A structure by conformal transformations of Finsler functions on the projectivised tangent bundle of Finsler spaces with the Chern connection

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Abstract. It is shown that the projectivised tangent bundle of Finsler spaces with the Chern connection has a contact metric structure under a conformal transformation with certain condition of the Finsler function and moreover it is locally isometric to $E^m \times S^{m-1}(4)$ for $m > 2$ and flat for $m = 2$ if and only if the Cartan tensor vanishes, i.e., the Finsler space is a Riemannian manifold.

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1 Preliminaries

Let M be an m-dimensional C^∞ manifold and x^i $(1 \leq i \leq m)$ local coordinates on M. It is said to be a Finsler manifold if the length s of any curve $t \mapsto (x^1(t), \dots, x^m(t))$ $(a \leq t \leq b)$ is given by an integral

$$s = \int_a^b F\left(x^1(t), \dots, x^m(t), \frac{dx^1}{dt}, \dots, \frac{dx^m}{dt}\right) dt,$$

where F has the first-degree homogeneity with respect to $\frac{dx^i}{dt}$.

Our convention for indices is as follows: Latin indices run from 1 to m (except m). Greek indices run from 1 to m. Greek indices with bar run from 1 to $m-1$.

A Finsler manifold M has a tangent bundle $\pi : TM \to M$. From TM we obtain the projectivised tangent bundle of M, PTM, by identifying the non-zero vectors differing from each other by a real factor. Geometrically PTM is the space of line elements on M. Then a non-zero tangent vector can be expressed as

$$X = y^i \partial_{x^i} \quad (y^i \text{ not all zero}),$$

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where we set $\partial_{x^i} := \dfrac{\partial}{\partial x^i}$ and $\partial_{y^i} := \dfrac{\partial}{\partial y^i}$. The x^i, y^i are local coordinates on TM. They are also local coordinates on PTM with y^i being homogeneous coordinates (determined up to a real factor). We can consider PTM as the base manifold of the vector bundle p^*TM, pulled back with the canonical projection map $p : PTM \to M$ defined by $p(x^i, y^i) = (x^i)$. The fibers of p^*TM are the vector spaces of dimension m and the base manifold PTM is of dimension $2m-1$.

From now on f_{y^i}, $f_{y^i y^j}$, …, etc. denote the partial derivative(s) of a smooth function f with respect to the coordinates y^i. Adopt a similar notation for the partial derivatives with respect to the coordinates x^i. From the first-degree homogeneity of F, we have

$$y^i F_{y^i} = F \quad \text{and} \quad y^i F_{y^i y^j} = 0.$$

A differential form on PTM can be represented as one on TM provided the latter is invariant under rescaling in the y^i and yields zero when contracted with $y^i \partial_{y^i}$. Our differential forms on PTM will be so represented, and exterior differentiation on PTM will be obtained by formal differentiation on TM. Then the Hilbert form

$$\omega = F_{y^i} dx^i$$

is intrinsically define on PTM.

Let

$$e_\alpha = u_\alpha{}^j \partial_{x^j}$$

be an orthonormal frame field on the bundle p^*TM, and

$$\omega^\alpha = v^\alpha{}_k dx^k$$

its dual coframe field, so that

$$(e_\alpha, e_\beta) = u_\alpha{}^l g_{lk} u_\beta{}^k = \delta_{\alpha\beta} \tag{1.1}$$

and

$$(e_\alpha, \omega^\beta) = \delta^\beta{}_\alpha. \tag{1.2}$$

(1.1) is the orthonormality condition with respect to the Finsler metric (positive definite)

$$\begin{aligned} G &= g_{ij} dx^i \otimes dx^j \\ &= \left(\frac{1}{2}F^2\right)_{y^i y^j} dx^i \otimes dx^j \\ &= \left(F F_{y^i y^j} + F_{y^i} F_{y^j}\right) dx^i \otimes dx^j \end{aligned}$$

defined intrinsically on PTM, and (1.2) is the duality condition, which is equivalent

$$u_\alpha{}^k v^\beta{}_k = \delta^\beta{}_\alpha.$$

We now distinguish the global sections

$$e_m = \frac{y^i}{F}\partial_{x^i} =: \ell^i \partial_{x^i} \quad \text{and} \quad \omega^m = F_{y^i} dx^i = \omega.$$

Then, taking the exterior derivative of the Hilbert form ω^m on PTM, we have ([4])

$$d\omega^m = \omega^{\bar{\alpha}} \wedge \omega_{\bar{\alpha}}{}^m, \tag{1.3}$$

where $\omega_{\bar{\alpha}}{}^m$ is

$$\begin{aligned}\omega_{\bar{\alpha}}{}^m &= -u_{\bar{\alpha}}{}^i F_{y^i y^j} dy^j + \frac{u_{\bar{\alpha}}{}^i}{F}\left(F_{x^i} - y^j F_{y^i x^j}\right)\omega^m \\ &\quad + u_{\bar{\alpha}}{}^i u_{\bar{\beta}}{}^j F_{x^i y^j}\omega^{\bar{\beta}} + \lambda_{\bar{\alpha}\bar{\beta}}\omega^{\bar{\beta}} \qquad (\text{see [4] for } \lambda_{\bar{\alpha}\bar{\beta}}).\end{aligned}$$

Define $N^i{}_j$ and δy^j as follows:

$$N^i{}_j = \frac{1}{F} G^i_{y^j} \quad \text{and} \quad \delta y^j = \frac{dy^j}{F} + N^j{}_k dx^k,$$

where G^i denotes

$$G^i = g^{il}\left\{ y^s \left(\frac{1}{2}F^2\right)_{y^l x^s} - \left(\frac{1}{2}F^2\right)_{x^l} \right\}.$$

