An alternative moving frame for tubular surfaces around timelike curves in the Minkowski 3-space

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Abstract. We describe a robust method for constructing a tubular surface surrounding a timelike space curve in Minkowski 3-Space. Our method is designed to eliminate undesirable twists and wrinkles in the tubular surface's skin at points where the curve experiences high torsion. In our construction the tubular surface's twist is bounded by the timelike curve's curvature and is independent of the timelike curve's torsion. This paper is a generalization of [4] to Minkowski 3–space.

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1 Preliminaries

Let $\mathbb{R}^3 = \{(x_1, x_2, x_3) | x_1, x_2, x_3 \in \mathbb{R}\}$ be a 3-dimensional vector space, and let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two vectors in $\mathbb{R}^3$. The Lorentz scalar product of x and y is defined by

$$\langle x, y \rangle_L = -x_1 y_1 + x_2 y_2 + x_3 y_3,$$

$\mathbb{E}_1^3 = \left(\mathbb{R}^3, \langle x, y \rangle_L\right)$ is called 3-dimensional Lorentzian space, Minkowski 3-Space or 3- dimensional Semi-Euclidean space. The vector x in $\mathbb{E}_1^3$ is called a spacelike vector, null vector or a timelike vector if $\langle x, x \rangle_L > 0$ or $x = 0$, $\langle x, x \rangle_L = 0$ or $\langle x, x \rangle_L < 0$, respectively. For $x \in \mathbb{E}_1^3$, the norm of the vector x defined by $\|x\|_L = \sqrt{|\langle x, x \rangle_L|}$, and x is called a unit vector if $\|x\|_L = 1$. For any $x,\ y \in \mathbb{E}_1^3$, Lorentzian vectoral product of x and y is defined by

$$x \wedge_L y = (x_3 y_2 - x_2 y_3, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{E}_1^3$ is locally spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s) = T(s)$ are respectively spacelike, timelike or null, for each $s \in I \subset \mathbb{R}$. The vectors $v, w \in \mathbb{E}_1^3$ are orthogonal if and only if

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$\langle v, w \rangle_L = 0$. The Lorentzian sphere of center $m = (m_1, m_2, m_3)$ and radius $r \in \mathbb{R}^+$ in the space $\mathbb{E}_1^3$ is defined by

$$S_1^2 = \left\{ a = (a_1, a_2, a_3) \in \mathbb{E}_1^3 \, \middle| \langle a - m, a - m \rangle_L = r^2 \right\}.$$

Basic notations and definitions in this section are taken from M. Petrovic and E. Sucurovic[3].

2 Inroduction

Consider a curve $\alpha(s)$ in $\mathbb{E}_1^3$, parameterized by its arc length s. Let $T(s)$ be its tangent vector, i.e., $T(s) = \alpha'(s) = \frac{d\alpha(s)}{ds}$. The arc length parameterization of the curve makes $T(s)$ a unit vector, i.e., $\|T(s)\|_L = 1$, therefore its derivative is orthogonal to T. The principal normal vector N is defined defined as $N = \frac{T'}{\|T'\|_L}$. The binormal vector B is defined as the cross product $B = T \wedge_L N$. The Frenet-Serret equations, express the rate of change of the moving orthonormal triad $\{T, N, B\}$ along the timelike curve $\alpha(s)$,

$$\left\{ \begin{array}{c} T' = \kappa N \\ N' = \kappa T + \tau B \\ B' = -\tau N \end{array} \right. \tag{2.1}$$

and

$$\begin{array}{rcl} \langle T, T \rangle_L & = & -1, \langle B, B \rangle_L = \langle N, N \rangle_L = 1 \ , \\ \langle T, N \rangle_L & = & \langle T, B \rangle_L = \langle B, N \rangle_L = 0 \, . \end{array}$$

The coefficients κ and τ are the curve's curvature and torsion [3].

A.Gray [1] has noted that wild gyrations of the Frenet-Serret system can be expected at points where the curvature κ is small and the torsion τ is large. An inspection of the Frenet-Serret equations (2.1) shows that at such points T' is small and N' and B' are large, in effect indicating that the Frenet-Serret system is spinning about its T axis.

