Some properties of a closed concircular almost contact manifold

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Abstract

We consider a skew symetric conformal vector field on a closed concircular almost contact manifold M and find its properties. Also, for a CR-product submanifold M' of M, the mean curvature vector field of the invariant submanifold and the flatness of the antiinvariant submanifold are studied, under the assumption that M' admits a skew symmetric Killing vector field tangent to the invariant submanifold, such that its generative is tangent to the antiinvariant submanifold.

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Introduction

Closed concircular almost contact manifolds $M(\Phi, \Omega, \eta, \xi, U, g)$ have been defined in [4]. For such a manifold, the Reeb vector field ξ satisfies two properties:
i) ξ is *concircular*, i.e. $\nabla\xi = \eta \otimes U$,
and
ii) the dual 1-form $U^\flat$ of U is closed,
where $U = \nabla_\xi \xi$.

In the present paper, it is first proved that the existence of an *horizontal* vector field C (i.e. $\eta(C) = 0$), which is skew symmetric conformal, is determined by an exterior differential system $\sum$ in involution (in the sense of E. Cartan). Since the structure 2-form Ω is purely symplectic (i.e. $\Omega^m \wedge \eta \neq 0, d\Omega = 0$), one may formulate the following properties:

i) C defines a weak automorphism of Ω and ΦC an infinitesimal automorphism of Ω;

ii) C and ΦC commute and C is exterior quasi concurrent;

iii) M is foliated by 3-codimensional submanifolds N and the immersion $x : N \longrightarrow M$ is of 1-geodesic index and 2-umbilical index;

iv) the following relation holds good

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$$2\mathcal{R}(Z,Z') = \left[(a-b)||U||^2 - (2m-1)\lambda\right] g(Z,Z') + bg(U,Z)g(U,Z') +$$

$$+a(\lambda + ||U||^2)g(\xi,Z)g(\xi,Z') + 2c\eta(Z)g(U,Z'),$$

where $\mathcal{R}$ is the Ricci tensor, Z, Z' are any vector fields and $a, b \in \wedge^\circ M, c = \text{const.}$

In the next section, we consider a CR-product M' of a closed concircular almost contact manifold M, i.e. $M' = M^\top \times M^\perp$, where $M^\top$(respective $M^\perp$) is the *invariant* submanifold (respective *antiinvariant* submanifold) of M. Then, if M' carries a mixed skew symmetric vector field X, it follows that the curvature vector field $H^\top$ of M in $M^\perp$ is, up to $-\frac{1}{2}$, equal to the generative V of X.

If the skew symmetric Killing vector fields X and Y are orthogonal, then Y defines an *infinitesimal conformal transformation* of X.

The following result is proved:

Theorem. *Let X be a skew symmetric Killing vector field of the antiinvariant submanifold $M^\perp$ of the CR-product submanifold $M' = M^\top \times M^\perp$.*

If the generative V of X is a closed torse forming, then the submanifold $M^\perp$ is flat.

1 Preliminaries

Let (M, g) be a n-dimensional oriented Riemannian manifold and let ∇ be the covariant differential operator defined by the metric tensor g.

Let ΓTM be the set of sections of the tangent bundle TM and $\flat : TM \longrightarrow T^*M, \# : T^*M \longrightarrow TM$ be the *musical isomorphisms* defined by g.

Following [10], we set $A^q(M, TM) = \Gamma Hom(\Lambda^q TM, M)$ and notice that the elements of $A^q(M, TM)$ are vector valued q-forms.

Denote by $d^\nabla : A^q(M, TM) \longrightarrow A^{q+1}(M, TM)$ the exterior covariant derivative operator with respect to ∇. If $p \in M$, then the vector valued 1-form of M, $dp \in A^1(M, TM)$, is the canonical vector valued 1-form of M and is called the *soldering form* [2].

