Some Representations of Affine Conformal Transformations of Minkowski Space

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Abstract

We consider the groups G_1, G_2, G_3 that are different from affine conformal group just because the space-like or (and) time-like symmetries are accompanied by the inversion I_0 (charge conjugation operation). For these groups there exist some fundamental spin representations (spin $s = \frac{1}{2}$) given by (14); the representations of the subgroup formed from the proper Lorentz group, the homotheties and the considered symmetries, for different couples λ, μ such that $\mu^2 - \lambda^2 = 1$, are equivalent.

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1. Let M be Minkowski space, x^i $(i = 1,2,3,4)$ being orthogonal coordinates, $\mathbf{x}^2 = -(x^1)^2 - (x^2)^2 - (x^3)^2 + (x^4)^2$. Let $M^\#$ be *compactified Minkowski space* [1], [2], [4] (M with a "null cone at infinity") and G the group of conformal transformations of M (more exactly, of $M^\#$). We know that the transformations of G may be considered as linear transformations in a six-dimensional pseudo-Euclidean space E_6^2 with the metric $G_{ab} = \varepsilon_a \delta_a^b$, $\varepsilon_a = -1, -1, -1, +1, -1, +1$ and let $O(4,2)$ the group of motions of E_6^2. Then $G \approx O(4,2)/\mathbf{Z}_2$, where $\mathbf{Z}_2$ is the center of $O(4,2)$ which, in the matrix (six-dimensional) representation, consists of I and $-I$. We can construct an epimorphism $\eta : O(4,2) \to G$ in the following manner: to every $\sigma \equiv \sigma(t_b^a) \in O(4,2)$, $(z'^a = t_b^a z^b$, $G_{ab} t_c^a t_d^b = G_{cd}$, $a,..,d = 1,..,6)$ corresponds $g \in G$

$$(1) \qquad g : x'^i = \frac{\sum t_j^i x^j + \frac{1}{2}(t_5^i + t_6^i)\mathbf{x}^2 + \frac{1}{2}(t_5^i - t_6^i)}{\sum (t_j^5 - t_j^6)x^j + \frac{1}{2}(t_6^5 - t_6^6 + t_5^5 - t_5^6)\mathbf{x}^2 + \frac{1}{2}(t_5^5 - t_5^6 - t_6^5 + t_6^6)};$$

$$\left(\frac{z^i}{z^5 - z^6} = x^i, \ \frac{z^5}{z^5 - z^6} = \frac{\mathbf{x}^2 + 1}{2}, \ \frac{z^6}{z^5 - z^6} = \frac{\mathbf{x}^2 - 1}{2} \right).$$

Note that $\sigma \equiv \sigma(t_b^a)$ and $\tilde{\sigma} \equiv \sigma(-t_b^a)$ have the same image, therefore $O(4,2)$ is a two-fold covering group of G. In particular, if O^c and G^c denote the identity-connected component of $O(4,2)$ and G respectively, we see that O^c is also a two-fold covering group of G^c.

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Let $G_0 \subset G$ be the *affine conformal group,* for which the "null cone at infinity" is preserved. G_0 is formed by the *full Lorentz group* $\mathcal{L}, (x'^i = t^i_j x^j$, 6 parameters), the *translation group* $\mathcal{T}$ $(T(\mathbf{a}) : x'^i = x^i - a^i$, 4 parameters) and the *dilatation group* $\mathcal{H}$ $(H(\theta) : x'^i = e^{-\theta} x^i$, 1 parameter). $\mathcal{L}$ is formed from $\mathcal{L}^{\uparrow}_{+}$ (identity-connected component of $\mathcal{L}$) by addition of two symmetries, for instance $\tau_1 : x^1 \to -x^1$ and $\tau_4 : x^4 \to -x^4$. The full conformal group G (with 15 parameters) is obtained from G_0 by addition of the inversion $I_0 : x'^i = \frac{x^i}{\mathbf{x}^2}$. One obtains a new subgroup with 4 parameters

$$G_s(a^i) : x'^i = \frac{x^i + a^i \mathbf{x}^2}{1 + 2a^i x_i + \mathbf{a}^2 \mathbf{x}^2}. \tag{2}$$

By direct computation one obtains $G_s(a^i) = I_0 T(-\mathbf{a}) I_0$ and also $G_s(a^i) = I(\mathbf{a})\tau_a$, where $I(\mathbf{a})$ is the inversion relative to the hypersphere of center $a^i/\mathbf{a}^2$ and radius $1/\sqrt{\mathbf{a}^2}$, and τ_a is the symmetry in the direction of nonisotropic vector $\mathbf{a}(\tau_a : x'^i = (\delta^i_j - 2\frac{a^i a_j}{\mathbf{a}^2}) x^j)$.

