Cauchy atlas on the manifold of all maximal solutions of an ODE system

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Abstract

In this paper we find the necessary and sufficient conditions that a family of functions to represent a change of coordinates in a Cauchy atlas over the manifold of all maximal solutions of an ODE system. The proof is constructive. The case of autonomous and the case of linear ODE system are discussed separately. The relation to the Sincov functional equation is clarified.

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1 Introduction

It is well-known that the set of all maximal solutions of Cauchy problems attached to homogeneous linear ODE system is isomorphic to $\mathbb{R}^n$. Also, an autonomous ODE system generates a local group with one parameter of diffeomorphisms. Our aim is to organize the set of all maximal solutions for Cauchy problems attached to a first-order non-autonomous ODE system, with n equations and n unknown functions, as a manifold of dimension n.

Section 2 proves the existence of a Cauchy atlas on the set of all maximal solutions of Cauchy problems attached to a first-order non-autonomous ODE system. For that we emphasize five necessary and sufficient conditions that must be satisfied by a family of functions $F = \{F_{\tau\sigma}\}_{\tau,\sigma\in\mathbb{R}}$ in order to be the coordinate transformations in a canonical atlas. The necessity is obtained from ODE theory (see, e.g., [2], [4], [7]), whereas the sufficiency is presented here, as far as the authors know, for the first time.

Sections 3, 4 and 5 contain special cases of first-order ODE systems. More precisely, we discuss here autonomous equations, linear non-homogeneous equations, and linear constant coefficient equations. Section 6 contains examples.

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2 The Cauchy atlas

Let M be a set, n a natural number, and I an index set. The set M is a differentiable manifold of dimension n if [3]:

(i) M is provided with a family of pairs $\{M_\tau, \varphi_\tau\}$, $\tau \in I$;

(ii) $\{M_\tau\}$ is a family of sets which cover M, i.e., $\bigcup_{\tau\in I} M_\tau = M$; each φ_τ is a bijection $\varphi_\tau : M_\tau \to U_\tau$ from M_τ to an open subset U_τ of $\mathbb{R}^n$;

(iii) given M_ρ, M_τ such that $M_\rho \cap M_\tau \neq \emptyset$, the map $\varphi_\tau \circ {\varphi_\rho}^{-1}$ from the subset $\varphi_\rho(M_\rho \cap M_\tau)$ of $\mathbb{R}^n$ to the subset $\varphi_\tau(M_\rho \cap M_\tau)$ of $\mathbb{R}^n$ is a C^∞ diffeomorphism.

The family $\{M_\tau, \varphi_\tau\}$ satisfying (i), (ii), (iii) is called a C^∞ *atlas*. The individual members (M_τ, φ_τ) of this family are called *coordinate charts*. The C^∞ map $\varphi_\tau \circ {\varphi_\rho}^{-1}$ is called a *change of coordinates*.

Let $X(\tau, x)$ be a τ-dependent C^∞ vector field on $\mathbb{R}^n$. More precisely, X is a C^∞ function, where $\mathrm{Dom}(X)$ is a non-empty open subset of $\mathbb{R} \times \mathbb{R}^n$ and $\mathrm{Codom}(X) = \mathbb{R}^n$. We attach the Cauchy problem

$$\frac{dx}{d\tau}(\tau) = X(\tau, x(\tau)), \quad x(\rho) = a, \tag{2.1}$$

where $(\rho, a) \in \mathrm{Dom}(X)$.

Let M be the set of all maximal C^∞ solutions of these Cauchy problems (the maximal solutions are points in M). To organize M like a differentiable manifold of dimension n, it is enough to build some mathematical ingredients satisfying (i)-(iii).

First, for each $\tau \in \mathbb{R}$, we define a map φ_τ as follows:

- $M_\tau = \{x \in M \mid \tau \in \mathrm{Dom}(x)\}$;
- $U_\tau = \{a \in \mathbb{R}^n \mid (\tau, a) \in \mathrm{Dom}(X)\}$;
- φ_τ maps a maximal solution x to the initial value $x(\tau)$.

The bijectivity of φ_τ follows from the existence and uniqueness of maximal solution of a Cauchy problem.

