

Biot-Savart-Laplace Dynamical Systems

C.Udriște and S.Udriște

Abstract. §1 recalls known facts about the magnetic field \vec{H} produced by the Biot-Savart-Laplace law for a massive conductor \bar{D} . §2 proves that generally the part in $\text{ext}\bar{D}$ of a magnetic line is a trajectory of a potential dynamical system of order two (a geodesic of the Riemann-Jacobi structure) and the part in $\text{int}\bar{D}$ is a trajectory of a nonpotential dynamical system of order two (new Lorentz world-force laws) describing new magnetic dynamics. This paragraph presents also some properties of magnetic traps, two significant examples and formulates an open problem. §3 describes the magnetic dynamical systems which can be reduced to 2-dimensional Hamiltonian systems. §4 analyses the magnetic fields which are symmetric or antisymmetric with respect to some symmetries.

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1 Biot-Savart-Laplace vector field

Let D be an open connected set of R^3 , with a piecewise smooth boundary ∂D . Denote by \vec{J} a C^∞ vector field on $\bar{D} = D \cup \partial D$.

The vector field

$$\vec{H}(M) = \frac{1}{4\pi} \int_D \frac{\vec{J}(P) \times P\vec{M}}{PM^3} dv_P, \quad M \in R^3$$

is called the *Biot-Savart-Laplace vector field*. The name comes from the fact that in case \bar{D} is a domain in which there exists a current density $\vec{J}(P), P \in \bar{D}$, then the magnetic field \vec{H} generated on R^3 by the electrical current is approximated by the preceding formula due to J.B.Biot, F.Savart, P.S.Laplace [2].

Remarks. 1) Since the measure (volume) of ∂D is zero, the preceding integral can be considered on $\bar{D} = D \cup \partial D$.

2) The integral defining $\vec{H}(M), M \in \bar{D}$ is an improper integral of the first type (both of the first and of the second type) if the domain D is bounded (unbounded).

3) The Biot-Savart-Laplace vector field \vec{H} is of class C^∞ on $R^3 \setminus \partial D$ and of class C^0 on ∂D .

4) The vector field \vec{J} can have zeros on \bar{D} .

The vector field \vec{H} is solenoidal. Hence it admits a vector potential

$$\vec{A}(M) = \frac{1}{4\pi} \int_D \frac{\vec{J}(P)}{PM} dv_P.$$

Indeed,

$$\begin{aligned} \text{rot} \vec{A}(M) &= \nabla_M \times \vec{A}(M) = \nabla_M \times \frac{1}{4\pi} \int_D \frac{\vec{J}(P)}{PM} dv_P = \\ &= \frac{1}{4\pi} \int_D \nabla_M \times \frac{\vec{J}(P)}{PM} dv_P = -\frac{1}{4\pi} \int_D \vec{J}(P) \times \nabla_M \frac{1}{PM} dv_P = \\ &= \frac{1}{4\pi} \int_D \frac{\vec{J}(P) \times P\vec{M}}{PM^3} dv_P = \vec{H}(M). \end{aligned}$$

On the other hand,

$$\begin{aligned} \text{div} \vec{A}(M) &= \left(\nabla_M, \vec{A}(M) \right) = \frac{1}{4\pi} \int_D \left(\nabla_M, \frac{\vec{J}(P)}{PM} \right) dv_P = \\ &= \frac{1}{4\pi} \int_D \left(\vec{J}(P), \nabla_M \frac{1}{PM} \right) dv_P = -\frac{1}{4\pi} \int_D \left(\vec{J}(P), \nabla_P \frac{1}{PM} \right) dv_P = \\ &= -\frac{1}{4\pi} \int_D \left(\nabla_P, \frac{\vec{J}(P)}{PM} \right) dv_P + \frac{1}{4\pi} \int_D \frac{1}{PM} \left(\nabla_P, \vec{J}(P) \right) dv_P = \\ &= -\frac{1}{4\pi} \int_{\partial D} \frac{(\vec{n}(P), \vec{J}(P))}{PM} dv_P + \frac{1}{4\pi} \int_D \frac{\text{div} \vec{J}(P)}{PM} dv_P, \end{aligned}$$

where $\vec{n}(P)$ is the unit normal vector field of the surface ∂D . If \vec{J} is a solenoidal vector field (a stationary electrokinetic field), and ∂D is a field surface of \vec{J} , i.e., $(\vec{n}(P), \vec{J}(P)) = 0$, then $\text{div} \vec{A}(M) = 0$, and so \vec{A} is a solenoidal vector field.