In E^2_6 we have

$$H(\theta) : \begin{cases} z'^i = z^i & (i = 1, ..., 4) \\ z'^5 = -\sinh\theta\ z^5 + \cosh\theta\ z^6 \\ z'^6 = \cosh\theta\ z^5 - \sinh\theta\ z^6, \end{cases} \qquad \mathcal{L} : \begin{cases} z'^i = l^i_j z^j \\ z'^5 = z^5 \\ z'^6 = z^6, \end{cases}$$

$$G_s(\mathbf{a}) : \begin{cases} z'^i = z^i + a^i(z^5 + z^6) \\ z'^5 = (1 + \frac{\mathbf{a}^2}{2}) z^5 + \frac{\mathbf{a}^2}{2} z^6 + a_i z^i \\ z'^6 = (1 - \frac{\mathbf{a}^2}{2}) z^6 - \frac{\mathbf{a}^2}{2} z^5 - a_i z^i, \end{cases}$$

$$T(a^i) : \begin{cases} z'^i = z^i + a^i(z^6 - z^5) \\ z'^5 = z^5 - \frac{1}{2}\mathbf{a}^2(z^6 - z^5) - a_i z^i \\ z'^6 = z^6 - \frac{1}{2}\mathbf{a}^2(z^6 - z^5) - a_i z^i. \end{cases}$$

Reflexion $\tau_i : x^i \to -x^i$ (fixed i) corresponds in E^2_6 to $z^i \to -z^i$. The reflexion $\tau_6 : z^6 \to -z^6$ corresponds, in $M^{\#}$,to the inversion I_0 and τ_5 is corresponding to I_0 followed of the " total reflection" of the axes $x'^i = -x^i$ $(i = 1, ...4)$.

To obtain the spin representations of G, it suffices to get the spin representations of $O(4, 2)$. For this, we consider the Clifford algebra C_6 associated to E^2_6.

2. Let C_6 be the associative algebra with unity generated (with complex coefficients) by six entities $\beta_1, ..., \beta_6$ satisfying

$$\beta_a \beta_b + \beta_b \beta_a = 2G_{ab} \cdot 1 \tag{3}$$

Then $\beta_1, ..., \beta_6$ are linearly independent and let $W \subset C_6$ be the linear space spanned by the β_a. For every two vectors $\mathbf{x},\mathbf{y} \in W$ we define $(\mathbf{x},\mathbf{y}) = \frac{1}{2}(\mathbf{xy}+\mathbf{yx})$. Writing down $\mathbf{x} = x^i \beta_i$, $\mathbf{y} = y^i \beta_i$ we will obtain $(\mathbf{x},\mathbf{y}) = \sum x^i y^i$ such that $(\mathbf{x},\mathbf{y})$ defines a scalar product into W and $\{\beta_a\}$ is an orthonormal basis. We agree to identify both spaces W and E^2_6 and their orthonormal considered bases. Let Ω be the multiplicative group of the nonisotropic vectors of E_6^2. Then for the symmetry τ_a in the direction of nonisotropic vector $\mathbf{a} \in E_6^2$ we get

$$\tau_a : \mathbf{z}' = -\mathbf{aza}^{-1} \quad \text{or} \quad \mathbf{z}' = (\mathbf{a}E)\mathbf{z}(\mathbf{a}E)^{-1} \quad (E = \beta_1 ... \beta_6), \quad \mathbf{z} \in E_6^2). \tag{4}$$

Because for each $\sigma \in O(4,2)$ we have $\sigma = \tau_{a_1} ... \tau_{a_k}$, there exists an epimorphism $h : \Omega \to O(4,2)$,

$$\sigma \mathbf{z} = (-1)^k \omega \mathbf{z} \omega^{-1} \quad \text{or} \quad \sigma \mathbf{z} = (\omega E)\mathbf{z}(\omega E)^{-1}, \; (\omega = \mathbf{a}_k ... \mathbf{a}_1 \in \Omega), \tag{5}$$