Second, we introduce the family of functions $F_{\tau\rho} = \varphi_\tau \circ {\varphi_\rho}^{-1}$

$$F_{\tau\rho}\colon \varphi_\rho(M_\tau \cap M_\rho) \ni a \mapsto \varphi_\tau({\varphi_\rho}^{-1}(a)) \in \varphi_\tau(M_\tau \cap M_\rho). \tag{2.2}$$

Theorem. *The maps $F_{\tau\rho}$ have the following properties:*

1) $K = \{(\tau, \rho, a) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \mid a \in \mathrm{Dom}(F_{\tau\rho})\}$ *is a non-empty open set;*

2) *each function from the family $F = \{F_{\tau\rho}\}$ is a bijection;*

3) *the map $K \ni (\tau, \rho, a) \mapsto F_{\tau\rho}(a) \in \mathbb{R}^n$ is of class C^∞;*

4) $J(\rho, a) = \{\tau \in \mathbb{R} \mid a \in \mathrm{Dom}(F_{\tau\rho})\}$ *is an open interval for each $\rho \in \mathbb{R}$, $a \in \mathbb{R}^n$;*

5) *the relation*

$$F_{\tau\sigma}(F_{\sigma\rho}(a)) = F_{\tau\rho}(a) \tag{2.3}$$

holds for each $\tau, \sigma, \rho \in \mathbb{R}$, $a \in {F_{\sigma\rho}}^{-1}(\mathrm{Codom}(F_{\sigma\rho}) \cap \mathrm{Codom}(F_{\sigma\tau}))$.

Proof. The map φ_τ is a coordinate chart on M and the C^∞ maps $F_{\tau\rho} = \varphi_\tau \circ {\varphi_\rho}^{-1}$, $\tau, \rho \in \mathbb{R}$ work as change of coordinates.

From (2) the map

$$x\colon J(\rho, a) \ni \tau \to F_{\tau\rho}(a) \in \mathbb{R}^n \tag{2.4}$$

is the maximal solution of the Cauchy problem (1). In such a case, the domain of the map in the statement 3) is open (see [7, Chapter 4, §23]). The statement 1) follows from the non-emptiness of $\mathrm{Dom}(X)$. The statement 2) is obtained from the definition of $F_{\tau\rho}$. The statement 3) follows from [2, Chapter 4, §32]. Since $J(\rho, a)$ is the domain of the maximal solution (4), the statement 4) follows from the openness of $\mathrm{Dom}(X)$. The statement 5) is coming from the definition of change of coordinates.□

Let $F = \{F_{\tau\rho}\}_{\tau,\rho\in\mathbb{R}}$ be a family of C^∞ functions, where the domain and the codomain of each function from F are subsets of $\mathbb{R}^n$. The following Theorem gives the necessary and sufficient conditions for F to represent the change of coordinates in a C^∞ atlas of the form $\{M_\tau, \varphi_\tau\}_{\tau\in\mathbb{R}}$.

Theorem. *For each family of functions $F = \{F_{\tau\sigma}\}_{\tau,\sigma\in\mathbb{R}}$ satisfying the properties* 1)-5) *from the previous Theorem there exists a C^∞ vector field $X(\tau, x)$ generating a manifold M of all maximal solutions such that F is a family of coordinate transformations of the $\{M_\tau, \varphi_\tau\}_{\tau\in\mathbb{R}}$ atlas on M.*

Proof. Assume that F is a family of coordinate transformations of the $\{M_\tau, \varphi_\tau\}_{\tau\in\mathbb{R}}$ atlas on M. Then the statements 1)-5) from Theorem 1 are satisfied.

Let us prove the sufficiency. From (3) we get $F_{\sigma\rho}(F_{\rho\sigma}(a)) = F_{\sigma\sigma}(a)$ for each $a \in \mathrm{Dom}(F_{\rho\sigma})$. If we put $\rho = \sigma$, then from the statement 2) of Theorem 1 we obtain $a = F_{\sigma\sigma}(a)$ for each $a \in \mathrm{Dom}(F_{\sigma\sigma})$. Since $F_{\sigma\sigma}$ is the identity and

$$\mathrm{Dom}(F_{\rho\sigma}) \subseteq \mathrm{Dom}(F_{\sigma\sigma}), \tag{2.5}$$

we obtain

$$F_{\rho\sigma} = {F_{\sigma\rho}}^{-1} \tag{2.6}$$

for each $\rho, \sigma \in \mathbb{R}$.

The family F defines the vector field X from the Cauchy problem (1):

$$X(\tau, a) = \frac{\partial F_{\tau\sigma}}{\partial \tau}(a)\bigg|_{\sigma=\tau}. \tag{2.7}$$

From the statement 3) of Theorem 1 we see that the function X is of class C^∞.