Let $\mathcal{O} = \text{vect}\{e_{\bar{A}} \mid \bar{A} = 1, ..., n\}$ be a local field of adapted vectorial frames over M and let $\mathcal{O}^* = \text{covect}\{\omega^{\bar{A}}\}$ its associated coframe. Then the soldering form dp is expressed by

$$dp = \omega^{\bar{A}} \otimes e_{\bar{A}}, \tag{1.1}$$

and E. Cartan's structure equations written in the indexless manner are

$$\nabla e = \theta \otimes e, \tag{1.2}$$

$$d\omega = -\theta \wedge \omega, \tag{1.3}$$

$$d\theta = -\theta \wedge \theta + \Theta. \tag{1.4}$$

In the above equations, θ (respective Θ) are the local *connection forms* in the tangent bundle TM (respective the *curvature 2-forms* of M). If M is endowed with a 2-form Ω, then we can define the morphism $\Omega^\flat : TM \longrightarrow T^*M$,

$$\Omega^\flat(Z) = {}^\flat Z = -i_Z\Omega, Z \in \Gamma TM, \tag{1.5}$$

where $i_Z\Omega(X) = \Omega(Z, X)$.

If $\mathcal{T}$ is a conformal vector field, then $\mathcal{T}$ satisfies

$$\mathcal{L}_{\mathcal{T}} g = \rho g \quad \text{or} \quad g(\nabla_Z \mathcal{T}, Z') + g(\nabla_{Z'} \mathcal{T}, Z) = \rho g(Z, Z'), \tag{1.6}$$

where the conformal scalar ρ is defined by

$$\rho = \frac{2}{\text{dim}M}(\text{div}\mathcal{T}). \tag{1.7}$$

Ørsted's lemma is expressed by

$$\mathcal{L}_{\mathcal{T}} Z^\flat = \rho Z^\flat + [\mathcal{T}, Z]^\flat, Z \in \Gamma TM. \tag{1.8}$$

Moreover, one has [14]

$$\mathcal{L}_{\mathcal{T}} K = (n-1)\Delta\rho - K\rho, \tag{1.9}$$

$$2\mathcal{L}_{\mathcal{T}}\mathcal{R}(Z, Z') = (\Delta\rho)g(Z, Z') - (n-2)(Hess_\nabla \rho)(Z, Z'), \tag{1.10}$$

where

$$(Hess_\nabla \rho)(Z, Z') = g(Z, H_\rho Z'); H_\rho Z' = \nabla_{Z'}(\nabla\rho), \tag{1.11}$$

$Z, Z' \in \Gamma TM, \nabla f = gradf, f \in \Lambda^\circ(M)$.

2 Skew-symmetric conformal vector fields on a closed concircular almost contact manifolds

Let $M(\Phi, \Omega, \eta, \xi, U, g)$ be a $(2m+1)$-dimensional *closed concircular almost contact* manifold, defined in [4]. The structure tensors on M satisfy

$$\begin{cases} \Phi^2 = -Id + \eta \otimes \xi, \eta \wedge \Omega^m \neq 0, \\ g(\Phi Z, \Phi Z') = g(Z, Z') - \eta(Z)\eta(Z') \\ \nabla\xi = \eta \otimes U, d\Omega = 0, d\eta = U^\flat \wedge \eta \\ \nabla U = \lambda dp - (\lambda + ||U||^2)\eta \otimes \xi, dU^\flat = 0, \lambda = \text{const.} \end{cases} \tag{2.1}$$

The vector fields ξ and U are called the *Reeb vector field* and its *generative*, respectively.

In the present paper we assume that M carries a skew symmetric conformal vector field C [11], i.e.

$$\nabla C = \rho dp + C \wedge U, \; \mathcal{L}_C g = \rho g, \tag{2.2}$$

where $\rho = \dfrac{2divC}{dimM}$ is the *conformal scalar* associated with C and dp the *soldering form* [2] of M (i.e. the basic vector valued 1-form).

If $\mathcal{O} = \{e_A, \xi\}$ is an orthonormal vector basis on M and $\mathcal{O}^* = \{\omega^A, \eta\}$ its associated cobasis, dp is expressed by

$$dp = \omega^A \otimes e_A + \eta \otimes \xi, A \in \{1, ..., 2m\}. \tag{2.3}$$

If Z is any vector field, we recall that Ørsted's lemma is expressed by

$$\mathcal{L}_C Z^\flat = \rho Z^\flat + [C, Z]^\flat, \tag{2.3}$$

for any conformal vector field C). Asumming that C is an horizontal vector field (i.e. $\eta(C) = 0$), one derives by the third equation (2.1)

$$\mathcal{L}_C \eta = -\alpha \Longrightarrow d(\mathcal{L}_C \eta) = 2\eta \wedge \mathcal{L}_C \eta, \tag{2.4}$$

which shows that $\mathcal{L}_C \eta$ is an *exterior recurrent form* [3], having 2η as recurrent factor.