Then the orthonormal vectors in $T(TM\backslash 0)$ and the dual orthonormal vectors in $T^*(TM\backslash 0)$ are given by

$$\widehat{e}_\alpha = u_\alpha{}^j \delta_{x^j} \iff \omega^\alpha = v^\alpha_{\ j} dx^j$$

and

$$\widehat{e}_{m+\alpha} = u_\alpha{}^j \delta_{y^j} \iff \omega_m{}^\alpha = v^\alpha_{\ j}\delta y^j,$$

where

$$\delta_{x^i} := \partial_{x^i} - F N^j{}_i \partial_{y^j}$$

and

$$\delta_{y^i} := F\partial_{y^i}.$$

The set $\left\{\delta_{x^j}, \delta_{y^i}\right\}$ is naturally dual to the set $\left\{dx^i, \delta y^i\right\}$, and these form local bases for $T(TM\backslash\{0\})$ and $T^*(TM\backslash\{0\})$, respectively.

Generally a (2n+1)-dimensional manifold $\widetilde{M}$ is said to have a contact structure and is called a contact manifold if it carries a global 1-form η such that

$$\eta \wedge (d\eta)^n \neq 0 \tag{1.4}$$

everywhere on $\widetilde{M}$, where the exponent denotes the n-th exterior power. We call η a contact form of $\widetilde{M}$. A structure tensor (ϕ, ξ, η, g) on $(2n+1)$-dimensional manifold $\widetilde{M}$ said to be an almost contact metric structure if a tensor field of type (1,1) ϕ, a vector field ξ, a 1-form η and a Riemannian metric g satisfy

$$\eta(\xi) = 1, \quad \phi^2 = -I + \xi \otimes \eta, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \tag{1.5}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \quad rank\ \phi = 2n$$

for any $X, Y \in \chi(\widetilde{M})$, where $\chi(\widetilde{M})$ is the Lie algebra of vector fields on $\widetilde{M}$.

Let $\widetilde{M}$ be a $(2n+1)$-dimensional manifold with a contact form η. If $\widetilde{M}$ has an almost contact metric structure (ϕ, ξ, η, g) such that

$$g(\phi X, Y) = d\eta(X, Y), \tag{1.6}$$

then $\widetilde{M}$ is said to have a contact metric structure and is called a contact metric manifold, that is

$$\eta(\xi) = 1, \quad \phi^2 = -I + \xi \otimes \eta, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \tag{1.7}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi),$$

$$rank\ \phi = 2n, \quad g(\phi X, Y) = d\eta(X, Y)$$

for any $X, Y \in \chi(\widetilde{M})$.

Let $\widetilde{M}$ be a $(2m-1)$-dimensional contact metric manifold with a contact metric structure (ϕ, ξ, η, g) and R the curvature tensor field on $\widetilde{M}$. It is well known that the condition $R(X, Y)\xi = 0$ for all X, Y has a strong and interesting implication for a contact metric manifold, namely that $\widetilde{M}$ is locally the product of Euclidean space E^m and a sphere of constant curvature $+4$. D. E. Blair proved the following theorem.

Theorem 1.1. [*2, 3*] *A contact metric manifold $\widetilde{M}^{2m-1}$ satisfying $R(X, Y)\xi = 0$ is locally isometric to $E^m \times S^{m-1}(4)$ for $m > 2$ and flat for $m = 2$.*

The following proposition is well known (cf. [2], [3], [6]).

Proposition 1.2. *Let $\widetilde{M}$ be a contact metric manifold with a contact metric structure (ϕ, ξ, η, g). Then $\widetilde{M}$ is a K-contact manifold if and only if*

$$\nabla_X \xi = \phi X$$

for any $X \in \chi(\widetilde{M})$.

The following lemma is well known (cf. [4]).

Lemma 1.3. *The Hilbert form on PTM given by*

$$\omega^m = F_{y^i} dx^i = \omega$$

satisfies the condition $\omega \wedge (d\omega)^{m-1} \neq 0$, that is PTM has a contact structure with respect to Hilbert form ω.

Then S. S. Chern proved the following theorem.

Theorem 1.4. [4] *There exists a torsion-free and an almost metric-compatible linear connection $p^*TM \to PTM$, that is the Chern connection*

$$D: \quad \Gamma(p^*TM) \to \Gamma(p^*TM \otimes PTM)$$

given by

$$De_\alpha = {\omega_\alpha}^\beta e_\beta, \qquad {\omega_m}^m = 0,$$

that is $d\omega^\alpha = \omega^\beta \wedge {\omega_\beta}^\alpha$ *and*

$$\omega_{\alpha\beta} + \omega_{\beta\alpha} = -2A_{\alpha\beta\gamma}{\omega_m}^\gamma. \tag{1.8}$$

In particular

$${\omega_\alpha}^m + {\omega_m}^\alpha = 0, \tag{1.9}$$

where $\omega_{\alpha\beta} = {\omega_\alpha}^\gamma \delta_{\gamma\beta}$ *and the Cartan tensor* $A = A_{\alpha\beta\gamma}\omega^\alpha \otimes \omega^\beta \otimes \omega^\gamma$ *is given by*

$$A_{\alpha\beta\gamma} = \frac{F}{2}(\frac{1}{2}F^2)_{y^iy^jy^k}{u_\alpha}^i{u_\beta}^j{u_\gamma}^k.$$

