{t_f} \left(u_1^2 + u_2^2\right) dt.$$

Then the controls which minimize $\mathcal{J}$ and steer the system from $X = X_0$ at $t = 0$ to $X = X_f$ at $t = t_f$ are given by $u_1 = x_4 - x_2$, $u_2 = x_1 + x_5$,   where x_i are the solutions of the SODE:

$$(3.1) \qquad \begin{cases} \dot{x_1} = -x_3 \left(x_2 - x_4\right) \\ \dot{x_2} = x_3 \left(x_1 + x_5\right) \\ \dot{x_3} = -x_2 x_5 - x_1 x_4 \\ \dot{x_4} = 0 \\ \dot{x_5} = 0 \end{cases}$$

It is not hard to see that the equations (3.1) can be put in the equivalent form:

$$(3.2) \qquad \begin{cases} \dot{x_1} = x_3 \left(k - x_2\right) \\ \dot{x_2} = x_3 \left(x_1 + l\right) \\ \dot{x_3} = -l x_2 - k x_1 \end{cases}$$

where $x_4 = k$, $x_5 = l$.

Proposition 3.2. *([13]) The system (3.2) has a Hamilton-Poisson realization with the phase space $\mathbb{R}^3$, the Poisson bracket given by the matrix $\Pi = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$ and the Hamiltonian H given by:*

$$(3.3) \qquad H(x_1, x_2, x_3) = \frac{1}{2}(k - x_2)^2 + \frac{1}{2}(l + x_1)^2.$$

The function C given by:

$$(3.4) \qquad C(x_1, x_2, x_3) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$$

is a Casimir of our configuration. It follows that the phase curves of our system are the intersections of the elliptic cylinder $H = const.$ with the sphere $C = const.$

4   Stability problems and periodical orbits

It is not hard to see that the equilibrium states of our dynamics are:

$$e_1^M = (-l, k, M), M \in \mathbb{R}, \quad \text{and} \quad e_2^M = \left(-\frac{l}{k}M, M, 0\right), M \in \mathbb{R}.$$

About the spectral stability of these equilibrium states we have the following results:

Proposition 4.1. *The equilibrium states e_1^M are spectrally stable for any $M \in \mathbb{R}$.*

Proposition 4.2. *The equilibrium states e_2^M are spectrally stable if one of the following conditions hold true: i) $k < 0, M > k$;   ii) $k > 0, M < k$.*

We can now pass to discuss the nonlinear stability of the equilibrium states $e_1^M, M \in \mathbb{R}$ and e_2^M in the conditions of the above Proposition. Using the energy-Casimir method [1], we are able to prove the following results:

Proposition 4.3. *The equilibrium states e_1^M, $M \in \mathbb{R}^*, l \in \mathbb{R}^*, k \in \mathbb{R}^*$ are nonlinearly stable.*

Proof. We use the energy-Casimir method (see [1]). Let

$$H_\varphi = H + \varphi(C) = \frac{(k - x_2)^2}{2} + \frac{(l + x_1)^2}{2} + \varphi\left(\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)\right)$$

be the energy-Casimir function, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a smooth real valued function defined on $\mathbb{R}$. Then, the first variation of H_φ is given by:

$$\delta H_\varphi = (l + x_1)\delta x_1 - (k - x_2)\delta x_2 + \dot{\varphi}\left(\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)\right) \cdot (x_1\delta x_1 + x_2\delta x_2 + x_3\delta x_3),$$

where $\dot{\varphi} = \frac{\partial \varphi}{\partial\left(\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)\right)}$. This equals zero at the equilibrium of interest if and only if $\dot{\varphi}(\frac{1}{2}(k^2 + M^2 + l^2)) = 0$. The second variation of H_φ is given by:

$$\delta^2 H_\varphi = (\delta x_1)^2 + (\delta x_2)^2 + \ddot{\varphi} \cdot (x_1\delta x_1 + x_2\delta x_2 + x_3\delta x_3)^2 + \dot{\varphi} \cdot ((\delta x_1)^2 + (\delta x_2)^2 + (\delta x_3)^2),$$

At the equilibrium of interest, the second variation becomes:

$$\delta^2 H_\varphi(-l, k, M) = (\delta x_1)^2 + (\delta x_2)^2 + \ddot{\varphi} \cdot (-l\delta x_1 + k\delta x_2 + M\delta x_3)^2.$$