$\ker\, h = \mathcal{C}$. Note that $\chi \equiv \eta \circ h$ is epimorphism of Ω onto G. Let $\bar{\Omega} \subset \Omega$ be the group [3] of normed elements of Ω such that $\bar{h} \equiv h|_{\bar{\Omega}}$ has $\ker \bar{h} = \{\pm 1\}$, the group $\bar{\Omega}$ being a two-fold covering group of $O(4,2)$ (cf. [3] there exist four normed groups $\Omega_{\varepsilon\varepsilon'} \subset \Omega$ that give us distinct representations of $O(4,2)$. Let $\bar{\Omega}$ be one of this.) Note that, denoting $\sigma = h(\pm\omega)$ for $\omega \in \bar{\Omega}$, we have $\tilde{\sigma} = h(\pm\omega E)$ whence, for $\bar{\chi} \equiv \chi|_{\bar{\Omega}}$ we have $\ker \bar{\chi} = \{\pm 1, \pm E\}$ and $\bar{\Omega}$ is a four-covering group of G. Obviously, the matrix representation of C_6 gives us a four-valued linear representation of G, called *fundamental spin representation* of G. For the infinitesimal generators of this representation we get

$$L_{ij} = \frac{1}{2}\beta_i\beta_j, D = \frac{1}{2}\beta_6\beta_5, P_i = \frac{1}{2}\beta_i(\beta_5 + \beta_6), K_i = -\frac{1}{2}\beta_i(\beta_5 - \beta_6),$$

and the operators $D(a^i) = e^{a^i P_i}, D(\theta) = e^\theta D, D(c^i) = e^{c^i K_i}$ of $T(\mathbf{a}), H(\theta), G_s(\mathbf{c})$ are

$$D(a^i) = 1 + \frac{1}{2}a^i\beta_i(\beta_5 + \beta_6), D(\theta) = \cosh\, \frac{\theta}{2} I + \sinh\, \frac{\theta}{2}\beta_6\beta_5, \tag{6}$$

$$D(c^i) = 1 - \frac{1}{2}c^i\beta_i(\beta_5 - \beta_6).$$

3. We will consider some of the subgroups of G, which contain G_0^c and which can have applications in the classification of the elementary particles.

Considering the decomposition of the groups G and G_0 in connected components

$$G = G^c \cup \tau_1 G^c \cup \tau_4 G^c \cup \tau_1\tau_4 G^c, \; G_0 = G_0^c \cup \tau_1 G_0^c \cup \tau_4 G_0^c \cup \tau_1\tau_4 G_0^c,$$

we notice that $G_1 = G^c \cup \tau_1\tau_4 G^c, \; G_2 = G^c \cup \tau_1 G^c, \; G_3 = G^c \cup \tau_4 G^c$, are subgroups of G. For a certain group $\mathrm{G}_r \subset G$ and $h_1, h_2 \in G$, we will denote with $\{\mathrm{G}_r; h_1, h_2\}$ the group formed from the elements of G_r and the composition of these, in a certain number of times, with the elements h_1, h_2.

Proposition 1. We have:

1. $G_1 = \{G_0^c; \tau_1 I_0, \tau_4 I_0\}$;
2. $G_2 = \{G_0^c; \tau_1, \tau_4 I_0\}$;
3. $G_3 = \{G_0^c; \tau_1 I_0, \tau_4\}$.

We will prove here only the affirmation relative to G_1.

Through a direct computation (which can be done in the space $M^\#$ or E_6^2) we get

$$\tau_i I_0 = I_0 \tau_i, \quad \tau_i T(\mathbf{a}) = T(\tau_i \mathbf{a})\tau_i, \; H(\theta) I_0 = I_0 H(-\theta).$$