Next we define the Chern connection in natural coordinates as follows:

$$D: \quad \Gamma(p^*TM) \to \Gamma(p^*TM \otimes T^*(TM\backslash 0))$$

given by

$$D\partial_{x^i} = {\omega_i}^j \partial_{x^j},$$

where ${\omega_i}^j$ are the components of the connection matrix in natural coordinates. Since the Chern connection is torsion-free, we can see that (see [1] and [4])

$$dx^i \wedge {\omega_i}^j = 0, \tag{1.10}$$

which is equivalent to the torsion-free condition of the Chern connection in natural coordinates. Wedge product of ${\omega_i}^j$ and dx^i is zero in (1.10), so they are linearly dependent. We can write ${\omega_i}^j$ in terms of dx^i as

$${\omega_i}^j = {\Gamma^j}_{il} dx^l,$$

where the quantities

$${\Gamma^i}_{jk} = \frac{g^{is}}{2}\left(\delta_{x^k} g_{sj} - \delta_{x^s} g_{jk} + \delta_{x^j} g_{ks}\right)$$

are called the Christoffel symbols of the first. Then we obtain

$${\Gamma^i}_{jk}\ell^j = {N^i}_k. \tag{1.11}$$

By using the Cartan formula, we obtain the following Lie bracket (cf. [1]):

$$[\delta_{x^k}, \delta_{y^l}] = \left\{ \dot{A}^i{}_{kl} + \frac{\ell^i}{F}(FF_{y^k})_{x^l} - \ell^i N_{kl} \right\} \delta_{y^i}, \tag{1.12}$$

where the quantities $\dot{A}^i{}_{kl}$ are

$$\dot{A}^i{}_{kl} := \left(\delta_{x^s} A^i{}_{kl} + A^h{}_{kl}\Gamma^i{}_{hs} - A^i{}_{hl}\Gamma^h{}_{ks} - A^i{}_{kh}\Gamma^h{}_{ls}\right)\ell^s.$$

On the other hand, by straightforward calculations we obtain

$$[\delta_{x^k}, \delta_{y^l}] = \frac{1}{2} G^i{}_{y^k y^l}\delta_{y^i} = \left\{ \dot{A}^i{}_{kl} + \Gamma^i{}_{kl} \right\} \delta_{y^i}. \tag{1.13}$$

On PTM, there are the quantities which are homogeneous of degree zero in the y^i. Let f be a smooth function on PTM. Using the Euler's theorem, we have

$$\ell^i \delta_{y^i} f = y^i f_{y^i} = 0. \tag{1.14}$$

From (1.11), (1.12), (1.13) and (1.14), it follows that

$$N^i{}_j \delta_{y^i} f = \ell^k \Gamma^i{}_{kj} \delta_{y^i} f = 0. \tag{1.15}$$

Then, by (1.15), we can see that the orthonormal vectors in $T(PTM)$ and the dual orthonormal vectors in $T^*(PTM)$ are given by

$$\widetilde{e}_\alpha = u_\alpha{}^j \partial_{x^j} \iff \omega^\alpha = v^\alpha{}_j dx^j \tag{1.16}$$

and

$$\widetilde{e}_{m+\bar{\alpha}} = u_{\bar{\alpha}}{}^j \delta_{y^j} \iff \omega_m{}^{\bar{\alpha}} = v^{\bar{\alpha}}{}_j \delta y^j. \tag{1.17}$$

2 Theorem

Now, let us consider the conformal transformation:

$$\overline{F} = e^{\sigma(x)} F, \tag{2.1}$$

of the fundamental function F, where $\sigma(x)$ is a local differentiable function on the base manifold M (cf. [5]).

With respect to (2.1) we have the conformal transformation:

$$\overline{g}_{ij} := \left(\frac{1}{2}\overline{F}^2\right)_{y^i y^j} = e^{2\sigma(x)} \left(\frac{1}{2}F^2\right)_{y^i y^j} =: e^{2\sigma(x)} g_{ij}, \tag{2.2}$$

of the fundamental tensor field.

On the manifold $TM\setminus\{0\}$ we locally define the tensor field :

$$g_{ij} dx^i \otimes dx^j + \overline{g}_{ij} \frac{\delta y^i}{F} \otimes \frac{\delta y^j}{F}. \tag{2.3}$$

For $\{\widetilde{e}_\alpha(\text{resp. } \omega^\alpha), \widetilde{e}_{m+\bar{\alpha}}(\text{resp. } {\omega_m}^{\bar{\alpha}})\}$ in $T(PTM)$ (resp. $T^*(PTM)$), we can rewrite it as

$$\delta_{\alpha\beta}\omega^\alpha \otimes \omega^\beta + e^{2\sigma(x)}\delta_{m+\bar{\alpha}\ m+\bar{\beta}}{\omega_m}^{\bar{\alpha}} \otimes {\omega_m}^{\bar{\beta}}. \tag{2.4}$$

We now distinguish the global sections

$$\bar{\widetilde{e}}_m := e^{-\sigma(x)}\widetilde{e}_m \quad \text{and} \quad \overline{\omega}^m := e^{\sigma(x)}\omega^m = e^{\sigma(x)}\omega \,(=: \overline{\omega})\,.$$

Putting $\overline{\omega}^{\bar{\alpha}} := \omega^{\bar{\alpha}}$, we locally define the tensor field:

$$\overline{g}^s = \delta_{\alpha\beta}\overline{\omega}^\alpha \otimes \overline{\omega}^\beta + e^{2\sigma(x)}\delta_{m+\bar{\alpha}\ m+\bar{\beta}}\omega_m^{\bar{\alpha}} \otimes \omega_m^{\bar{\beta}}. \tag{2.5}$$

We consider the following tensor field ϕ of (1,1) type:

$$\phi\widetilde{e}_{\bar{\alpha}} = -e^{-\sigma(x)}\widetilde{e}_{m+\bar{\alpha}}, \quad \phi\,\bar{\widetilde{e}}_m = 0 \quad \text{and} \quad \phi\widetilde{e}_{m+\bar{\alpha}} = e^{\sigma(x)}\widetilde{e}_{\bar{\alpha}}. \tag{2.6}$$

For the conformal transformation, we get the following theorem.