Under the hypothesis $\text{div} \vec{A}(M) = 0$, we compute

$$\begin{aligned} \text{rot} \vec{H}(M) &= \nabla_M \times (\nabla_M \times \vec{A}(M)) = \nabla_M (\nabla_M, \vec{A}(M)) - (\nabla_M, \nabla_M) \vec{A}(M) = \\ &= \nabla_M \text{div} \vec{A}(M) - \nabla_M^2 \vec{A}(M) = -\nabla_M^2 \vec{A}(M) = -\Delta_M \vec{A}(M), \end{aligned}$$

so that $\text{rot} \vec{H}(M) = 0$ for $M \in R^3 \setminus \bar{D}$, and $\text{rot} \vec{H}(M) = \vec{J}(M)$ for $M \in \bar{D}$; also, we have

$$\text{div} \vec{H}(M) = \text{div} \text{rot} \vec{A} = 0.$$

Consequently, the vector field \vec{H} is not irrotational, but the restriction of \vec{H} to $R^3 \setminus \bar{D}$ is an irrotational vector field. This restriction admits a local scalar potential.

Remarks.

1) We notice that

$$\nabla_M F(PM) = F'(PM) \frac{P\vec{M}}{PM} = -\nabla_P F(PM), \forall F : R \rightarrow R, \text{ derivable.}$$

2) If the point M has the coordinates (x, y, z) and the point P has the coordinates (ξ, η, ζ) , then

$$\frac{\partial}{\partial x} \vec{A}(M) = - \int_D \frac{\partial}{\partial \xi} \left(\frac{\vec{J}(P)}{PM} \right) dv_P + \int_D \frac{1}{PM} \frac{\partial}{\partial \xi} \vec{J}(P) dv_P, \text{ etc.}$$

3) If \vec{J} is a constant vector field on \bar{D} , then the magnetic field \vec{H} generated on R^3 by \vec{J} is a biscalar field, i.e., $(\vec{H}, \text{hbox} \text{rot} \vec{H}) = 0$.

4) The domain D can be replaced by a surface or a curve. In case of a curve, the current density \vec{J} must be nonzero everywhere along the curve.

2 Magnetic lines and surfaces. Magnetic traps

Let $\vec{H} = H_x \vec{i} + H_y \vec{j} + H_z \vec{k}$ be a magnetic field defined on R^3 . Denote by $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ the position vector of the point $M(x, y, z)$.

The magnetic line α starting of $M_0(x_0, y_0, z_0)$ at moment $t = 0$ is the oriented curve $\vec{r} = \vec{r}(t), t \in (-\epsilon, \epsilon)$, which satisfies the Cauchy problem

$$\frac{d\vec{r}}{dt} = \vec{H}(\vec{r}), \vec{r}(0) = \vec{r}_0.$$

The magnetic surface $\Sigma : h(x, y, z) = c$ relying on a curve $\beta : (a, b) \rightarrow R^3$ is the solution of the Cauchy problem

$$(\vec{H}, \nabla h) = 0, h(\beta(u)) = h(\beta(0)), \forall u \in (a, b).$$

A magnetic surface is generated by magnetic lines, and, in the absence of symmetries, a magnetic line is an open curve, and sometimes its image is densely in the magnetic surface.

Let U be an open connected set of R^3 with a piecewise smooth boundary ∂U and T_t the flow generated by the magnetic vector field \vec{H} . The flow T_t conserves the spaces volume since $\text{div} \vec{H} = 0$. The set U or its closedness \bar{U} is called a *trap region* of the magnetic field \vec{H} (*magnetic trap*) if $T_t(\bar{U}) \subset \bar{U}, \forall t \geq 0$. Particularly, any T_t -invariant set is a trap region. A magnetic trap is characterized by the fact that a magnetic line starting inside cannot leave it (because such a line cannot attend the boundary ∂U). The magnetic line starting in the exterior of the magnetic trap can enter or not inside the trap.