Then, the relation $G(\mathbf{a}) = I_0 T(-\mathbf{a}) I_0$ can be written

$$G(\mathbf{a}) = (\tau_4 I_0) \circ T(-\mathbf{a}) \circ (\tau_4 I_0),$$

and therefore $\{G_0^c; \tau_4 I_0\} \supseteq G^c$. But $\tau_4 I_0 \in G^c$ (because in E_6^2 it represents the transformation $\tau_4\tau_6$; or in $M^\#$ we can verify that $\tau_4 I_0 = T(\mathbf{b}_0)G(\mathbf{b}_0)T(\mathbf{b}_0) \in G^c$ with $\mathbf{b}_0 = (0,0,0,1)$). It results

$$G^c = \{G_0^c; \tau_4 I_0\} \tag{7}$$

On the other hand, $\tau_1 I_0 \notin G^c$ because, in E_6^2 it is the product of the space-like or time-like symmetries τ_1 and τ_6 (in $M^\#$ we can verify that $\tau_1 I_0 = \bar{\tau} \circ T(\mathbf{a}_0) \circ G(-\mathbf{a}_0) \circ T(\mathbf{a}_0)$ with $\mathbf{a}_0 = (-1,0,0,0)$ and $\bar{\tau} \equiv \tau_1\tau_2\tau_3\tau_4 \notin G^c$). For $g \in G^c$ it results $(\tau_1 I_0)g = \bar{\tau}T(\mathbf{a}_0)G(-\mathbf{a}_0)T(\mathbf{a}_0)g = \tau_1\tau_4 g', g' \in G^c$. Using (7) follows $G_1 = \{G_0^c; \tau_1 I_0, \tau_4 I_0\}$.

For the group G_1 we will give the decomposition in connected components:

$$G_1 = G_0^c \cup G_0^c(\tau_4 I_0)G_0^c G_0^c(\tau_1 I_0)G_0^c \cup \tau_1\tau_4 G_0^c. \tag{8}$$

Indeed, (see above) we have $G_s(a^i) = I(\mathbf{a})\tau_a$. But, for $\mathbf{a}^2 > 0$ we can write $\tau_a = g_0\tau_4$ (because through a translation and a dilatation, therefore through a proper Lorentz transformation, we can transform the vector $\mathbf{a}$ into $\mathbf{e}_4$) and for $\mathbf{a}^2 < 0$ we can write $\tau_a = g_0\tau_1$, $g_0 \in G_0^c$. Because the inversion

$$\mathcal{I}_{\mathcal{C},\mathcal{R}^\in} : §'^\rangle - \rfloor^\rangle = \mathcal{R}^\in \frac{§^\rangle - \rfloor^\rangle}{\textbf{x-c}^\in}$$

can be written, for $R^2 > 0$ as $\mathcal{I}_{\mathcal{C},\mathcal{R}^\in} = \mathcal{T}(-\rfloor^\rangle)\mathcal{H}(\mathcal{R}^\in)\mathcal{I}_\prime\mathcal{T}(\rfloor^\rangle)$ and for $R^2 < 0$ as $\mathcal{I}_{\mathcal{C},\mathcal{R}^\in} = \mathcal{T}(-\rfloor^\rangle)\not{\mathcal{I}}\mathcal{H}(\mathcal{R}^\in)\mathcal{I}_\prime\mathcal{T}(\rfloor^\rangle)$, it results that for any nonisotropic vector $\mathbf{a}$, we can write $G_s(a^i) = g_0(\tau_4 I_0)g_0'$, $g_0, g_0' \in G_0^c$. From $\tau_4 I_0 \in G^c$ it follows

$$G^c = G_0^c \cup G_0^c(\tau_4 I_0)G_0^c.$$

Using the relation which defines G_1, we obtain (8).

We notice that the groups G_1, G_2, G_3 are different from G_0 just because of the fact that the symmetries are accompanied by the inversion I_0. Thus, it is possible that the elementary particles (for example neutrino) for which the symmetry operations are accompanied in an obligatory way by the operation of charge conjugation, to be the representations of one of these subgroups.

4. Obviously, every representation of G gives us a representation of G_1 (respectively of G_2, G_3). We will prove that there exist some representations of G_1 (also, of G_2, G_3) that are not obtained by this way.

Lemma 2. *Let* $\mathbf{K}=\lambda\beta_5 + \mu\beta_6, \mu^2 - \lambda^2 \equiv \varepsilon = \pm 1$. *Then the mapping* $E_6^2 \to E_6^2$ *given by*

$$I(\lambda,\mu): \quad \mathbf{z}' = -\mathbf{K}\mathbf{z}\mathbf{K}^{-1}, \quad \mathbf{z} = z^a\beta_a \in E_6^2,\ (a = 1,...,6; \lambda, \mu \in \mathcal{R})$$

is the inversion with respect to the hypersphere with center O(0,...,0) and the radius $R = \sqrt{\frac{\lambda+\mu}{\mu-\lambda}}$.