Theorem 2.1. *A structure tensor $(\phi, \bar{\widetilde{e}}_m, \overline{\omega}, \overline{g}^s)$ is an almost contact metric structure on PTM. Moreover $\overline{\omega}$ is a contact form on PTM and $(\phi, \bar{\widetilde{e}}_m, \overline{\omega}, \overline{g}^s)$ is a contact metric structure if and only if $\sigma(x)$ is a function satisfying $d\sigma = \omega^m$.*

Proof. It is evident that $\overline{\omega}(\bar{\widetilde{e}}_m) = 1$. From (1.16), (1.17) and (2.5) we have

$$\overline{g}^s(\bar{\widetilde{e}}_m, \bar{\widetilde{e}}_m) = \delta_{\alpha\beta}\overline{\omega}^\alpha \otimes \overline{\omega}^\beta(\bar{\widetilde{e}}_m, \bar{\widetilde{e}}_m) = \delta_{mm} = 1,$$

from which

$$\overline{g}^s(\bar{\widetilde{e}}_m, \bar{\widetilde{e}}_m) = \omega(\bar{\widetilde{e}}_m) = 1. \tag{2.7}$$

Using the argument similar to (2.7), we get

$$\overline{g}^s(\widetilde{e}_{\bar{\alpha}}, \bar{\widetilde{e}}_m) = \overline{\omega}(\widetilde{e}_{\bar{\alpha}}) = 0 \tag{2.8}$$

and

$$\overline{g}^s(\widetilde{e}_{m+\bar{\alpha}}, \bar{\widetilde{e}}_m) = \overline{\omega}(\widetilde{e}_{m+\bar{\alpha}}) = 0. \tag{2.9}$$

By (2.7), (2.8) and (2.9), we get

$$\overline{g}^s(X, \bar{\widetilde{e}}_m) = \overline{\omega}(X) \tag{2.10}$$

for any $X \in \chi(PTM)$.

From (2.6) we see that

$$\phi^2\widetilde{e}_{\bar{\alpha}} = -\phi e^{-\sigma(x)}\widetilde{e}_{m+\bar{\alpha}} = -\widetilde{e}_{\bar{\alpha}}, \ \phi^2\bar{\widetilde{e}}_m = 0$$

and

$$\phi^2\widetilde{e}_{m+\bar{\alpha}} = \phi e^{\sigma(x)}\widetilde{e}_{\bar{\alpha}} = -\widetilde{e}_{m+\bar{\alpha}}.$$

Then it follows that

$$\phi^2 X = -X + \overline{\omega}(X)\overline{\widetilde{e}}_m \tag{2.11}$$

for any $X \in \chi(PTM)$. Moreover, we get

$$\phi \longleftrightarrow \begin{bmatrix} 0 & \cdots & 0 & 0 & e^{\sigma(x)} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & e^{\sigma(x)} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ -e^{-\sigma(x)} & \ldots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -e^{-\sigma(x)} & 0 & 0 & \cdots & 0 \end{bmatrix},$$

from which we have

$$rank\ \phi = 2(m-1). \tag{2.12}$$

It is clear that $\overline{\omega}(\phi\,\overline{\widetilde{e}}_m) = 0$. Moreover we have

$$\overline{\omega}(\phi\widetilde{e}_{\bar{\alpha}}) = -e^{-\sigma(x)}\overline{\omega}^m(\widetilde{e}_{m+\bar{\alpha}}) = 0$$

and

$$\overline{\omega}(\phi\widetilde{e}_{m+\bar{\alpha}}) = e^{\sigma(x)}\overline{\omega}^m(\widetilde{e}_{\bar{\alpha}}) = e^{2\sigma(x)}\delta^m_{\bar{\alpha}} = 0.$$

It follows that

$$\overline{\omega}(\phi X) = 0 \tag{2.13}$$

for any $X \in \chi(PTM)$.

From (2.5), (2.6) and (2.8) we see that

$$\begin{aligned}\overline{g}^s(\phi\widetilde{e}_{\bar{\gamma}}, \phi\widetilde{e}_{\bar{\mu}}) &= e^{-2\sigma(x)}\overline{g}^s(\widetilde{e}_{m+\bar{\gamma}}, \widetilde{e}_{m+\bar{\mu}}) \\ &= \delta_{m+\bar{\alpha}\ m+\bar{\beta}}\ \overline{\omega}_m{}^{\bar{\alpha}} \otimes \overline{\omega}_m{}^{\bar{\beta}}(\widetilde{e}_{m+\bar{\gamma}}, \widetilde{e}_{m+\bar{\mu}}) \\ &= \delta_{m+\bar{\alpha}\ m+\bar{\beta}}\ \delta^{\bar{\alpha}}{}_{\bar{\gamma}}\delta^{\bar{\beta}}{}_{\bar{\mu}} = \ \delta_{m+\bar{\gamma}\ m+\bar{\mu}}.\end{aligned}$$

Since we have

$$\overline{g}^s(\widetilde{e}_{\bar{\gamma}}, \widetilde{e}_{\bar{\mu}}) = \delta_{\alpha\beta}\overline{\omega}^\alpha \otimes \overline{\omega}^\beta(\widetilde{e}_{\bar{\gamma}}, \widetilde{e}_{\bar{\mu}}) = \delta_{\bar{\gamma}\bar{\mu}},$$

we get

$$\overline{g}^s(\phi\widetilde{e}_{\bar{\gamma}}, \phi\widetilde{e}_{\bar{\mu}}) = \overline{g}^s(\widetilde{e}_{\bar{\gamma}}, \widetilde{e}_{\bar{\mu}}) - \overline{\omega}(\widetilde{e}_{\bar{\gamma}})\overline{\omega}(\widetilde{e}_{\bar{\mu}}). \tag{2.14}$$