Now setting

$$U(C) = t, \tag{2.5}$$

one derives from (2.1) by a direct computation

$$\mathcal{L}_C \eta = t\eta, t \in \Lambda^\circ(M). \tag{2.6}$$

Then one may write,

$$C^\flat = -t\eta. \tag{2.7}$$

On the other hand, recall that the covariant differential of any vector field Z is expressed by

$$\nabla Z = (dZ^A + Z^B \theta_B^A) e_A + (dZ^0 - g(Z, U)\eta) \otimes \xi + \eta(Z)\eta \otimes U, \tag{2.8}$$

where θ_B^A means the connection forms associated with $\mathcal{O}^* = \{\omega^A, \eta\}$.

Because $\eta(C) = 0$, one derives from (2.2)

$$dC^A + C^B \theta_B^A = \rho\omega^A + C^A \eta, \tag{2.9}$$

and since

$$C^\flat = \sum C^A \omega^A, \tag{2.10}$$

one infers from (2.1) and (2.7)

$$dC^\flat = 2\eta \wedge C^\flat. \tag{2.11}$$

The above relation is, in fact, the **Rosca's Lemma** [8]. In addition, by (2.7) and (2.11), it follows that, in the case under consideration,

$$dC^\flat = 0, \tag{2.12}$$

i.e. C is a closed vector field.

Setting $||C||^2 = 2l$, one gets from (2.2)

$$dl = \rho C^\flat + 2l\eta, \tag{2.13}$$

and by exterior differential and (2.7) and (2.12) one may get

$$d\rho = aU + c\eta, \tag{2.14}$$

where

$$a = \frac{2l}{t}, c = \text{const.} \tag{2.15}$$

Further, by (2.7) and (2.15), one quickly finds

$$\mathcal{L}_C \alpha = (\rho - 4a)\alpha, \tag{2.16}$$

where $\alpha = C^\flat$.

As is known, we denote by $*$ the star isomorphism; then

$$*\alpha = \sum(-1)^A C^A \omega^1 \wedge ... \wedge \hat{\omega}^A \wedge ... \wedge \omega^{2n} \wedge \eta.$$

(where we denote by $\hat{}$ the missing term).

Then, by (2.2) and the structure equations (1.3), one derives by a standard calculation

$$\mathcal{L}_C * \alpha = 2m\rho * \alpha. \tag{2.17}$$

Therefore, one may say that C defines an *infinitesimal self-conformal transformation* and this property is preserved by star isomorphism.

On the other hand, since the existence of C is determined by

$$C^\flat = -t\eta, d\eta = U \wedge \eta, dC^\flat = 0, \tag{2.18}$$

the above equations define an exterior differential system $\sum$, whose *characteristic numbers* are $r = 3, s_0 = 1, s_1 = 2$. Since $r = s_0 + s_1$, $\sum$ is in *involution* (in the sense of E. Cartan [1]) and, consequently, the existence of C is determined by 2 arbitrary functions of 1 argument.

In another order of ideas, one may write the symplectic form Ω carried by the closed concircular almost contact manifold under consideration as

$${}^\flat\Omega = \omega^a \wedge \omega^{a^*}, a \in \{1, ..., m\}, a^* = a + m. \tag{2.19}$$

Let $\Omega^\flat(Z) = {}^\flat Z = -i_Z\Omega, Z \in \Gamma TM$, be the symplectic isomorphism defined by Ω (see also [8]).

One has ${}^\flat C = -i_C\Omega = \sum(C^{a^*}\omega^a - C^a\omega^{a^*})$ and, since Ω is closed, one derives by (2.2) and the structure equation (1.3)

$$\mathcal{L}_C\Omega = \rho\Omega - \eta \wedge {}^\flat C. \tag{2.20}$$

This proves that C is a *weak automorphism* of Ω. In addition, by reference to [ERC], one has

$$\nabla \Phi C = \rho \Phi dp + \eta \otimes \Phi C + g(\Phi U, C)\eta \otimes \xi \tag{2.21}$$

and by (2.1) one derives

$$[C, \Phi C] = 0. \tag{2.22}$$

One quickly finds

$$^\flat(\Phi C) = C^\flat, \tag{2.23}$$

and since $dC^\flat = 0$, it follows

$$\mathcal{L}_{\Phi C}\Omega = 0, \tag{2.24}$$

i.e. Ω is *invariant* by ΦC.