The curve's torsion is a function of the curve's third derivative as evidenced by the explicit formulas

$$\kappa = \frac{\|x' \wedge_L x''\|_L}{\|x'\|_L^3} \quad , \quad \tau = \frac{\det(x', x'', x''')}{\|x' \wedge_L x''\|_L^2}.$$

3 Characterization of Canal and Tubular Surfaces

Theorem 3.1. *Suppose the center curve of a canal surface is a unit speed timelike curve $\alpha : I \to \mathbb{E}_1^3$ with nonzero curvature. Then the canal surface can be parametrized*

$$C(s,t) = \alpha(s) + r(s) r'(s) T(s) \mp r(s)\sqrt{1 + r'^2(s)} N(s) \cos t \pm r(s) \sqrt{1 + r'^2(s)} B(s) \sin t, \tag{3.1}$$

where T, N and B denote the tangent, normal and binormal of α.

Proof: Let C denote a patch that parametrizes the envelope of Lorentzian spheres defining the canal surface. Since the curvature of α is nonzero, the Frenet- Serret frame $\{T, N, B\}$ is well-defined, and we can write

$$C(s,t) - \alpha(s) = a(s,t)T + b(s,t)N + c(s,t)B \tag{3.2}$$

where a, b and c are differentiable on the interval on which α is defined. We must have

$$\|C(s,t) - \alpha(s)\|_L^2 = r^2(s) \tag{3.3}$$

Equation (3.3) expresses analytically the geometric fact that $C(s,t)$ lies on a Lorentzian sphere $S_1^2(s)$ of radius $r(s)$ centered at $\alpha(s)$. Furthermore, $C(s,t) - \alpha(s)$ is a normal vector to the canal surface; this fact implies that

$$\langle C(s,t) - \alpha(s), C_s \rangle_L = 0, \tag{3.4}$$

$$\langle C(s,t) - \alpha(s), C_t \rangle_L = 0, \tag{3.5}$$

Equations (3.3) and (3.4) say that the vectors C_s and C_t are tangent to $S_1^2(s)$. From (3.2) and (3.3) we get

$$\begin{cases} -a^2 + b^2 + c^2 = r^2 \\ -aa_s + bb_s + cc_s = rr' \end{cases} \tag{3.6}$$

When we differentiate (3.2) with respect to t and use the Frenet-Serret formulas, we obtain

$$C_s = (1 + a_s + b\kappa)\, T + (a\kappa - c\tau + b_s)\, N + (c_s + b\tau)\, B. \tag{3.7}$$

Then (3.6), (3.7), (3.2) and (3.4) imply that

$$-a + rr' = 0, \tag{3.8}$$

and from (3.6) and (3.7) we get

$$b^2 + c^2 = r^2 \left(1 + r'^2\right). \tag{3.9}$$

We can write

$$\begin{aligned} b &= \mp r\sqrt{1 + r'^2} \cos t, \\ c &= \pm r\sqrt{1 + r'^2} \sin t. \end{aligned}$$

Thus (3.2) becomes

$$C(s,t) = \alpha(s) + r(s)r'(s)T(s) \mp r(s)\sqrt{1 + r'^2(s)}N(s)\cos t \pm r(s)\sqrt{1 + r'^2(s)}B(s)\sin t.$$

With the Frenet-Serret system in hand, we can construct a "tubular surface" of radius $r = const.$ about the curve by defining a surface with parameters s and t:

$$Tube(s,t) = \alpha(s) + r\,(N(s)\cos t + B(s)\sin t) \tag{3.10}$$

$\square$

4 An Alternative Moving Frame

We introduce an alternative, more tamely behaved moving triad for timelike curves. For this, we let $\alpha(s)$ be a regular timelike curve in $\mathbb{E}_1^3$ parameterized by its arc length, and $T(s) = \alpha'(s)$ be its unit tangent vector the same as before. We will define unit vector fields $P(s)$ and $Q(s)$ such that $\{T, P, Q\}$ is orthonormal at each point along the timelike curve. We will chose $P(s)$and $Q(s)$ in such a way as to minimize its gyrations as the triad moves along the timelike curve.