Similarly we obtain

$$\overline{g}^s(\phi\widetilde{e}_{\bar{\gamma}}, \phi\widetilde{e}_{m+\bar{\mu}}) = \overline{g}^s(\widetilde{e}_{\bar{\gamma}}, \widetilde{e}_{m+\bar{\mu}}) - \overline{\omega}(\widetilde{e}_{\bar{\gamma}})\overline{\omega}(\widetilde{e}_{m+\bar{\mu}}) \tag{2.15}$$

and

(2.16) $$\overline{g}^s(\phi\widetilde{e}_{m+\bar{\gamma}}, \phi\widetilde{e}_{m+\bar{\mu}}) = \overline{g}^s(\widetilde{e}_{m+\bar{\gamma}}, \widetilde{e}_{m+\bar{\mu}}) - \overline{\omega}(\widetilde{e}_{m+\bar{\gamma}})\overline{\omega}(\widetilde{e}_{m+\bar{\mu}}).$$

By means of (2.14)∼(2.16) it follows that

(2.17) $$\overline{g}^s(\phi X, \phi Y) = \overline{g}^s(X, Y) - \overline{\omega}(X)\overline{\omega}(Y)$$

for any $X, Y \in \chi(PTM)$, so that we find that ϕ is skew-symmetric. Hence we see that a structure tensor $(\phi, \overline{\widetilde{e}}_m, \overline{\omega}, \overline{g}^s)$ is an almost contact metric structure on PTM.

From the exterior derivative of the form $\overline{\omega}$ on PTM, we see that

(2.18) $$d\overline{\omega} = d\left(e^{\sigma(x)}\omega^m\right) = de^{\sigma(x)} \wedge \omega^m + e^{\sigma(x)}\omega^{\bar{\alpha}} \wedge \omega_{\bar{\alpha}}{}^m,$$

from which

$$\overline{\omega} \wedge (d\overline{\omega})^{m-1} = e^{m\sigma(x)}\omega \wedge (d\omega)^{m-1} \neq 0.$$

Hence $\overline{\omega}$ is a contact form of PTM.

Using (1.9), (2.6) and (2.18) we have

$$\overline{g}^s(\phi\widetilde{e}_{\bar{\gamma}}, \widetilde{e}_{\bar{\mu}}) = -e^{-\sigma(x)}\overline{g}^s(\widetilde{e}_{m+\bar{\gamma}}, \widetilde{e}_{\bar{\mu}}) = 0$$

and

$$d\overline{\omega}(\widetilde{e}_{\bar{\gamma}}, \widetilde{e}_{\bar{\mu}}) = (de^{\sigma(x)} \wedge \omega^m - e^{\sigma(x)}\omega^{\bar{\alpha}} \wedge \omega_m{}^{\bar{\alpha}})(\widetilde{e}_{\bar{\gamma}}, \widetilde{e}_{\bar{\mu}}) = 0.$$

Thus we find that

(2.19) $$\overline{g}^s(\phi\widetilde{e}_{\bar{\gamma}}, \widetilde{e}_{\bar{\mu}}) = d\overline{\omega}(\widetilde{e}_{\bar{\gamma}}, \widetilde{e}_{\bar{\mu}}).$$

Using the similar techniques, we obtain

(2.20) $$\overline{g}^s(\phi\widetilde{e}_{m+\bar{\gamma}}, \widetilde{e}_{\bar{\mu}}) = d\overline{\omega}(\widetilde{e}_{m+\bar{\gamma}}, \widetilde{e}_{\bar{\mu}}),$$

(2.21) $$\overline{g}^s(\phi\widetilde{e}_{\bar{\gamma}}, \widetilde{e}_{m+\bar{\mu}}) = d\overline{\omega}(\widetilde{e}_{\bar{\gamma}}, \widetilde{e}_{m+\bar{\mu}})$$

and

(2.22) $$\overline{g}^s(\phi\widetilde{e}_{m+\bar{\gamma}}, \widetilde{e}_{m+\bar{\mu}}) = d\overline{\omega}(\widetilde{e}_{m+\bar{\gamma}}, \widetilde{e}_{m+\bar{\mu}}).$$

Using (2.5) we get

$$\overline{g}^s(\phi X, \overline{\widetilde{e}}_m) = 0.$$

On the other hand, by (2.18), we obtain

$$d\overline{\omega}(X, \overline{\widetilde{e}}_m) = Xe^{\sigma(x)} - \omega^m(X)\overline{\widetilde{e}}_m e^{\sigma(x)}.$$

By (2.19)∼(2.22), we get

(2.23) $$\overline{g}^s(\phi X, Y) = d\overline{\omega}(X, Y)$$

for any $X, Y \in \chi(PTM)$ if and only if

$$d\overline{\omega}(X, \overline{\widetilde{e}}_m) = 0,$$

or equivalently,

$$Xe^{\sigma(x)} = \omega^m(X)\overline{\widetilde{e}}_m e^{\sigma(x)} \iff de^{\sigma(x)} = \omega^m.$$

This proves the theorem.

□

We assume that $\sigma(x)$ is a function satisfying $d\sigma = \omega^m$. We calculate the Levi-Civita connection ∇ on PTM with respect to $\overline{g}^s$, which is given by

$$\begin{aligned}2\overline{g}^s(\nabla_X Y, Z) =& X(\overline{g}^s(Y,Z)) + Y(\overline{g}^s(X,Z)) - Z(\overline{g}^s(X,Y)) \\ &+\overline{g}^s([X,Y],Z) + \overline{g}^s([Z,X],Y) - \overline{g}^s([Y,Z],X)\end{aligned} \tag{2.24}$$

for any $X, Y, Z \in \chi(PTM)$.