Proof. Obviously, $\mathbf{K}$ is nonisotropic vector and $\mathbf{K}^{-1} = \varepsilon\mathbf{K}$. We have

$$\mathbf{z}' = -\mathbf{K}\mathbf{z}\mathbf{K}^{-1} = z^i\beta_i + \frac{1}{\varepsilon}[(\mu^2+\lambda^2)z^5 - 2\lambda\mu z^6]\beta_5 + + \frac{1}{\varepsilon}[-(\mu^2+\lambda^2)z^6 + 2\lambda\mu z^5],$$

and therefore

$$z'^i = z^i, z'^5 = \frac{1}{\varepsilon}[(\mu^2+\lambda^2)z^5 - 2\lambda\mu z^6], z'^6 = \frac{1}{\varepsilon}[-(\mu^2+\lambda^2)z^6 + 2\lambda\mu z^5].$$

Since $z^a z_a = z'^a z'_a$, the function $I(\lambda,\mu)$ is an isometry in E^2_6. Then in $M^{\#}$ we have

$$x'^i = \frac{z'^i}{z'^5 - z'^6} = \frac{\mu+\lambda}{\mu-\lambda} \cdot \frac{x^i}{\mathbf{x}^2}$$

and the lemma is proved.

Denoting h_0 the homothety with the center O and power $\left|\frac{\mu+\lambda}{\mu-\lambda}\right|$, it results that

$$(9) \qquad I(\lambda,\mu) = h_0 \circ I_0, \text{ if } \frac{\mu+\lambda}{\mu-\lambda} > 0;\ I(\lambda,\mu) = h_0 \circ \bar{\tau} \circ I_0, \text{ if } \frac{\mu+\lambda}{\mu-\lambda} < 0.$$

We denote Ω^c_0 the subgroup of $\bar{\Omega}$ for which $\chi(\Omega^c_0) = G^c_0 \subset G^c$. The group Ω^c_0 does not depend on the choice done for $\bar{\Omega}$. It is the subgroup of the elements like $\omega = \mathbf{a}_1\mathbf{a}_2...\mathbf{a}_{2k}$, where $\mathbf{a}_i$ are unit vectors, such that an even number of vectors are space-like among $\mathbf{a}_1, \mathbf{a}_2, ...\mathbf{a}_{2k}$.

Proposition 3. *Let* $\mathbf{K}=\lambda\beta_5 + \mu\beta_6$, $\mu, \lambda \in \mathcal{R}$, $\mu^2 - \lambda^2 = \pm 1$. *Let* $\mathbf{s}=\nu\beta_1\mathbf{K}$, $\mathbf{t} = \nu'\beta_4\mathbf{K}$, *where* ν, ν' *are defined by* $\mathbf{s}^2 = \varepsilon, \mathbf{t}^2 = \varepsilon'$ *and* $\varepsilon = \pm 1, \varepsilon' = \pm 1$ *(therefore* $\nu, \nu' = \pm 1, \pm i$*). Let*

$$(10) \qquad \Omega^*_{\varepsilon\varepsilon'}(\lambda,\mu) = \Omega^c_0 \cup \Omega^c_0\mathbf{s}\Omega^c_0 \cup \Omega^c_0\mathbf{t}\Omega^c_0 \cup \mathbf{st}\Omega^c_0.$$

Then:

1. $\Omega^*_{\varepsilon\varepsilon'}(\lambda,\mu)$ *is group.*

2. $\chi_1 = \chi|_{\Omega^*_{\varepsilon\varepsilon'}(\lambda,\mu)}$ *is homomorphism of* $\Omega^*_{\varepsilon\varepsilon'}(\lambda,\mu)$ *onto G.*

3. There exists $U \in \Omega$ *such that* $U\Omega^*_{\varepsilon\varepsilon'}(\lambda,\mu)U^{-1} = \begin{cases} \Omega^*_{\varepsilon\varepsilon'}(0,1) \text{ for } \mu^2-\lambda^2 = 1, \\ \Omega^*_{\varepsilon\varepsilon'}(1,0) \text{ for } \mu^2-\lambda^2 = -1. \end{cases}$