Choosing φ such that $\dot{\varphi}(\frac{1}{2}(k^2 + M^2 + l^2)) = 0$ and $\ddot{\varphi}(\frac{1}{2}(k^2 + M^2 + l^2)) > 0$, we conclude that the second variation of H_φ at the equilibrium of interest is positive definite, and thus e_1 is nonlinearly stable. $\square$

Using the same arguments we are able to prove

Proposition 4.4. *The equilibrium states e_2^M are nonlinearly stable if one of the following conditions holds true: i) $k < 0, M < 0, M > k$; ii) $k > 0, M > 0, M < k$.*

As consequence, under the conditions of Proposition 4.4, we can prove

Proposition 4.5. *Near to $e_2^M = \left(-\frac{l}{k}M, M, 0\right)$, the reduced dynamics has, for each sufficiently small value of the reduced energy, at least 1-periodic solution whose period is close to:*

$$2\pi\sqrt{\frac{k}{(k - M)(l^2 + k^2)}}.$$

Proof. We can easily check the following:

(i) restriction of our dynamics (3.2) to the coadjoint orbit $x_1^2 + x_2^2 + x_3^2 = \frac{l^2+k^2}{k^2}M^2$ gives rise to a classical Hamiltonian system;

(ii) the matrix of the linear part of the reduced dynamics has purely imaginary roots. More exactly, $\lambda_{2,3} = \pm i\sqrt{(k-M)(l^2+k^2)/k}$;

(iii) $\mathrm{span}(\nabla C(e_2^M)) = V_0 = span\left\{(-\frac{l}{k}, 1, 0)^t\right\}$, where $V_0 = \ker(A(e_2^M))$;

(iv) the smooth function $F_{\frac{k-M}{M}} \in C^\infty(\mathbb{R}^3, \mathbb{R})$ given by

$$F_{\frac{k-M}{M}}(x_1, x_2, x_3) = \frac{1}{2}(k-x_2)^2 + \frac{1}{2}(l+x_1)^2 + \frac{k-M}{2M}(x_1^2 + x_2^2 + x_3^2)$$

has the following properties:

- it is a constant of motion of the dynamics (3.1);
- $\nabla F_{\frac{k-M}{M}}(e_2^M) = 0$;
- if $k < 0, M < 0, M > k$ or $k > 0, M > 0, M < k$ then

$$\left.\nabla^2 F_{\frac{k-M}{M}}(e_2^M)\right|_{W \times W} > 0,$$

where $W := ker\, dC(e_2^M) = span_{\mathbb{R}}\left\{(\frac{k}{l}, 1, 0)^t, (0, 0, 1)^t\right\}.$

Then our assertion follows via the Moser-Weinstein theorem with zero eigenvalue (for details, see [2]). □

5　Numerical integration of the equations (3.2)

It is easy to see that, for the equations (3.2), Kahan's integrator can be written in the following form:

(5.1)
$$\begin{cases} x_1^{n+1} - x_1^n = \frac{h}{2}(kx_3^{n+1} + kx_3^n - x_3^{n+1}x_2^n - x_2^{n+1}x_3^n) \\[2mm] x_2^{n+1} - x_2^n = \frac{h}{2}(x_1^{n+1}x_3^n + x_3^{n+1}x_1^n + lx_3^{n+1} + lx_3^n) \\[2mm] x_3^{n+1} - x_3^n = -\frac{h}{2}(lx_2^{n+1} + lx_2^n + kx_1^{n+1} + kx_1^n) \end{cases}$$

A long but straightforward computation or alternatively, by using MATHEMATICA, leads us to:

Proposition 5.1. *Kahan's integrator* (5.1) *has the following properties:*

(i) It is not Poisson preserving.

(ii) It does not preserve the Casimir C (3.4) of our Poisson configuration $(\mathbb{R}^3, \Pi)$.

(iii) It does not preserve the Hamiltonian H (3.3) of our system (3.2).