Let f be a smooth function on PTM. By the definition of Lie bracket and

$$\omega_\alpha{}^\beta = v^\beta{}_i(du_\alpha{}^i + u_\alpha{}^j\omega_j{}^i) = v^\beta{}_i(du_\alpha{}^i + u_\alpha{}^j\Gamma^i{}_{jk}dx^k),$$

we get

$$\begin{aligned}\left[\widetilde{e}_{\bar\alpha}, \widetilde{e}_{\bar\beta}\right](f) =& \left[u_{\bar\alpha}{}^i\partial_{x^i}, u_{\bar\beta}{}^j\partial_{x^j}\right](f) \\ =& u_{\bar\alpha}{}^i u_{\bar\beta}{}^j\left[\partial_{x^i}, \partial_{x^j}\right](f) + u_{\bar\alpha}{}^i(\partial_{x^i}u_{\bar\beta}{}^j)\partial_{x^j}(f) - u_{\bar\beta}{}^j(\partial_{x^j}u_{\bar\alpha}{}^i)\partial_{x^i}(f) \\ =& \left(u_{\bar\alpha}{}^j(\partial_{x^j}u_{\bar\beta}{}^i) - u_{\bar\beta}{}^j(\partial_{x^j}u_{\bar\alpha}{}^i)\right)\partial_{x^i}(f) \\ =& \left(u_\gamma{}^i\omega_{\bar\beta}{}^\gamma(\widetilde{e}_{\bar\alpha}) - u_\gamma{}^i\omega_{\bar\alpha}{}^\gamma(\widetilde{e}_{\bar\beta})\right)v^\delta{}_i\widetilde{e}_\delta(f) \\ =& \left(\omega_{\bar\beta}{}^\gamma(\widetilde{e}_{\bar\alpha}) - \omega_{\bar\alpha}{}^\gamma(\widetilde{e}_{\bar\beta})\right)\widetilde{e}_\gamma(f),\end{aligned}$$

from which

$$\left[\widetilde{e}_{\bar\alpha}, \widetilde{e}_{\bar\beta}\right] = \left(\omega_{\bar\beta}{}^\gamma(\widetilde{e}_{\bar\alpha}) - \omega_{\bar\alpha}{}^\gamma(\widetilde{e}_{\bar\beta})\right)\widetilde{e}_\gamma. \tag{2.25}$$

Similarly, by straightforward calculations, using (1.16) and (1.17), we have the followings:

$$\left[\widetilde{e}_{\bar\alpha}, \widetilde{e}_{m+\bar\beta}\right] = \omega_{\bar\beta}{}^{\bar\gamma}(\widetilde{e}_{\bar\alpha})\widetilde{e}_{m+\bar\gamma} - \omega_{\bar\alpha}{}^\gamma(\widetilde{e}_{m+\bar\beta})\widetilde{e}_\gamma \tag{2.26}$$

and

$$\left[\widetilde{e}_{m+\bar\alpha}, \widetilde{e}_{m+\bar\beta}\right] = \left(\omega_{\bar\beta}{}^{\bar\gamma}(\widetilde{e}_{m+\bar\alpha}) - \omega_{\bar\alpha}{}^{\bar\gamma}(\widetilde{e}_{m+\bar\beta})\right)\widetilde{e}_{m+\bar\gamma}, \tag{2.27}$$

in particular

$$\begin{aligned}[\widetilde{e}_{\bar\alpha}, \widetilde{e}_m] =& -\omega_{\bar\alpha}{}^{\bar\gamma}(\widetilde{e}_m)\widetilde{e}_{\bar\gamma}, \\ [\widetilde{e}_m, \widetilde{e}_{m+\bar\alpha}] =& \omega_{\bar\alpha}{}^{\bar\gamma}(\widetilde{e}_m)\widetilde{e}_{m+\bar\gamma} - \widetilde{e}_{\bar\alpha}.\end{aligned}$$

Moreover, by the definition of Lie bracket, we get

$$\begin{aligned}\left[\widetilde{e}_{\bar\alpha}, \overline{\widetilde{e}}_m\right](f) =& \left[\widetilde{e}_{\bar\alpha}, e^{-\sigma(x)}\widetilde{e}_m\right](f) \\ =& e^{-\sigma(x)}\left[\widetilde{e}_{\bar\alpha}, \widetilde{e}_m\right](f) + (\widetilde{e}_{\bar\alpha}e^{-\sigma(x)})\widetilde{e}_m(f) \\ =& -e^{-\sigma(x)}\omega_{\bar\alpha}{}^{\bar\gamma}(\widetilde{e}_m)\widetilde{e}_{\bar\gamma}(f) - e^{-2\sigma(x)}(de^{\sigma(x)}(\widetilde{e}_{\bar\alpha}))\widetilde{e}_m(f),\end{aligned}$$

from which

$$\left[\widetilde{e}_{\bar{\alpha}}, \overline{\widetilde{e}}_m\right] = -e^{-\sigma(x)}\omega_{\bar{\alpha}}{}^{\bar{\gamma}}(\widetilde{e}_m)\widetilde{e}_{\bar{\gamma}}. \tag{2.28}$$

Similarly, by straightforward calculations we have

$$\left[\overline{\widetilde{e}}_m, \widetilde{e}_{m+\bar{\alpha}}\right] = e^{-\sigma(x)}\omega_{\bar{\alpha}}^{\;\bar{\gamma}}(\widetilde{e}_m)\widetilde{e}_{m+\bar{\gamma}} - e^{-\sigma(x)}\widetilde{e}_{\bar{\alpha}}. \tag{2.29}$$