Unlike the Frenet-Serret triad which is defined locally on the timelike curve, our triad is denned in terms of the solution of a differential equation, hence it depends not only on thetimelike curve's local properties but also on the location and value of the differential equation's initial conditions.

The construction of the new triad is based on the set of solutions of the system of differential equations

$$\frac{d}{ds}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & \tau(s) \\ -\tau(s) & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \tag{4.1}$$

where a and b are functions of s and $\tau(s)$ is the torsion of the timelike curve $\alpha(s)$. Let denote the semigroup of solutions of this differential equation, that is, $(a(s), b(s)) = \phi_{s_0,s}(a(s_0), b(s_0))$ is the solution of the differential equation corresponding to the initial data $(a(s_0), b(s_0))$ at $s = s_0$. If the timelike curve $\alpha(s)$ is sufficiently regular, then the standard theory of linear systems, guarantees the existence, uniqueness and maximal continuation of its solutions.

Definition 4.1. *Let $U(s)$ denote the normal plane of the curve $\alpha(s)$ at any s. That is, $U(s)$ is perpendicular to $T(s)$ and contains the principal normal and binormal vectors $N(s)$ and $B(s)$. We equip each $U(s)$ with the inner product inherited from $\mathbb{E}_1^3$. We define the mapping $\phi_{s_0,s} : U(s_0) \to U(s)$ with*

$$\phi_{s_0,s} : V_0 \in U(s_0) \to a(s)N(s) + b(s)B(s)$$

where $(a(s), b(s)) = \phi_{s_0,s}\left(\langle V_0, N_0(s_0)\rangle_L, \langle V_0, B_0(s_0)\rangle_L\right)$.

Lemma 4.1. *Let $\phi_{s_0,s}$ be as in Definition 4.1. Then for any $V_0 \in U(s_0)$ the vector valued function $V(s)$ defined by $V(s) = \phi_{s_0,s}(V_0)$ satisfies the differential equation $V' = \langle V, T'\rangle_L T$, where T is the timelike curve's unit tangent vector.*

Proof: According to the definition of $\phi_{s_0,s}$, the components $\langle V(s), N(s)\rangle_L$ and $\langle V(s), B(s)\rangle_L$ of $V(s)$ satisfy the differential equation (4.1), therefore

$$\langle V, N\rangle_L' = \tau \langle V, B\rangle_L$$

and

$$\langle V, B\rangle_L' = -\tau \langle V, N\rangle_L .$$

We now compute $V'(s)$ while making use of the Frenet-Serret equations:

$$
\begin{aligned}
V'(s) &= [\langle V,N\rangle_L N + \langle V,B\rangle_L B]' \\
&= \langle V,N\rangle_L' N + \langle V,N\rangle_L N' + \langle V,B\rangle_L' B + \langle V,B\rangle_L B' \\
&= \tau\langle V,B\rangle_L N + \langle V,N\rangle_L(\kappa T + \tau B) - \tau\langle V,N\rangle_L B - \tau\langle V,B\rangle_L N \\
&= \kappa\langle V,N\rangle_L T \\
&= \langle V,\kappa N\rangle_L T \\
&= \langle V,T'\rangle_L T
\end{aligned}
$$

as asserted. □

Theorem 4.1. *The mapping $\phi_{s_0,s} : U(s_0) \to U(s)$ is an isometry.*

Proof: Take any two vectors P_0 and Q_0 in $U(s_0)$ and let $P(s) = \phi_{s_0,s}(P_0)$ and $Q(s) = \phi_{s_0,s}(Q_0)$. Then applying the result of Lemma 4.1 we get

$$
\begin{aligned}
\langle P(s), Q(s)\rangle_L' &= \langle P', Q\rangle_L + \langle P, Q'\rangle_L \\
&= \langle \langle P,T'\rangle_L T, Q\rangle_L + \langle \langle Q,T'\rangle_L T, P\rangle_L \\
&= 0
\end{aligned}
$$

because $\langle T,P\rangle_L = 0$ and $\langle T,Q\rangle_L = 0$. Therefore $\phi_{s_0,s}$ preserves the inner product hence is an isometry. □