On the other hand, since the q-th covariant differential ∇^q of a vector field Z on a Riemannian or pseudo-Riemannian manifold is defined inductively [8], i.e. $\nabla^q Z = d^\nabla(\nabla^{q-1}Z)$, one derives from (2.1), (2.2) and (2.12)

$$d^\nabla(\nabla C) = \nabla^2 C = (dp - \rho\eta) \wedge dp + d\eta \otimes C. \tag{2.25}$$

The above equation says that C is *exterior quasi-concurrent* [8]. Moreover, the distribution $\mathcal{D}_C$ annihilated by the canonical 2-form $C^\flat \wedge (dp - \rho\eta)$ defines a $(2m-1)$-dimensional foliation and η is an element of the first class of cohomology $H^1(\mathcal{D}_C, \mathbf{R})$.

We consider now on M the 3-form

$$\Psi = \eta \wedge U^\flat \wedge C^\flat. \tag{2.26}$$

Then, from (2.1) and (2.12), it follows

$$d\Psi = 0. \tag{2.27}$$

Hence the vector fields $i_Z\Psi = 0$ form a Lie algebra and M receives a foliation determined by the 3-distribution $\mathcal{D} = \{C, U, \xi\}$ and by (2.1) and (2.2) it is easily seen that M is foliated by $(2m-2)$–dimensional submanifolds N of 1-geodesic index and 2-umbilical index (see, for instance, [8]).

Further, from (2.14) one has

$$\nabla\rho = aU + c\xi, \quad c = \text{const.}. \tag{2.28}$$

Then, since one finds

$$da = c\eta + bU, b = \frac{c}{t}, \tag{2.29}$$

one derives by (2.21)

$$\nabla^2\rho = a\lambda dp + (bU + 2c\eta) \otimes U - a(\lambda + ||U||^2)\eta \otimes \xi, \tag{2.30}$$

and therefore, by the known formula $\Delta\rho = -div(\nabla\rho)$, one infers

$$\Delta\rho = (a-b)||U||^2 - 2m\lambda a. \tag{2.31}$$

If K means the scalar curvature of M, then by Yano's formula [14] one may write

$$\mathcal{L}_C K = (2m-1)\left[(a-b)||U||^2 - 2m\lambda a\right] - K\rho.$$

By using the formula

$$2\mathcal{L}_C \mathcal{R}(Z, Z') = (\Delta\rho) g(Z, Z') - (2m-1) Hess_{\nabla}\rho(Z, Z'), \tag{2.32}$$

we obtain

$$2\mathcal{R}(Z, Z') = \left[(a-b)||U||^2 - (2m-1)\lambda\right] g(Z, Z') +$$

$$+ bg(U, Z)g(U, Z') + a(\lambda + ||U||^2)g(\xi, Z)g(\xi, Z') + 2c\eta(Z)g(U, Z'),$$

for any Z, Z' vector fields on M.

We state the:

Theorem. *Let $M(\Phi, \Omega, \eta, \xi, U, g)$ be a $(2m+1)$-dimensional closed concircular almost contact manifold. Then the existence of a skew symmetric conformal vector field C, having the Reeb vector field ξ as generative, is determined by an exterior differential system $\sum$ in involution. The following properties are proved:*

(i) the vector field C defines a weak authomorphism of the structure form Ω and ΦC defines an infinitesimal automorphism of Ω, i.e. $\mathcal{L}_C \Omega = \rho\Omega - \eta \wedge C, \mathcal{L}_{\Phi C}\Omega = 0$;

(ii) C and ΦC commute and C is an exterior quasi concurent vector field;

(iii) M is foliated by 3-codimensional submanifolds N, and the immersion $x : N \longrightarrow M$ is of 1-geodesic index and 2-umbilical index;

(iv) if $\mathcal{R}$ is the Ricci tensor and Z, Z' are any vector fields, the following relation holds good

$$2\mathcal{R}(Z, Z') = \left[(a-b)||U||^2 - (2m-1)\lambda\right] g(Z, Z') +$$

$$+ bg(U, Z)g(U, Z') + a(\lambda + ||U||^2)g(\xi, Z)g(\xi, Z') + 2c\eta(Z)g(U, Z'),$$

where $a, b \in \wedge^{\circ} M, c = const.$

3 Skew-symmetric Killing vector fields on CR-product submanifolds

Let M' be an m-dimensional CR-submanifold of M, i.e. there exists a differentiable distribution $\mathcal{D}^{\top} : p \longrightarrow \mathcal{D}_p \subset T_p M'$ such that

(i) $\mathcal{D}^{\top}$ is holomorphic on M', i.e. $\Phi\mathcal{D}_p = \mathcal{D}_p$;

(ii) its complementary orthogonal distribution $\mathcal{D}^{\perp} : p \longrightarrow \mathcal{D}_p^{\perp}$ is antiinvariant, i.e. $\Phi(\mathcal{D}_p^{\perp}) \subset T_p^{\perp} M'$.