Suppose that the unit normal vector field \vec{n} of the surface ∂D is oriented towards $\text{ext}U$. If U is a magnetic trap, then on the boundary ∂U we have $(\vec{n}, \vec{H}) \leq 0$. Conversely, if $(\vec{n}, \vec{H}) = 0$ on ∂U , then ∂U is a magnetic surface; if $(\vec{n}, \vec{H}) > 0$ on ∂U , then $R^3 \setminus U$ is a trap region of the magnetic field \vec{H} , and if $(\vec{n}, \vec{H}) < 0$ on ∂U , then U is a trap region of the magnetic field \vec{H} .

Suppose there exist two magnetic traps U_1 and U_2 such that their boundaries $\partial U_1, \partial U_2$ have a common part which is a surface Σ or a curve γ . Then Σ is a magnetic surface, and γ is a magnetic line, respectively.

2.1. Theorem. *Let U be an open connected set of R^3 with a piecewise smooth boundary ∂U . If U is a magnetic trap and $\bar{U} = U \cup \partial U$ is compact, then the closed surface ∂U is a magnetic surface and $T_t(\bar{U}) = \bar{U}$.*

Proof. Suppose that the unit normal vector field \vec{n} of the surface ∂U is oriented towards ext U . Since $\operatorname{div} \vec{H} = 0$, by Gauss Theorem we obtain $\int_{\partial U} (\vec{n}, \vec{H}) d\sigma = 0$. If ∂U is not a magnetic surface, i.e., $(\vec{n}, \vec{H}) \neq 0$, then (\vec{n}, \vec{H}) must change sign on the closed surface ∂U and consequently U is not a magnetic trap.

The last assertion of the Theorem is a consequence of the conservation of the volume.

Remark. If \vec{H} is a magnetic field and \bar{U} is not a compact set, then it is possible that U is a magnetic trap for \vec{H} without ∂U be a magnetic surface. For example, the magnetic field $\vec{H}(x, y, z) = \vec{i}$, the region $U : x + y + z + 1 > 0$, the boundary $\partial U : x + y + z + 1 = 0$, the unit normal $\vec{n} = \frac{-\vec{i}-\vec{j}-\vec{k}}{\sqrt{3}}$ imply $(\vec{n}, \vec{H}) < 0$; hence U is a magnetic trap. Also, this example shows that in the preceding context the T_t -invariant set $\cap_{t \geq 0} T_t(\bar{U})$ can be the void set.

In the following part of this paragraph we refer only to the Biot-Savart-Laplace vector field. Is the region D in the Biot-Savart-Laplace formula a magnetic trap or not? A possible answer was given in the preceding theorem. The next theorem presents another alternative.

2.2. Theorem. *Suppose that the unit normal vector field \vec{n} of the surface ∂D is oriented towards ext D . Let*

$$\varphi : \bar{D} \times \partial D \rightarrow R, \varphi(P, M) = (\vec{n}(M), \vec{J}(P) \times P\vec{M}).$$

1) *If $\varphi(P, M) = 0, \forall P \in \bar{D}, \forall M \in \partial D$, i.e., the vector fields $n(\vec{M}), J(\vec{P}), P\vec{M}$ are coplanar on $\bar{D} \times \partial D$, then ∂D is a magnetic surface, and $R^3 \setminus D$ and D are magnetic traps.*

2) *If $\varphi(P, M) > 0, \forall P \in \bar{D}, \forall M \in \partial D$, then $R^3 \setminus D$ is a magnetic trap.*

3) *If $\varphi(P, M) < 0, \forall P \in \bar{D}, \forall M \in \partial D$, then D is a magnetic trap.*

Proof. Consequences of the relation

$$(\vec{n}(M), \vec{H}(M)) = \int_D \frac{(\vec{n}(M), \vec{J}(P) \times P\vec{M})}{PM^3} dv_P, \quad \forall M \in \partial D.$$

Remark. In the case 1) the vector fields \vec{J} and \vec{H} are tangent to ∂D . In the cases 2)-3), the vector field \vec{J} is tangent to ∂D and the vector field \vec{H} is transversal to ∂D .

Let

$$f : R^3 \rightarrow R, f = \frac{\mu_0}{2} (H_x^2 + H_y^2 + H_z^2)$$

be the energy of \vec{H} , where μ_0 is the absolute magnetic permeability of the medium.