Definition 4.2. *Let $\alpha(s)$ be a regular timelike curve in $\mathbb{E}_1^3$ parameterized by its arc length, and let $T(s)$ denote its unit tangent. Arbitrarily fix a parameter value s_0 and let $T_0 = T(s_0)$. Choose any pair of vectors P_0 and Q_0 such that $\{T_0, P_0, Q_0\}$ forms and orthonormal set. Then let $P(s) = \phi_{s_0,s}(P_0)$ and $Q(s) = \phi_{s_0,s}(Q_0)$. According to Theorem 4.2, the triad $\{T(s), P(s), Q(s)\}$ is orthonormal for all s. We call $\{T(s), P(s), Q(s)\}$ a tubular surface triad for the timelike curve.*

Remark 4.1. *If the initial triad $\{T_0, P_0, Q_0\}$ is chosen such that it is positively oriented, i.e.,* $\det\{T_0, P_0, Q_0\} = 1$, *then the tubular surface triad will be positively oriented for all s. This is an immediate consequence of the continuity of solutions of the differential equations (4.1).*

The rate of change of the tube triad is expressed in equations akin to Frenet-Serret's:

Theorem 4.2. *Let $\{T(s), P(s), Q(s)\}$ be a positively oriented tubular surface triad. Then its rate of change is expressed by:*

$$
\left\{
\begin{aligned}
T' &= k_1 P + k_2 Q \\
P' &= k_1 T \\
Q' &= k_2 T
\end{aligned}
\right.
\tag{4.2}
$$

where $k_1(s)$ and $k_2(s)$ are scalar functions defined along the timelike curve.

Proof: The second and third equations in (4.2) are consequences of Lemma 4.1. To obtain the first equation we note that the positive orientation of the tubular surface triad implies $-T = P \wedge_L Q$ and $P = Q \wedge_L T$ and $Q = T \wedge_L P$. Therefore we have

$$\begin{aligned}
-T' &= P \wedge_L Q \\
-T' &= P' \wedge_L Q + P \wedge_L Q' \\
T' &= -P' \wedge_L Q - P \wedge_L Q' \\
T' &= -\langle P, T' \rangle_L T \wedge_L Q - P \wedge_L \langle Q, T' \rangle_L T \\
T' &= -\langle P, T' \rangle_L (T \wedge_L Q) - (P \wedge_L T) \langle Q, T' \rangle_L \\
T' &= -\langle P, T' \rangle_L (-P) - \langle Q, T' \rangle_L (-Q) \\
T' &= \langle P, T' \rangle_L P - \langle Q, T' \rangle_L Q \\
T' &= k_1 P + k_2 Q
\end{aligned}$$

as asserted. □

Remark 4.2. *Rather than computing both P and Q components of the tubular surface triad with $P(s) = \phi_{s_0,s}(P_0)$ and $Q(s) = \phi_{s_0,s}(Q_0)$ as in Definition 4.2, it is more practical to compute one, say $P(s)$, then define $Q(s)$ as $Q(s) = T(s) \wedge_L P(s)$, which also implies the permutations $P = Q \wedge_L T$ and $-T = P \wedge_L Q$. We can verify directly that Q, defined this way, satisfies the differential equation (4.1):*

$$\begin{aligned}
Q' &= (T \wedge_L P)' \\
&= T' \wedge_L P + T \wedge_L P' \\
&= T' \wedge_L P + T \wedge_L (\langle P, T' \rangle_L T) \\
&= T' \wedge_L P + (T \wedge_L T) \langle P, T' \rangle_L \\
&= T' \wedge_L P \\
&= T' \wedge_L (Q \wedge_L T) \\
&= -\langle T', T \rangle_L Q + \langle T', Q \rangle_L T \\
&= \langle Q, T' \rangle_L T
\end{aligned}$$

therefore $Q(s) = \phi_{s_0,s}(Q_0)$, as required. In this derivation we have made use of the fact that . We have also used the general vector algebra identities $a \wedge_L a = 0$ and $a \wedge_L (b \wedge_L c) = -\langle a, c \rangle_L b + \langle a, b \rangle_L c$.