Proof. Indeed, it is easy to see that the factorizations of $H_{n+1} - H_n$, (resp. $C_{n+1} - C_n$ and $J\Pi_n J^t - \Pi_{n+1}$, where J is the Jacobian of the transformation described by the above integrator) do not vanish, and then our result immediately follows. □

We shall discuss now the numerical integration of the dynamics (3.2) via the Lie-Trotter integrator, see for details [12], [15]. To begin with, let us observe that the Hamiltonian vector field X_H splits as follows: $X_H = X_{H_1} + X_{H_2}$, where

$$H_1 = \frac{1}{2}(l + x_1)^2, \ H_2 = \frac{1}{2}(k - x_2)^2.$$

Following [15], we obtain the Lie-Trotter integrator:

$$(5.2) \quad \begin{cases} x_1^{n+1} = & x_1^n \cos(x_2(0) - k)t + x_3^n \sin(x_2(0) - k)t \\ x_2^{n+1} = & -x_1^n \sin(x_1(0) + l)t \sin(x_2(0) - 1)t + x_2^n \cos(x_1(0) + l)t \\ & +x_3^n \sin(x_1(0) + l)t \cdot \cos(x_2(0) - k)t \\ x_3^{n+1} = & -x_1^n \sin(x_1(0) - k)t \cos(x_2(0) + l)t - x_2^n \sin(x_1(0) + l)t \\ & +x_3^n \cos(x_2(0) - k)t \cdot \cos(x_1(0) + l)t \end{cases}$$

Now, a direct computation or using MATHEMATICA 7 leads us to:

Proposition 5.2. *The Lie-Trotter integrator* (5.2) *has the following properties:*

(i) It preserves the Poisson structure Π.
(ii) It preserves the Casimir C of our Poisson configuration $(\mathbb{R}^3, \Pi)$.
(iii) It doesn't preserve the Hamiltonian H of our system (3.2).
(iv) Its restriction to the coadjoint orbit $(\mathcal{O}_k, \omega_k)$, where

$$\mathcal{O}_k = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 = const.\}$$

and ω_k is the Kirilov-Kostant-Souriau symplectic structure on $\mathcal{O}_k$ gives rise to a symplectic integrator.

Proof. The items (i), (ii) and (iv) hold because Lie-Trotter is a Poisson integrator. The last assertion is due to the fact that $\{H_1, H_2\} \neq 0$. □

If we compare this method to the 4th-step Runge-Kutta method, we can see that the Lie-Trotter's integrator gives us a good approximation of our dynamics. For this time, Kahan' s integrator provides a weak approximation.
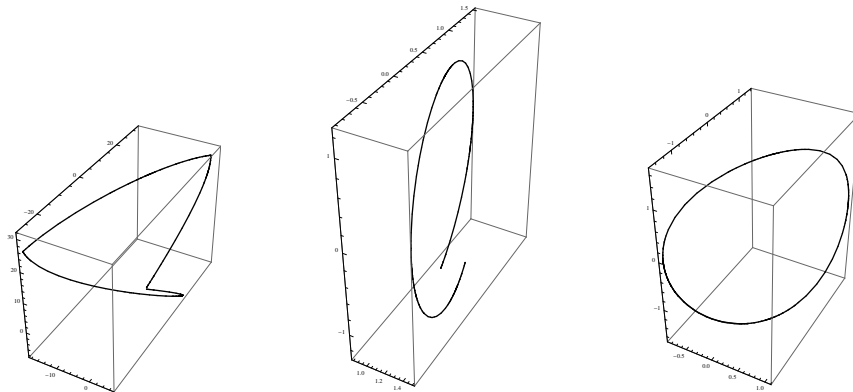

Figure 1: Kahan, Lie-Trotter and Runge-Kutta 4th integrator, respectively, for the equations (3.2) $(k = 2, l = 3, x_1(0) = x_2(0) = x_3(0) = 1)$

6   Conclusion

The paper presents the ball-plate problem from the mechanical geometry point of view. An optimal control problem associated to the ball-plate dynamics is the subject of the second paragraph. Realizing this system as a Hamilton-Poisson system we can study it from the standard Hamilton-Poisson geometry points of view. The third paragraph presents the spectral and nonlinear stability problems and the periodical orbits. In the last paragraph we have presented a comparison between three numerical integration methods: Runge-Kutta 4th steps, Lie-Trotter algorithm and Kahan algorithm. This time, the three methods provided different results: Lie-Trotter algorithm's results are close enough to Runge-Kutta's one while Kahan's method gives us a weak approximation of trajectory movement.

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