2.3. Theorem. *Any magnetic line in $\operatorname{int}(R^3 \setminus D)$ is a trajectory of a potential dynamical system with three degrees of freedom associated to the potential $-f\mu_0^{-1}$ on $\operatorname{int}(R^3 \setminus D)$.*

Proof. Let α be a magnetic line included in $\operatorname{int}(R^3 \setminus D)$. Deriving $\frac{d\vec{r}}{dt} = \vec{H}$ along α we find the prolongation

$$\frac{d^2 \vec{r}}{dt^2} = \mu_0^{-1} \nabla f.$$

which is a potential dynamical systems of order two.

Remark. If α contains at least one point of ∂D , then the previous assertion fails.

Denoting

$$\frac{d\vec{r}}{dt} = -\vec{v},$$

the differential system of order two

$$\frac{d^2\vec{r}}{dt^2} = \mu_0^{-1}\nabla f$$

can be transcribed as a Hamiltonian system

$$\frac{d\vec{r}}{dt} = -\vec{v}, \quad \frac{d\vec{v}}{dt} = -\mu_0^{-1}\nabla f,$$

on the phase space R^6 , with the Hamiltonian

$$\mathcal{H}(\vec{r}, \vec{v}) = \frac{1}{2} \|\vec{v}\|^2 - \mu_0^{-1}f(\vec{r}).$$

The Hamiltonian flow conserves the phase space volume. The Hamiltonian \mathcal{H} is a first integral of the Hamiltonian dynamical system. The theory of Hamiltonian systems shows that

2.4. Theorem. *The trajectory of the dynamical system*

$$\frac{d^2\vec{r}}{dt^2} = \mu_0^{-1}\nabla f, \quad \vec{r}(0) = \vec{r}_0 \in R^3 \setminus \bar{D},$$

having $\mathcal{H} > -\mu_0^{-1}\infty$ { as constant total energy and staying in $R^3 \setminus \bar{D}$ is a reparametrized geodesic of the Riemann–Jacobi metric

$$g_{ij} = (\mathcal{H} + \mu_0^{-1}f)\delta_{ij}, \quad i, j = 1, 2, 3.$$

Remarks. 1) If the domain D is reduced to a curve γ , then the theorems 2.3 and 2.4 hold true on $R^3 \setminus \gamma$.

2) Theorem 2.4 shows that the preceding prolongation is a new Lotentz law based on a geometrical structure which incorporates the magnetic field.

2.5. Theorem. *Any magnetic line included in D is a trajectory of a nonpotential dynamical system with three degrees of freedom for which the energy*

$$\mathcal{H}(\vec{r}, \vec{v}) = \frac{1}{2} \|\vec{v}\|^2 - \mu_0^{-1}f(\vec{r})$$

is conserved.

Proof. If $\vec{r}(0) = \vec{r}_0 \in D$ and α rests in D , then deriving $\frac{d\vec{r}}{dt} = \vec{H}$ along α we get a prolongation

$$\frac{d^2\vec{r}}{dt^2} = \mu_0^{-1}\nabla f + \vec{J} \times \frac{d\vec{r}}{dt},$$

which is a nonpotential dynamical system of order two. If we take the scalar product with $\frac{d\vec{r}}{dt}$, we find $\frac{d}{dt}\mathcal{H} = 0$, i.e., the energy \mathcal{H} is conserved.

Remarks.

1) The vector field $\vec{J} \times \frac{d\vec{r}}{dt}$ does not produce a dissipation of energy along the solution α since it is orthogonal to the curve α .

2) Recently [8], [9] we have discovered a new geometrical structure showing that the preceding conservative nonpotential dynamical system of order two describes a new Lorentz world-force law (new magnetic dynamics).

3) The trajectories of the preceding conservative (potential or nonpotential) dynamical system of order two divide into three classes:

- the set of original magnetic lines with the energy $\mathcal{H} = \iota$;
- a set of trajectories with the energy $\mathcal{H} = \text{const.} < \iota$,
- a set of trajectories with the energy $\mathcal{H} = \text{const.} > \iota$.