Using (2.24)∼(2.29) and (1.8), we obtain

$$\begin{aligned} 2\overline{g}^s(\nabla_{\widetilde{e}_{m+\bar{\alpha}}}\widetilde{e}_{m+\bar{\beta}}, \widetilde{e}_{\bar{\gamma}}) &= \widetilde{e}_{m+\bar{\alpha}}(\overline{g}^s(\widetilde{e}_{m+\bar{\beta}}, \widetilde{e}_{\bar{\gamma}})) + \widetilde{e}_{m+\bar{\beta}}(\overline{g}^s(\widetilde{e}_{m+\bar{\alpha}}, \widetilde{e}_{\bar{\gamma}})) \\ &\quad -\widetilde{e}_{\bar{\gamma}}(\overline{g}^s(\widetilde{e}_{m+\bar{\alpha}}, \widetilde{e}_{m+\bar{\beta}})) \\ &\quad +\overline{g}^s([\widetilde{e}_{m+\bar{\alpha}}, \widetilde{e}_{m+\bar{\beta}}], \widetilde{e}_{\bar{\gamma}}) - \overline{g}^s([\widetilde{e}_{m+\bar{\beta}}, \widetilde{e}_{\bar{\gamma}}], \widetilde{e}_{m+\bar{\alpha}}) \\ &\quad +\overline{g}^s([\widetilde{e}_{\bar{\gamma}}, \widetilde{e}_{m+\bar{\alpha}}], \widetilde{e}_{m+\bar{\beta}}) \\ &= e^{2\sigma(x)}\left(\omega_{\bar{\beta}\bar{\alpha}}(\widetilde{e}_{\bar{\gamma}}) + \omega_{\bar{\alpha}\bar{\beta}}(\widetilde{e}_{\bar{\gamma}})\right) = -2e^{2\sigma(x)}A_{\bar{\alpha}\bar{\beta}\delta}\omega_m{}^{\delta}(\widetilde{e}_{\bar{\gamma}}) \\ &= 0. \end{aligned}$$

Moreover we get

$$\overline{g}^s(\nabla_{\widetilde{e}_{m+\bar{\alpha}}}\widetilde{e}_{m+\bar{\beta}}, \overline{\widetilde{e}}_m) = -e^{\sigma(x)}\delta_{\bar{\alpha}\bar{\beta}}$$

and

$$\overline{g}^s(\nabla_{\widetilde{e}_{m+\bar{\alpha}}}\widetilde{e}_{m+\bar{\beta}}, \widetilde{e}_{m+\bar{\gamma}}) = e^{2\sigma(x)}\left\{\omega_{\bar{\beta}\bar{\gamma}}(\widetilde{e}_{m+\bar{\alpha}}) + A_{\bar{\alpha}\bar{\beta}\bar{\gamma}}\right\}.$$

Thus we find that

$$\nabla_{\widetilde{e}_{m+\bar{\alpha}}}\widetilde{e}_{m+\bar{\beta}} = -e^{\sigma(x)}\delta_{\bar{\alpha}\bar{\beta}}\overline{\widetilde{e}}_m + \omega_{\bar{\beta}}^{\;\bar{\gamma}}(\widetilde{e}_{m+\bar{\alpha}})\widetilde{e}_{m+\bar{\gamma}} + A^{\bar{\gamma}}{}_{\bar{\alpha}\bar{\beta}}\widetilde{e}_{m+\bar{\gamma}}. \tag{2.30}$$

Using the similar techniques, we have

$$\nabla_{\widetilde{e}_{m+\bar{\alpha}}}\widetilde{e}_{\beta} = \omega_{\beta}{}^{\gamma}(\widetilde{e}_{m+\bar{\alpha}})\widetilde{e}_{\gamma} + A^{\gamma}{}_{\bar{\alpha}\beta}\widetilde{e}_{\gamma}, \tag{2.31}$$

$$\nabla_{\widetilde{e}_{\alpha}}\widetilde{e}_{m+\bar{\beta}} = A^{\gamma}{}_{\alpha\bar{\beta}}\widetilde{e}_{\gamma} + \omega_{\bar{\beta}}^{\;\bar{\gamma}}(\widetilde{e}_a)\widetilde{e}_{m+\bar{\gamma}} \tag{2.32}$$

and

$$\nabla_{\widetilde{e}_{\alpha}}\widetilde{e}_{\beta} = \omega_{\beta}{}^{\gamma}(\widetilde{e}_{\alpha})\widetilde{e}_{\gamma} - A^{\bar{\gamma}}{}_{\alpha\beta}\widetilde{e}_{m+\bar{\gamma}}. \tag{2.33}$$

From (2.24)∼(2.29) and (1.8), it follows that

$$\overline{g}^s(\nabla_{\widetilde{e}_{\bar{\alpha}}}\overline{\widetilde{e}}_m, \widetilde{e}_{\gamma}) = \overline{g}^s(\nabla_{\widetilde{e}_{\bar{\alpha}}}\overline{\widetilde{e}}_m, \widetilde{e}_{m+\bar{\gamma}}) = 0.$$

Thus we find that

$$\nabla_{\widetilde{e}_{\bar{\alpha}}}\overline{\widetilde{e}}_m = 0. \tag{2.34}$$

Using the similar techniques, we have

$$\nabla_{\widetilde{\overline{e}}_m}\widetilde{\overline{e}}_m = 0 \quad\text{and}\quad \nabla_{\widetilde{e}_{m+\bar{\alpha}}}\widetilde{\overline{e}}_m = e^{-\sigma(x)}\widetilde{e}_{\bar{\alpha}}. \tag{2.35}$$

From (2.35) it follows that

$$\nabla_X\widetilde{\overline{e}}_m = e^{-3\sigma(x)}\sum_{\bar{\alpha}}\overline{g}^s(X,\widetilde{e}_{m+\bar{\alpha}})\widetilde{e}_{\bar{\alpha}} \tag{2.36}$$

for any $X \in \chi(PTM)$.