In order to simplify, we agree to denote the elements induced by different immersions by the same letters. Without loss of generality, we assume that the orthogonal vector basis $\mathcal{O}(M)$ is defined such that

$$\mathcal{D}_p^{\top} = vect\{e_i, e_{i^*} \mid i = 1, ..., m-l; i^* = i + m\}, \tag{3.1}$$

which implies

(3.2) $$\mathcal{D}_p^{\perp} = vect\{e_r, e_0 \mid r = m-l+1, ..., m; e_0 = \xi\}.$$

If $\mathcal{O}^*(M) = \{\omega\}$ denotes the dual basis of $\mathcal{O}(M) = \{e\}$, we cosider

(3.3) $$\Psi^{\top} = \omega^1 \wedge ... \wedge \omega^{m-l} \wedge \omega^{1^*} \wedge ... \wedge \omega^{(m-l)^*},$$

(3.4) $$\Psi^{\perp} = \omega^{m-l+1} \wedge ... \wedge \omega^{m} \wedge \eta$$

the simple unit forms corresponding to $\mathcal{D}_p^{\top}$ and $\mathcal{D}_p^{\perp}$, respectively. Let $\gamma_{BC}^A(A, B, C \in \{0, 1, ..., m\})$ be the coefficients of the connection forms θ_B^A, associated with the moving frame $\mathcal{O}(M) = \{e\}$. We recall that the antiinvariant distribution $\mathcal{D}^{\perp}$ is always *involutive.* Since the Reeb vector field ξ is normal to $\mathcal{D}^{\top}$, the distribution $\mathcal{D}^{\top}$ is also called the *ξ-normal horizontal distribution.*

Denote now by $M'^{\perp}$ the leaf of $\mathcal{D}^{\perp}$ and consider the immersion $x^{\perp} : M'^{\perp} \longrightarrow \mathcal{D}^{\top}$.

Since one has $\omega_0^A = U^A \eta$, the mean curvature vector field corresponding to the immersion $x^{\perp}$ is expressed by

(3.5) $$H^{\perp} = \sum (\gamma_{ii^*}^a + U^a) e_a; a \in \{i, i^*\}.$$

Since the volume element τ of the submanifold M' is written as $\tau = \Psi^{\top} \wedge \Psi^{\perp}$, it follows by Frobenius theorem that the necessary and sufficient condition for the distribution $\mathcal{D}^{\top}$ to be involutive is that the simple unit form $\Psi^{\perp}$ to be exterior recurrent. In this case, the CR-submanifold M' is called a *CR-product* submanifold.

In these conditions, we consider the immersion $x^{\top} : M'^{\top} \longrightarrow M'^{\perp}$ and denote by $H^{\top}$ the mean curvature vector field corresponding to $x^{\top}$. One derives

(3.6) $$H^{\top} = \sum \gamma_{aa}^r e_r.$$

Using the structure equation (1.3) one obtains

(3.7) $$\begin{cases} d\Psi^{\top} = -(H^{\top})^{\flat} \wedge \Psi^{\top} \\ d\Psi^{\perp} = -(H^{\perp})^{\flat} \wedge \Psi^{\perp}. \end{cases}$$

If $M^{\top}$ and $M^{\perp}$ are minimal, we are in the situation of Tachibana's theorem [13].

Recall now that a closed cosymplectic almost contact manifold M admits a skew symmetric Killing vector field Y [R], i.e.

(3.8) $$\nabla Y = Y \wedge U = U^{\flat} \otimes Y - Y^{\flat} \otimes U.$$

We assume that the CR-product submanifold $M' = M^{\top} \times M^{\perp}$ of M carries a skew symmetric Killing vector field X tangent to $M^{\top}$ such that its generative V is tangent to $M^{\perp}$

(3.9) $$\nabla X = X \wedge V, V \in T_p M^{\perp}, X \in T_p M^{\top}.$$

We agree to call such a vector field X a *mixed* skew symmetric Killing vector field on a CR-submanifold.