4) Another prolongation is the nonpotential nonconservative dynamical system

$$\frac{d^2\vec{r}}{dt^2} = \mu_0^{-1}\nabla f + \vec{J} \times \vec{H}.$$

Since $\text{rot}(\vec{J} \times \vec{H}) \neq 0$, the vector field $\vec{J} \times \vec{H}$ corresponds to a dissipation of energy along the solutions α which are not orthogonal to $\vec{J} \times \vec{H}$.

5) In both cases the projection of the acceleration $\frac{d^2\vec{r}}{dt^2}$ on \vec{J} depends only of the projection of ∇f on \vec{J} .

Open problem. What is the physical signification for trajectories of the preceding conservative differential systems, with positive or negative constant energy ?

If the magnetic line α traverses ∂D , then its part contained in D is a trajectory of a nonpotential dynamical system of order two, and its part in $R^3 \setminus \bar{D}$ is a trajectory of a potential dynamical system of order two. Obviously, α can be or not smooth at the traversing point of ∂D , being a field line of the vector field \vec{H} which is of class C^∞ on $R^3 \setminus \partial D$ and of class C^o on ∂D .

Examples.

1) Suppose that $\bar{D} = D \cup \partial D$ is a circular cylinder of radius a carrying a steady current I . We fix the Cartesian frame $Oxyz$ such that Oz is the axis of the cylinder and $\vec{J} = \frac{I}{\pi a^2}\vec{k}$ is the current density. The electrical current generates the magnetic field [2] (Fig.1) given by

$$\vec{H}(M) = \frac{-y\vec{i} + x\vec{j}}{2\pi a^2}I, \text{ for } M(x, y, z) \in \bar{D},$$

and

$$\vec{H}(M) = \frac{-y\vec{i} + x\vec{j}}{2\pi(x^2 + y^2)}I, \text{ for } M(x, y, z) \in R^3 \setminus \bar{D}.$$

The Oz -axis consists of zeros of \vec{H} . The nonconstant magnetic lines are circles with the centers on Oz and situated in planes orthogonal to Oz . The boundary ∂D is a magnetic surface, though \vec{H} is only continuous on ∂D . The regions D and $R^3 \setminus \bar{D}$ are magnetic traps. Even in this simple case we have no answer for the preceding open problem.

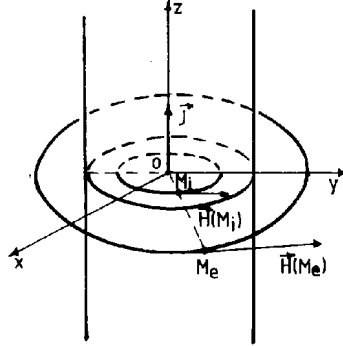


Fig.1

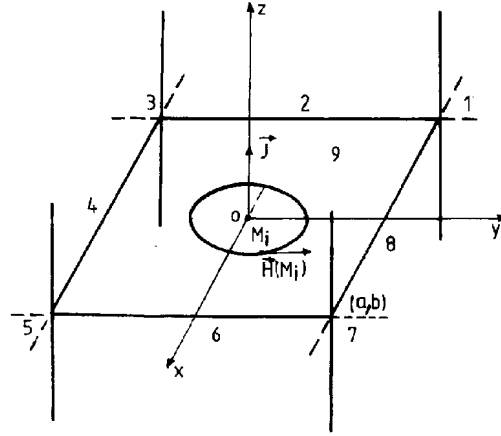


Fig.2

2) Suppose that $\bar{D} = D \cup \partial D$ is an infinite prism of rectangular section $[-a, a] \times [-b, b]$ carrying a steady current I . We choose a Cartesian frame $Oxyz$ such that Oz is the axis of the prism and $\vec{J} = \frac{I}{4ab}\vec{k}$ is the current density (Fig.2). The magnetic field generated by \vec{J} on R^3 has the components $(H_x, H_y, 0)$, where [2]

$$H_x = -\frac{I}{2ab}[(y-b)(\arctan \frac{x-a}{y-b} - \arctan \frac{x+a}{y-b}) - (y+b)(\arctan \frac{x-a}{y+b} - \arctan \frac{x+a}{y+b}) - \frac{x+a}{2} \ln \frac{(x+a)^2 + (y-b)^2}{(x+a)^2 + (y+b)^2} + \frac{x-a}{2} \ln \frac{(x-a)^2 + (y-b)^2}{(x-a)^2 + (y+b)^2}],$$