Let $(\rho, a) \in \mathrm{Dom}(X)$ be fixed. Then $a \in \mathrm{Dom}(F_{\rho\rho})$. Moreover, $J(\rho, a)$ is non-empty according to the statement 4) of Theorem 1. Let $\sigma \in J(\rho, a)$. Then $a \in \mathrm{Dom}(F_{\sigma\rho})$. Let $\tau \in J(\sigma, F_{\sigma\rho}(a))$. This set is non-empty, since from (2.5), (2.6) we have $\sigma \in J(\sigma, F_{\sigma\rho}(a))$. From (2.6) $a \in {F_{\sigma\rho}}^{-1}(\mathrm{Codom}(F_{\sigma\rho}) \cap \mathrm{Codom}(F_{\sigma\tau}))$. Therefore the condition (3) holds for such a. Differentiating (3) with respect to τ, putting $\sigma = \tau$ and using (7) we get

$$\frac{\partial F_{\tau\rho}}{\partial \tau}(a) = X(\tau, F_{\tau\rho}(a)).$$

That is why the map x defined by (4) is the solution of the Cauchy problem (1). We must check that x is a maximal solution. Let us suppose x is not maximal. Then there exists a maximal solution $\bar{x}$ such that $x = \bar{x}|_{J(\rho,a)}$. At least one of the values $\sup(J(\rho,a)), \inf(J(\rho,a))$ is an element of $\mathrm{Dom}(\bar{x})$. Let us suppose $\omega = \sup(J(\rho,a)) \in \mathrm{Dom}(\bar{x})$ (the case $\inf(J(\rho,a)) \in \mathrm{Dom}(\bar{x})$ is analogous). Then $(\omega, \bar{x}(\omega)) \in \mathrm{Dom}(X)$. Further, from (2.7), we obtain $\bar{x}(\omega) \in \mathrm{Dom}(F_{\omega\omega})$. Therefore $(\omega, \omega, \bar{x}(\omega)) \in K$. By the statement 1) of Theorem 1, there exists $\varepsilon > 0$ such that, for each σ from an open interval $(\omega - \varepsilon, \omega)$, we have $(\omega, \sigma, \bar{x}(\sigma)) = (\omega, \sigma, x(\sigma)) = (\omega, \sigma, F_{\sigma\rho}(a)) \in K$. Using (2.6) we get $a \in {F_{\sigma\rho}}^{-1}(\mathrm{Codom}(F_{\sigma\rho}) \cap \mathrm{Codom}(F_{\sigma\omega}))$. From (3) we obtain $F_{\omega\sigma}(F_{\sigma\rho}(a)) = F_{\omega\rho}(a)$. Therefore $\sup(J(\rho,a)) = \omega \in J(\rho,a)$. Nevertheless, from the statement 4) of Theorem 1, $J(\rho,a)$ is an open set. This contradiction proves that x is the maximal solution. Thus for each Cauchy problem (1) with $(\rho,a) \in \mathrm{Dom}(X)$, the unique maximal solution is given by (2.4). Then we can construct the $\{M_\tau, \varphi_\tau\}_{\tau\in\mathbb{R}}$ atlas and $F_{\tau\rho}$ identifies to $\varphi_\tau \circ {\varphi_\rho}^{-1}$ as maximal solutions of the same Cauchy problem. From (4), (6) and from the statement 4) of Theorem 1, the condition (2) holds. □

The C^∞ atlas on M defined by the conditions in the previous Theorems will be called *Cauchy atlas.*

Corollary. *The set M of all maximal solutions of a non-autonomous ODE system* (1) *is a manifold of dimension n.*

3 Case of autonomous ODE system

Let $U \subset \mathbb{R}^n$ be an open set. Let X be a C^∞ function with $\mathrm{Dom}(X) = \mathbb{R} \times U$, where $X\colon (\tau, a) \mapsto \xi(a)$ and $\xi\colon U \to \mathbb{R}^n$. Then the map $J(\rho,a) \ni \tau \mapsto F_{\tau-\rho,0}(a) \in \mathbb{R}^n$ is the maximal solution of the Cauchy problem (1), satisfying the same Cauchy condition as the solution (2.4). Denoting $G_\tau = F_{\tau 0}$, we have

$$G_{\tau-\rho} = F_{\tau\rho}. \tag{3.8}$$

From (3), (3.8) we obtain

$$G_\alpha(G_\beta(a)) = G_{\alpha+\beta}(a)$$

for each $a \in G_\beta^{-1}(\mathrm{Codom}(G_\beta) \cap \mathrm{Codom}(G_{-\alpha}))$. The map $G\colon (\alpha, a) \mapsto G_\alpha(a)$, where $\mathrm{Dom}(G) = \{(\alpha, a) \in \mathbb{R} \times \mathbb{R}^n \mid a \in \mathrm{Dom}(G_\alpha)\}$, is the *maximal flow* of the vector field ξ (see, e.g., [5, Chapter 17]). The maps G_τ form a *local one-parameter group* of transformations (see, e.g., [6, Section 1.2]). If $\mathrm{Dom}(G_\tau) = U$ for each $\tau \in \mathbb{R}$, then G_τ's form a *group of transformations* of U.