Proof. First, we notice that the four subsets which appear in (10) are disjunctive. So, for example, an equality in the form $\omega_0\beta_1\mathbf{K}\omega'_0 = \omega_1\beta_4\mathbf{K}\omega'_1$, $\omega_0, \omega'_0, \omega_1, \omega'_1 \in \Omega_0$ implies $\omega\beta_1\mathbf{K} = \beta_4\mathbf{K}\omega'$, $\omega, \omega' \in \Omega_0$. Then, through the homomorphism χ, we have $g\tau_1 h_0 I_0 = \tau_4 h_0 I_0 g'$, $g, g' \in G^c_0$. Because $h_0\tau_1 = \tau_1 h_0$, $gh_0 = h_0 g_1$, it results $g_1\tau_1 I_0 = \tau_4 I_0 g', g_1, g' \in G^c_0$.

Considering the corresponding transformations in E^2_6, it result $g\tau_1\tau_6 = \tau_4\tau_6 g'$, where g, g' belong to the identity-connected component of the group O(4,2); but this relation is impossible because the right member belongs to this connected component and the left member does not belong to it.

To prove that $\Omega^*_{\varepsilon\varepsilon'}(\lambda,\mu)$ is subgroup of Ω, it is sufficient to show that the product of any two of its elements belongs to $\Omega^*_{\varepsilon\varepsilon'}(\lambda,\mu)$. Firstly, we mention that if $\omega \in \Omega^*_{\varepsilon\varepsilon'}(\lambda,\mu)$, then also $\omega E \in \Omega^*_{\varepsilon\varepsilon'}(\lambda,\mu)$. Indeed, $\omega \in \Omega^*_{\varepsilon\varepsilon'}(\lambda,\mu)$ implies $\omega_0 E \equiv \omega_0\beta_1...\beta_6 \in \Omega^c_0$, because $\chi(\omega_0 E) = \chi(\omega_0)\chi(E) = \chi(\omega_0) \in G^c_0$. It follows that each of the four components of $\Omega^*_{\varepsilon\varepsilon'}(\lambda,\mu)$ (see (10)) contains, together with the element ω, also the elements $-\omega, \pm\omega E$. Thus, if $\omega \in \Omega^c_0(\beta_4\mathbf{K})\Omega^c_0$, therefore $\omega = \omega_0(\beta_4\mathbf{K})\omega'_0$, then $\omega E = \omega_0(\beta_4\mathbf{K})\omega'_0 E$ belongs to $\Omega^c_0(\beta_4\mathbf{K})\Omega^c_0$, because $\omega_0, \omega_0 E \in \Omega^c_0$. Whence $\chi^{-1}(G_1) = \Omega^*_{\varepsilon\varepsilon'}(\lambda,\mu)$. Then, for $\omega_1, \omega_2 \in \Omega^*_{\varepsilon\varepsilon'}(\lambda,\mu)$ we have $\chi(\omega_1\omega_2) = \chi(\omega_1)\chi(\omega_2)$; since $\chi(\omega_1), \chi(\omega_2)$ belong to the group G_1,it follows $\omega_1\omega_2 \in \chi^{-1}(G_1) = \Omega^*_{\varepsilon\varepsilon'}(\lambda,\mu)$.

To prove the last point, we denote

$$\mathbf{K}_1 = \cosh\varphi\ \beta_5 + \sinh\varphi\ \beta_6,\ \ \mathbf{K}_2 = \sinh\varphi\ \beta_5 + \cosh\varphi\ \beta_6.$$

We have

$$\mathbf{K}_1^2 = -1, \mathbf{K}_2^2 = +1,\ \mathbf{K}_1\mathbf{K}_2 + \mathbf{K}_2\mathbf{K}_1 = 0,\ \mathbf{K}_\alpha\beta_i + \beta_i\mathbf{K}_\alpha = 0, \alpha = 1, 2.$$

Therefore the systems of vectors $\{\beta_i, \beta_5, \beta_6\}$ and $\{\beta_i, \mathbf{K}_1, \mathbf{K}_2\}$ satisfy the same commutation rules, and hence there exists U$\in \Omega$ such that

$$U\beta_i U^{-1} = \beta_i,\ U\mathbf{K}_1 U^{-1} = \beta_5,\ U\mathbf{K}_2 U^{-1} = \beta_6.$$