Using the structure equations, one finds by a standard calculation

(3.10) $$dX^\flat = 2V^\flat \wedge X^\flat,$$

(3.11) $$g(X,U)\eta = \eta(V)X^\flat,$$

(3.12) $$X^a\theta_a^r = -V^r X^\flat,$$

$(a \in \{i, i^*\}, r \in \{m-l+1, ..., m\})$.

Since η is not colinear to $X^\flat$, it follows from (3.11)

(3.13) $$\begin{cases} g(X,U) = 0, \\ \eta(V) = 0, \end{cases}$$

and this shows that necessarily X is orthogonal to the structure vector field U. From (3.12), one obtains

(3.14) $$\begin{cases} 2\theta_i^r + V^r(\omega^i - \omega^{i^*}) = 0, \\ 2\theta_{i^*}^r + V^r(\omega^i + \omega^{i^*}) = 0, \end{cases}$$

where $i \in \{m-l+1, ..., m\}; e_0 = \xi, i \in \{1, 2, ..., m-l\}, i^* = i+m$.

On the other hand, by referrence to [4], one has $\theta_0^A = U^A\eta$ and the mean curvature vector field regarding the immersion $x^\top : M^\top \longrightarrow M^\perp$ is expressed by $H^\top = \dfrac{1}{dimM^\top}\sum(\gamma_{ii}^r + \gamma_{i^*i^*}^r)e_r$. Then, using (3.14), one derives $H^\top = -\frac{1}{2}V$.

This proves the fact that $H^\top$ is, up to $-\frac{1}{2}$, equal to the generative vector field V of the mixed skew symmetric Killing vector field X.

Also, by the first equation (3.12) and by (3.8) and (3.9), one finds $[Y,X] = g(Y,V)X + g(X,Y)(U-V)$.

One may say that if the skew symmetric Killing vector fields Y and X are orthogonal, then Y defines an infinitesimal conformal transformation of X.

We state the:

Theorem. *Let M' be a CR-product submanifold of a closed concircular almost cosymplectic manifold M, i.e. $M' = M^\top \times M^\perp$, where $M^\top$ (respective $M^\perp$) is the invariant submanifold (respective antiinvariant) submanifold of M'. Then, if M' carries a mixed skew symmetric Killing vector field X ,it follows that the mean curvature vector field $H^\top$ of $M^\top$ in M is, up to $-\frac{1}{2}$, equal to the generative V of X.*

Also, if the skew symmetric Killing vector fields X and Y are orthogonal, then Y defines an infinitesimal conformal tranformation of X.

Assume that the generative V of X is a closed *torse forming*. Then, following [4], the covariant differential of V is expressed by

(3.15) $$\nabla V = \lambda dp - v \otimes V, \lambda \in \wedge^\circ M,$$

where $v = V^\flat$ is closed [R]. One derives

(3.16) $$d^\nabla(\nabla X) = \nabla^2 X = \lambda X^\flat \wedge dp,$$

which means that X is an exterior concurrent vector field [R], having λ as conformal scalar . Hence, by reference to [8], the Ricci tensor field $\mathcal{R}(X,Z)$ (where Z is any vector field on $M^\perp$) is expressed by

$$\mathcal{R}(Y,Z) = -(l-1)\lambda g(X,Z). \tag{3.17}$$

One easily get

$$d\lambda \wedge v = 0, \tag{3.18}$$

$$dV^r + V^s\theta^r_s = \lambda\theta^r - V^r v, s, r \in \{m-l+1, ..., m\}, \tag{3.19}$$

$$V^a\theta^r_a = 0, a \in \{i, i^*\}. \tag{3.20}$$

By exterior differentiation of (3.19), one derives

$$\Theta^r_b = 0. \tag{3.21}$$

Hence, since the curvature forms of the submanifold $M^\perp$ are vanishing, it follows that $M^\perp$ is a flat submanifold.

Then, we state the

Theorem. *Let X be the skew symmetric Killing vector field of the antiinvariant submanifold $M^\perp$ of the CR-product submanifold $M' = M^\top \times M^\perp$ of a closed concircular almost contact manifold M.*

If the generative V of the X is a closed torse forming, then the submanifold $M^\perp$ is flat.

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