$$H_y = -\frac{I}{2ab}[(x-a)(\arctan \frac{y+b}{x-a} - \arctan \frac{y-b}{x-a}) - (x+a)(\arctan \frac{y+b}{x+a} - \arctan \frac{y-b}{x+a}) - \frac{y-b}{2} \ln \frac{(x-a)^2 + (y-b)^2}{(x+a)^2 + (y-b)^2} + \frac{y+b}{2} \ln \frac{(x-a)^2 + (y+b)^2}{(x+a)^2 + (y+b)^2}].$$

The field \vec{H} is defined on D and $R^3 \setminus D$ by the same formulas and can be extended by continuity on ∂D . The boundary ∂D is piecewise smooth, and is not a magnetic surface. The field \vec{H} is continuous on ∂D . If $S : R^3 \rightarrow R^3$ is the symmetry with respect to Oz and \vec{H} is regarded as $\vec{H} : R^3 \rightarrow R^3$, then $\vec{H} \circ S = S \circ \vec{H}$.

The Oz -axis consists of zeros of \vec{H} . The field lines are closed curves (symmetrical with respect to Oz) included in the family of curves

$$\int_{x_0}^x H_y(x, y) dx - \int_{y_0}^y H_x(x_0, y) dy = b_k, \quad z = c.$$

Computing the integrals, the first equation becomes

$$\frac{(x-a)^2 + (y+b)^2}{2} \arctan \frac{y+b}{x-a} + (y+b)^2 \arctan \frac{x-a}{y+b} - \frac{(x-a)^2 + (y-b)^2}{2} \arctan \frac{y-b}{x-a} - (y-b)^2 \arctan \frac{x-a}{y-b} +$$

$$\begin{aligned}
& + \frac{(x+a)^2 + (y-b)^2}{2} \arctan \frac{y-b}{x+a} + (y-b)^2 \arctan \frac{x+a}{y-b} - \\
& - \frac{(x+a)^2 + (y+b)^2}{2} \arctan \frac{y+b}{x+a} - (y+b)^2 \arctan \frac{x+a}{y+b} - \\
& - \frac{(x-a)(y-b)}{2} \ln((x-a)^2 + (y-b)^2) + \frac{(x+a)(y-b)}{2} \ln((x+a)^2 + (y-b)^2) + \\
& + \frac{(x-a)(y+b)}{2} \ln((x-a)^2 + (y+b)^2) - \frac{(x+a)(y+b)}{2} \ln((x+a)^2 + (y+b)^2) + \\
& + \frac{(y+b)^2}{2} \left(\arctan \frac{y+b}{x_o+a} + \arctan \frac{x_o+a}{y+b} - \arctan \frac{y+b}{x_o-a} - \arctan \frac{x_o-a}{y+b} \right) + \\
& \frac{(y-b)^2}{2} \left(\arctan \frac{y-b}{x_o-a} + \arctan \frac{x_o-a}{y-b} - \arctan \frac{y-b}{x_o+a} - \arctan \frac{x_o+a}{y-b} \right) = c_k.
\end{aligned}$$

The parameters b_k, c_k are constants depending on the regions $k = 1, 2, \dots, 9$ in the Fig.2, containing the fixed point (x_0, y_0) . Also, we must have in mind that

$$\arctan u + \arctan \frac{1}{u} = (\text{sign } u) \frac{\pi}{2}.$$

Some of these field lines traverse the boundary of the rectangle $[-a, a] \times [-b, b]$.

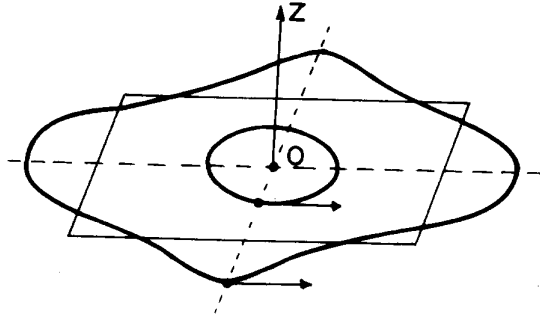


Fig. 3

Let us consider Runge-Kutta approximations of Cauchy problems

$$\begin{aligned}
\frac{dx}{dt} &= H_x, \quad \frac{dy}{dt} = H_y, \quad \frac{dz}{dt} = 0, \\
x(0) &= x_0, \quad y(0) = 0, \quad z(0) = 0,
\end{aligned}$$

obtained by us using a personal computer, referring to the case $a = 1, b = 2$, the step $p = 0,02$ and the following table