Corollary. *The set M of all maximal solutions of an autonomous ODE system* (1) *is a manifold of dimension n.*

4 Case of linear ODE system

Let I be an open interval. Let us consider the affine functions $F_{\tau\sigma}$, where

$$\mathrm{Dom}(F_{\tau\sigma}) = \begin{cases} \mathbb{R}^n & \text{for } \tau, \sigma \in I, \\ \emptyset & \text{otherwise.} \end{cases}$$

The condition (3) was suggested by the *Sincov's functional equation* (see [1, section 8.1])

$$F_{\tau\sigma} \circ F_{\sigma\rho} = F_{\tau\rho},\ \tau, \sigma, \rho \in I,$$

with the general solution

$$F_{\tau\sigma}(a) = W_\tau(W_\sigma^{-1}(a) + h_\tau - h_\sigma),$$

where $W_\tau \colon \mathbb{R}^n \to \mathbb{R}^n$ is an arbitrary linear automorphism and h_τ is an arbitrary element of $\mathbb{R}^n$ for each $\tau \in I$. If the conditions from Theorems are satisfied, then the vector field $X(\tau, x)$ is also affine and $\mathrm{Dom}(X) = I \times \mathbb{R}^n$. Moreover, $\tau \mapsto W_\tau$ is the *Wronski matrix* and $\tau \mapsto W_\tau h_\tau$ is the *particular solution* of this equation.

Corollary. *The set M of all maximal solutions of an affine ODE system* (1) *is an affine manifold of dimension n.*

5 Case of linear constant coefficient ODE system

Let the map $G_\tau \colon \mathbb{R}^n \to \mathbb{R}^n$ defined by (3.8) be affine for each $\tau \in \mathbb{R}$. We can rewrite the condition (3) as

$$G_\alpha \circ G_\beta = G_{\alpha+\beta}. \tag{5.9}$$

Therefore G_τ's form a group of affine transformations of $\mathbb{R}^n$. Let us suppose the map $\beta \mapsto G_\beta$ is continuous. We define

$$H \colon \varepsilon \mapsto \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} G_\beta \, d\beta.$$

Since $\lim_{\varepsilon\to 0} H_\varepsilon = \mathrm{id}_{\mathbb{R}^n}$, from continuity, there exists $\varepsilon > 0$ such that H_ε is invertible. By integrating (5.9) and substituting $\gamma = \alpha + \beta$ we obtain

$$G_\alpha = \frac{1}{2\varepsilon} \int_{\alpha-\varepsilon}^{\alpha+\varepsilon} G_\gamma \circ H_\varepsilon^{-1} \, d\gamma.$$

From this $\alpha \mapsto G_\alpha$ is a C^1 map and from (2.7), (3.8) the equation (1) is linear nonhomogeneous with constant coefficients. From Theorems we see that functions from the family $F = \{G_{\tau-\sigma}\}_{\tau,\sigma\in\mathbb{R}}$ are the coordinates transformations of the Cauchy atlas on the manifold M of the maximal solutions of Cauchy problems attached to a linear nonhomogeneous ODEs system with constant coefficients.

Corollary. *The set M of all maximal solutions of a linear constant coefficient ODE system* (1) *is a linear manifold of dimension n.*

6 Examples

1) Let us consider the Cauchy problem (1), where $X \colon \mathbb{R}^2 \ni (\tau, b) \mapsto b^2 \in \mathbb{R}$. It is easy to see that

$$F_{\tau\sigma}(a) = \frac{a}{1 + (\sigma - \tau)a},$$

where $\mathrm{Dom}(F_{\tau\sigma}) = \{a \in \mathbb{R} \mid (\tau - \sigma)a < 1\}$, $\mathrm{Codom}(F_{\tau\sigma}) = \{a \in \mathbb{R} \mid (\tau - \sigma)a > -1\}$. The map $G\colon (\tau, a) \mapsto a/(1 - \tau a)$ is the maximal flow of the vector field $\xi\colon b \mapsto b^2$.

2) The Cauchy problem (1), where $X\colon \mathbb{R}^2 \ni (\tau, b) \mapsto 2\tau b \in \mathbb{R}$, has the maximal solution $x\colon \mathbb{R} \ni \tau \mapsto a e^{\tau^2 - \rho^2} \in \mathbb{R}$. This determines a Cauchy atlas on the set M.

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