Let us consider two choices of the vectors $\mathbf{K}$, namely $\mathbf{K} = \mathbf{K}_2$, $\mathbf{K}' = \beta_6$. Then $\mathbf{s}=\nu\beta_1\mathbf{K}_2$, $\mathbf{t}=\nu'\beta_4\mathbf{K}_2$ and $\mathbf{s}' = \nu\beta_1\beta_6, \mathbf{t}' = \nu'\beta_4\beta_6$. Because $U\mathbf{s}U^{-1} = \mathbf{s}'$, $U\mathbf{t}U^{-1} = \mathbf{t}'$, it results

$$U\Omega^*_{\varepsilon\varepsilon'}(\sinh\varphi, \cosh\varphi)U^{-1} = \Omega^*_{\varepsilon\varepsilon'}(0, 1). \tag{11}$$

Analogously,

$$U\Omega^*_{\varepsilon\varepsilon'}(\cosh\varphi, \sinh\varphi)U^{-1} = \Omega^*_{\varepsilon\varepsilon'}(1, 0). \tag{12}$$

Indeed, the operators $D(l^i_j), D(a^i)$ and $D(\theta)$ for the proper Lorentz transformation, translation $T(a^i)$ respectively for the homothety $H(\theta)$ in the matrix representation of C_6 are given by (6) and using the relations

$$\mathbf{K}_1 + \mathbf{K}_2 = e^{\varphi}(\beta_5 + \beta_6), \mathbf{K}_1\mathbf{K}_2 = \beta_5\beta_6$$

we obtain

$$UD(l^i_j)U^{-1} = D(l^i_j), UD(\theta)U^{-1} = D(\theta), UD(a^i)U^{-1} = D(e^{-\varphi}a^i), \tag{13}$$

and therefore $U\Omega^c_0 U^{-1} = \Omega^c_0$.

Analogously, since $\{\beta_i, \beta_5, \beta_6\}$ and $\{\beta_i, i\beta_6, i\beta_5\}$ satisfy (3) there exists $V \in \Omega$ such that

$$V\beta_i V^{-1} = \beta_i,\ V\beta_5 V^{-1} = i\beta_6, V\beta_6 V^{-1} = i\beta_5.$$

Then

$$V\beta_i\beta_5 V^{-1} = i\beta_i\beta_6, V\beta_5\beta_6 V^{-1} = \beta_5\beta_6, V(\beta_5 + \beta_6)V^{-1} = i(\beta_5 + \beta_6).$$

Therefore

$$V\Omega^*_{\varepsilon\varepsilon'}(1, 0)V^{-1} = \Omega^*_{-\varepsilon,-\varepsilon'}(0, 1).$$

The relations (13) suggest to consider the subgroup $G_1' \equiv \{\mathcal{L}^{\uparrow}_{+}, \mathcal{H}, \tau_1 I_0, \tau_4 I_0\}$. It is the *Weyl group* $\{\mathcal{L}, \mathcal{H}\}$, where the symmetries are accompanied by the inversion I_0.
Conclusion. Considering the matrix (spin) representation of C_6, one obtains:

For given $\varepsilon, \varepsilon'$ ($\varepsilon, \varepsilon' = 1$ or -1) there exist four classes of fundamental spin representations (spin $s = \frac{1}{2}$) of the group G_1 such that to the symmetries in the direction of vectors $\mathbf{a}$,$\mathbf{b}$$\in M^{\#}$ ($\mathbf{a}^2 = 1$, $\mathbf{b}^2 = -1$, $\mathbf{a}= a^i\beta_i$, $\mathbf{b}= b^i\beta_i$) correspond the operators

(14) $$\mathbf{s}=\nu\mathbf{a}(\lambda\beta_5+\mu\beta_6),\ \mathbf{t}=\nu'\mathbf{b}(\lambda\beta_5+\mu\beta_6),\ \mu^2-\lambda^2=1.$$

For different couples λ, μ such that $\mu^2 - \lambda^2 = 1$ one obtain equivalent representations of the subgroup $G_1' \subset G_1$ (for given $\varepsilon, \varepsilon'$). For this subgroup, the substitution $\mu^2-\lambda^2 = 1 \to \mu^2 - \lambda^2 = -1$ is equivalent to $\varepsilon, \varepsilon' \to -\varepsilon, -\varepsilon'$.

It is possible that an elementary particle with isospin corresponds to such a representation.

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