The following theorem establishes bounds on the rate of change of the tubular surface triad:

Theorem 4.3. *Let $\{T, P, Q\}$ be the tubular surface triad as in Definition 4.2 and let κ be the curvature of the timelike curve. We have the following pointwise bounds on the rate of change of the triad:*

$$\begin{aligned}
&\|T'(s)\|_L = \kappa(s), \\
(4.3) \qquad &\|P'(s)\|_L \leq \kappa(s), \\
&\|Q'(s)\|_L \leq \kappa(s).
\end{aligned}$$

Proof: The first of the estimates (4.3) is merely the definition of curvature. To verify the second, we refer to equation (4.1):

$$\|P'(s)\|_L = |\langle P, T'\rangle_L| \ \|T\|_L \le \|P\|_L \, \|T'\|_L = \|T'\|_L = \kappa(s).$$

The third equation is verified in the same way. □

Unlike the Frenet-Serret system, the rate of change of the tubular surface triad $\{T, P, Q\}$ is bounded by the curvature but is independent of timelike curve's torsion. A tubular surface based on the tubular surface triad, i.e.,

$$Tube\,(s,t) = \alpha(s) + r\,(P(s)\cos t + Q(s)\sin t)$$

will have fewer twists and wrinkles in its skin compared to one based on the Frenet-Serret formulas as in (3.10).

Example 4.1. *Let $c^2 = a^2 - b^2 > 0$. For illustration, we compute the P and Q vectors of the tubular surface triad for the unit speed timelike helix:*

$$\alpha(s) = \left(a\sinh\frac{s}{c}, a\cosh\frac{s}{c}, \frac{bs}{c}\right) \tag{4.4}$$

with $a \neq 0$. For the corresponding Frenet–Serret triad is:

$$\begin{aligned}
T(s) &= \left(\frac{a}{c}\cosh\frac{s}{c}, \frac{a}{c}\sinh\frac{s}{c}, \frac{b}{c}\right)\\
N(s) &= \left(\sinh\frac{s}{c}, \cosh\frac{s}{c}, 0\right)\\
B(s) &= \left(\frac{b}{c}\cosh\frac{s}{c}, \frac{b}{c}\sinh\frac{s}{c}, \frac{a}{c}\right)
\end{aligned}$$

The curvature and torsion are constants, independent of s, and are given by

$$\begin{aligned}
\|T'\|_L &= \kappa = \frac{a}{a^2 - b^2}\ ,\\
\|B'\|_L &= \tau = \frac{b}{a^2 - b^2}.
\end{aligned}$$

Since torsion is constant, the system (4.1) reduces to a differential equation with constant coefficients which can be solved readily once a set of initial conditions is supplied. Suppose that we take $P(0) = B(0)$. Note that $P(0)$ is of unit length and orthogonal to $T(0)$, as required. Then

$$a(0) = \langle P(0), N(0)\rangle_L = 0$$

and

$$b(0) = \langle P(0), B(0)\rangle_L = 1.$$

Solving (4.1) with these initial conditions we obtain $a(s) = \langle P, N\rangle_L = \sin\tau s$ and $b(s) = \langle P, B\rangle_L = \cos\tau s$, whence, according to Definition 4.1:

$$P(s) = N(s)\sin\tau s + B(s)\cos\tau s,$$

where $N(s), B(s)$ and τ are given in terms of s by the explicit formulas above. The vector Q can also be computed in the same manner, or simply through $Q(s) = T(s) \wedge_L P(s)$. Since $T \wedge_L N = B$ and $T \wedge_L B = -N$, we get:

$$Q(s) = -N(s)\cos\tau s + B(s)\sin\tau s.$$

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