Initial points x_0	0.6	0.9	1.2	1.5	1.8	2.00	2.5
Number of Iterations	220	220	220	275	350	520	900

We remark that Runge-Kutta approximations of magnetic lines through points corresponding to $x_0 \in (0; 0.9)$ have a shape similar to those of an ellipse, and the curvature of the magnetic lines corresponding to $x_0 \in (1.2; 2.5)$ has non-constant sign (Fig.3).

Generally, the sign of the curvature of a magnetic line α coincides with the sign of the function

$$H_x^2 \frac{\partial H_y}{\partial x} + H_x H_y \left(\frac{\partial H_y}{\partial y} - \frac{\partial H_x}{\partial x} \right) - H_y^2 \frac{\partial H_x}{\partial y} = H_x^2 \frac{\partial H_y}{\partial x} + 2H_x H_y \frac{\partial H_y}{\partial y} - H_y^2 \frac{\partial H_x}{\partial y}$$

along $\alpha(I)$, but this sign cannot be constant throughout the plane xOy .

The shape of the preceding magnetic lines around the points on $xOz(yOz)$ becomes obvious if we take into account that component $H_x(x, y)$ vanishes for $y = 0$ (the component $H_y(x, y)$ vanishes for $x = 0$).

In fact, if $\alpha(t) = (x(t), y(t), 0), t \in R$ is the maximal solution of the preceding Cauchy problem, then $x_0 \neq 0$ is an extremum value of the function $x(t), t \in R$.

3 Magnetic dynamical systems which are bidimensional Hamiltonian systems

In this paragraph we shall generalize an idea of [4]. Let $\vec{H} = H_x \vec{i} + H_y \vec{j} + H_z \vec{k}$ be the magnetic field of Biot-Savart-Laplace and

$$\frac{dx}{dt} = H_x, \quad \frac{dy}{dt} = H_y, \quad \frac{dz}{dt} = H_z$$

the associated magnetic dynamical system. Suppose $H_z = 0, H_x = H_x(x, y), H_y = H_y(x, y)$. Since

$$0 = \operatorname{div} \vec{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y},$$

there exists a function $\mathcal{H} : R^2 \rightarrow R$ such that

$$\frac{\partial \mathcal{H}}{\partial y} = H_x, \quad \frac{\partial \mathcal{H}}{\partial x} = -H_y$$

and hence the magnetic dynamical system reduces to the bidimensional Hamiltonian system

$$\frac{dx}{dt} = \frac{\partial \mathcal{H}}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial \mathcal{H}}{\partial x}.$$

(see the examples of §2).

Suppose that H_z is not zero function and that

$$\frac{H_x}{H_z}, \quad \frac{H_y}{H_z}$$

are functions of x and y only. The magnetic dynamical system transcribes

$$\frac{dx}{dz} = \frac{H_x}{H_z}, \quad \frac{dy}{dz} = \frac{H_y}{H_z}.$$

If there exists $\mathcal{H} : R^2 \rightarrow R$ such that

$$\frac{\partial \mathcal{H}}{\partial y} = \frac{H_x}{H_z}, \quad \frac{\partial \mathcal{H}}{\partial x} = -\frac{H_y}{H_z},$$

then

$$\begin{aligned} 0 &= \frac{\partial}{\partial x} \left(\frac{H_x}{H_z} \right) + \frac{\partial}{\partial y} \left(\frac{H_y}{H_z} \right) = \operatorname{div} \frac{\vec{H}}{H_z} = \frac{1}{H_z} \operatorname{div} \vec{H} + \left(\vec{H}, \operatorname{grad} \frac{1}{H_z} \right) = \\ &= - \frac{(\vec{H}, \operatorname{grad} H_z)}{H_z^2} \end{aligned}$$

and hence H_z is either a constant function or a first integral of the magnetic dynamical system (i.e., $H_z(x, y, z) = c$ are field surfaces of \vec{H}).