From Proposition 1.2 and (2.36), we obtain the following theorem.

Theorem 2.2. *PTM has a non-K-contact, contact metric structure $(\phi, \widetilde{\overline{e}}_m, \overline{\omega}, \overline{g}^s)$ with respect to $\overline{g}^s$ satisfying $d\sigma = \omega^m$.*

Remark 2.3. *PTM gives us a example of non-K-contact, contact metric manifold with respect to $\overline{g}^s$ satisfying $d\sigma = \omega^m$.*

The curvature tensor filed $\overline{R}$ on PTM is given by

$$\overline{R}(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \tag{2.37}$$

for any $X, Y, Z \in \chi(PTM)$. From (2.36) and (2.37) it follows that

$$\begin{aligned}
&\overline{R}(X,Y)\widetilde{\overline{e}}_m \\
&= -3e^{-4\sigma(x)}\sum_{\bar{\alpha}}\left(\omega^m(X)\overline{g}^s(Y,\widetilde{e}_{m+\bar{\alpha}}) - \omega^m(Y)\overline{g}^s(X,\widetilde{e}_{m+\bar{\alpha}})\right)e_{\bar{\alpha}} \\
&+e^{-3\sigma(x)}\sum_{\bar{\alpha}}\left(\overline{g}^s(Y,\nabla_X\widetilde{e}_{m+\bar{\alpha}}) - \overline{g}^s(X,\nabla_Y\widetilde{e}_{m+\bar{\alpha}})\right)e_{\bar{\alpha}} \\
&+e^{-3\sigma(x)}\sum_{\bar{\alpha}}\{\overline{g}^s(Y,\widetilde{e}_{m+\bar{\alpha}})\nabla_X\widetilde{e}_{\bar{\alpha}} - \overline{g}^s(X,\widetilde{e}_{m+\bar{\alpha}})\nabla_Y\widetilde{e}_{\bar{\alpha}}\}
\end{aligned} \tag{2.38}$$

for any $X, Y \in \chi(PTM)$.

Setting $X = \widetilde{e}_\alpha$ and $Y = \widetilde{e}_\beta$ in (2.38), by (2.32), we get

$$\begin{aligned}
\overline{R}(\widetilde{e}_\alpha,\widetilde{e}_\beta)\widetilde{\overline{e}}_m &= e^{-3\sigma(x)}\sum_{\bar{\alpha}}\left(\overline{g}^s(\widetilde{e}_\beta,\nabla_{\widetilde{e}_\alpha}\widetilde{e}_{m+\bar{\alpha}}) - \overline{g}^s(\widetilde{e}_\alpha,\nabla_{\widetilde{e}_\beta}\widetilde{e}_{m+\bar{\alpha}})\right)\widetilde{e}_{\bar{\alpha}} \\
&\quad +e^{-3\sigma(x)}\sum_{\bar{\alpha}}\left\{\overline{g}^s(\widetilde{e}_\beta,\widetilde{e}_{m+\bar{\alpha}})\nabla_{\widetilde{e}_\alpha}\widetilde{e}_{\bar{\alpha}} - \overline{g}^s(\widetilde{e}_\alpha,\widetilde{e}_{m+\bar{\alpha}})\nabla_{\widetilde{e}_\beta}\widetilde{e}_{\bar{\alpha}}\right\} \\
&= e^{-3\sigma(x)}\sum_{\bar{\alpha}}\left(A^\gamma{}_{\alpha\bar{\alpha}}\delta_{\gamma\beta} - A^\gamma{}_{\beta\bar{\alpha}}\delta_{\gamma\alpha}\right)\widetilde{e}_{\bar{\alpha}} = 0.
\end{aligned}$$

Similarly, replacing X by $\widetilde{e}_\alpha$ and Y by $\widetilde{e}_{m+\bar{\alpha}}$ and using (2.30) and (2.32), we obtain

$$\overline{R}(\widetilde{e}_\alpha,\widetilde{e}_{m+\alpha})\widetilde{\overline{e}}_m = -e^{-\sigma(x)}A^{\bar{\gamma}}{}_{\alpha\bar{\alpha}}\widetilde{e}_{m+\bar{\gamma}}.$$

Also, setting $X = \widetilde{e}_{m+\bar{\alpha}}$ and $Y = \widetilde{e}_{m+\bar{\beta}}$ in (2.38), by (2.30) and (2.31), we have

$$\overline{R}(\widetilde{e}_{m+\bar{\alpha}},\widetilde{e}_{m+\bar{\beta}})\widetilde{\overline{e}}_m = 0.$$

Hence we obtain

$$\overline{R}(X,Y)\overline{\widetilde{e}}_m = -e^{-3\sigma(x)}\sum_{\alpha}\sum_{\bar{\beta}}\{\overline{g}^s(X,\widetilde{e}_\alpha)\overline{g}^s(Y,\widetilde{e}_{m+\bar{\beta}}) - \overline{g}^s(Y,\widetilde{e}_\alpha)\overline{g}^s(X,\widetilde{e}_{m+\bar{\beta}})\}\, A^{\bar{\gamma}}{}_{\alpha\bar{\beta}}\widetilde{e}_{\bar{\gamma}} \tag{2.39}$$

for all $X, Y \in \chi(PTM)$.

From Theorem 1.1 and (2.39), we obtain

Theorem 2.4. *A $(2m-1)$-dimensional contact metric manifold PTM with respect to $\overline{g}^s$ satisfying $d\sigma = \omega^m$ is locally isometric to $E^m \times S^{m-1}(4)$ for $m > 2$ and flat for $m = 2$ if and only if the Cartan tensor $A = 0$, i.e., M is a Riemannian manifold.*

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