Conversely, if H_z is either a first integral of the magnetic dynamical system or a constant function and the ratios

$$\frac{H_x}{H_z}, \quad \frac{H_y}{H_z}$$

are functions of x, y only, then the Hamiltonian \mathcal{H} does exist. Consequently the following theorem is true.

3.1. Theorem. *Suppose that none of the components (H_x, H_y, H_z) is the function zero. The corresponding magnetic dynamical system is reducible to a bidimensional Hamiltonian system if and only if at least one of the components (H_x, H_y, H_z) is either a constant or a first integral of the magnetic dynamical system and the ratios of the other two by this component depend effectively only of the variables who index them.*

Remark. Suppose that the magnetic dynamical system reduces to a Hamiltonian system on xOy . Then the magnetic lines are geodesics of a Riemannian-Jacobi structure which is conformal to the Euclidean structure on R^2 .

4 Symmetric and antisymmetric magnetic fields

The examples in §2 suggest the following considerations.

Let $S : R^3 \rightarrow R^3$ be a symmetry and \vec{H} be the Biot–Savart–Laplace field regarded as a function of the type $\vec{H} : R^3 \rightarrow R^3$.

If $\vec{H} \circ S = S \circ \vec{H}$, then the magnetic field \vec{H} is called *symmetric*. If $\vec{H} \circ S = -S \circ \vec{H}$, then the magnetic field \vec{H} is called *antisymmetric*.

4.1. Theorem. *Let D, \vec{J}, \vec{H} the mathematical entities from the formula of Biot–Savart–Laplace, and S the symmetry of R^3 with respect to the origin. Suppose $S(D) = D$.*

- 1) *If $\vec{J} \circ S = S \circ \vec{J}$, then $\vec{H} \circ S = -S \circ \vec{H}$.*
- 2) *If $\vec{J} \circ S = -S \circ \vec{J}$, then $\vec{H} \circ S = S \circ \vec{H}$.*

Proof. 1) We take into account the change of variables in the triple integral and the action of an orthogonal transformation upon the vector product. Since $S(x, y, z) = (-x, -y, -z)$, successively we have

$$\begin{aligned} \vec{H} \circ S(M) &= \int_{S(D)} \frac{\vec{J} \circ S(P) \times S(P) \vec{S}(M)}{S(P)S(M)^3} dv_{S(P)} = \int_D \frac{S \circ \vec{J}(P) \times S \circ P \vec{M}}{PM^3} dv_P = \\ &= (\operatorname{sign} S) S \circ \int_D \frac{\vec{J}(P) \times P \vec{M}}{PM^3} dv_P = -S \circ \vec{H}(M). \end{aligned}$$

4.2. Theorem. Let D, \vec{J}, \vec{H} the mathematical entities from the formula of Biot-Savart-Laplace, and S be the symmetry of R^3 with respect to Oz . Suppose $S(D) = D$.

- 1) If $\vec{J} \circ S = S \circ \vec{J}$, then $\vec{H} \circ S = S \circ \vec{H}$.
- 2) If $\vec{J} \circ S = -S \circ \vec{J}$, then $\vec{H} \circ S = -S \circ \vec{H}$.

Proof.1) The symmetry with respect to Oz is $S(x, y, z) = (-x, -y, z)$. It follows

$$\begin{aligned} \vec{H} \circ S(M) &= \int_{S(D)} \frac{\vec{J} \circ S(P) \times S \circ P\vec{M}}{S(P)S(M)^3} dv_{S(P)} = \int_D \frac{S \circ \vec{J}(P) \times S \circ P\vec{M}}{PM^3} dv_P = \\ &= \int_D (\text{sign}S) S \circ \frac{\vec{J}(P) \times P\vec{M}}{PM^3} dv_P = S \circ \int_D \frac{\vec{J}(P) \times P\vec{M}}{PM^3} dv_P = S \circ \vec{H}(M). \end{aligned}$$

Remark. The theory in this paragraph can be extended to any isometry.

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University Politehnica of Bucharest
Department of Mathematics I
Splaiul Independentei 313
RO-060042, Bucharest, Romania
e-mail:udriste@mathem.pub.ro