Blowing up in Rigid Analytic Geometry

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Abstract

We define the concept of blowing up map in rigid analytic geometry and show that such maps exist in full generality by giving an explicit construction. We then derive some elementary properties of blowing up maps, similar to those in the classical case.

0 Introduction and preliminaries

0.1. Introduction. The purpose of this work is to give a concise treatment on the existence of blowing up in rigid analytic geometry. (For an introduction into rigid analytic geometry, we refer to the book [BGR].)

To our knowledge, such an exposé did not yet appear in the literature and therefore, we found ourselves without any source for proper referencing. We had been using blowing up maps in rigid analytic geometry already in our Ph.D. thesis ([Sch 0]) and in our papers on the uniformization of (strongly) rigid subanalytic sets ([Sch 2]) and on the semianalyticity of these subanalytic sets lying in the plane ([Sch 3]). In the former article, we used a rigid analytic version of Hironaka's Embedded Resolution of Singularities (in the zero characteristic case), without having any reference for this theorem either. This was the motivation of recently writing down yet another paper called Embedded Resolution of Singularities in Rigid Analytic Geometry ([Sch 4]), in which we heavily relied on the existence and various elementary properties of blowing up maps in rigid analytic geometry. We hope therefore, that the

Received by the editors September 1994
Communicated by A. Verschoren
AMS Mathematics Subject Classification : 32S45, 32P05.
Keywords : Rigid Analytic Geometry, Blowing Up.

current work can establish a thorough work of reference for these and other papers on the subject.

In algebraic geometry, one is familiar with the general definition of a blowing up map as a solution to the universal problem of rendering a coherent sheaf of ideals invertible. We will take this too as the definition of a blowing up map in rigid analytic geometry. In the algebraic case, the existence of a blowing up, is proved by taking the **Proj** of the graded algebra associated to this ideal (see for instance [Hart,§7, p.163]). General as this construction might be, it is not a very illusive one to work with.

However, when one works over a field, more transparent constructions are available. In particular, we want to mention the construction proposed by Hironaka in his paper [Hi], in the complex analytic case, this time. It is exactly this construction we will mimic in our case, proving not only the existence of a blowing up of any rigid analytic variety by a closed analytic subvariety, but also indicating how to calculate this blowing up. We like to mention that we show the existence of a blowing up map without any reducedness requirement.

Let K be an algebraic closed field endowed with a complete non-archimedean norm. We denote by R the corresponding valuation ring. In the first paragraph, we give the definition of a blowing up of a rigid analytic variety with center a closed analytic subvariety. We prove the existence in case our variety is just the closed unit (poly-)disk R^n and, as center of the blowing up, we take the origin. Then we show, that given a blowing up map, base change over a flat map gives us again a blowing up map (see proposition (1.4.1)). In particular, we derive from this that the blowing up of R^{k+n} with center R^k exists.

In the second paragraph we then give our main theorem (2.2.2) on the existence of a blowing up. We show first, that, given a blowing up map, we can define the strict transform $\tilde{W}$ of any closed analytic subvariety W under this map. Moreover, this strict transform $\tilde{W}$ turns out to be the blowing up of W with center the original center intersected with W (see proposition (2.2.1)). The general construction goes now as follows. Let X be a general rigid analytic variety and take as center a closed analytic subvariety Z of X. By the flat base change property, we show that, by taking an admissible affinoid cover of X, we may already assume from the start that X is affinoid. Then we embed X as a closed analytic subvariety in R^{k+n}, in such way that the intersection of X with R^k is exactly the center Z. By supra, the blowing up $\pi : \tilde{Y} \to R^{k+n}$ of R^{k+n} with center R^k exists, so that the desired blowing up of X is given by the strict transform of X under π.

In the last paragraph, we show some elementary properties of blowing up, which hold also in the algebraic case. More precisely, in (3.2.1) we prove that a blowing up map is proper (in the sense of Kiehl). Moreover, if our variety is irreducible, we obtain as a corollary in (3.2.2) that this blowing up map is also surjective (excluding the extreme and uninteresting case that (the support of) the center is the whole space).

0.2. Conventions. We fix, once and for all an algebraic closed field K endowed with a complete non-archimedean norm. We denote the corresponding valuation ring by R. In the sequel we will adopt the notation and the terminology of [BGR].

In particular, let X be a rigid analytic variety. We will denote its structure sheaf by $\mathcal{O}_X$. Let $i : Y \to X$ be a closed immersion of rigid analytic varieties. Then we call Y a *closed analytic subvariety* of X. Let $i^{\#} : \mathcal{O}_X \to i_*(\mathcal{O}_Y)$ denote the corresponding surjective homomorphism of $\mathcal{O}_X$-modules. The kernel $\mathcal{I} = \ker(i^{\#})$ is a coherent $\mathcal{O}_X$-ideal and we call it the *$\mathcal{O}_X$-ideal defining* Y, or alternatively, we say that Y is the *closed analytic subvariety of X associated* to the $\mathcal{O}_X$-ideal $\mathcal{I}$.

The underlying set $|i(Y)|$ of the image $i(Y)$ is an *analytic subset* of X. By abuse of notation, we will sometimes consider Y itself as an analytic subset of X, especially when we consider the admissible open given by $X \setminus Y$, where the correct notation should be $X \setminus |i(Y)|$. Note that on an analytic subset Y of X, we can define many structures of a closed analytic subvariety. Namely one for each coherent $\mathcal{O}_X$-ideal $\mathcal{I}$, such that $V(\mathcal{I}) = Y$. Recall that

$$V(\mathcal{I}) = \{\, x \in X \mid \mathcal{I}_x \neq \mathcal{O}_{X,x} \,\},$$

and we call this analytic subset the *zero-set* of $\mathcal{I}$. (Any analytic subset is realized in such way). In particular there is exactly one structure of a reduced analytic subvariety on Y, given by the coherent $\mathcal{O}_X$-ideal $\mathfrak{id}(Y)$, which is a radical ideal.

0.3. Definition. Given a map $f : Y \to X$ and a coherent sheaf of $\mathcal{O}_X$-ideals $\mathcal{I}$, we call the *inverse image ideal sheaf* of $\mathcal{I}$, the image of the canonical map $f^*\mathcal{I} \to \mathcal{O}_Y$, and we denote this coherent sheaf of $\mathcal{O}_Y$-modules by $f^{-1}(\mathcal{I})\mathcal{O}_Y$, or, when no confusion can arise, simply by $\mathcal{I}\mathcal{O}_Y$.

If Z is the closed analytic subvariety of X defined by $\mathcal{I}$, then we define $f^{-1}(Z)$ to be the closed analytic subvariety of Y associated to $\mathcal{I}\mathcal{O}_Y$. In other words, we have that $f^{-1}(Z) = Z \times_X Y$. Of course, if Z is only considered as an analytic subset of X, we mean by $f^{-1}(Z)$ only the analytic subset, which is the set-theoretical inverse image of Z.

In particular, if both X and Y are affinoid, with corresponding affinoid algebra A, respectively B, and if $\mathfrak{a}$ is the ideal of A corresponding to $\mathcal{I}$, then $\mathfrak{a}B$ corresponds to $\mathcal{I}\mathcal{O}_Y$.

If $y \in Y$, then we will sometimes denote the stalk of $f^{-1}(\mathcal{I})\mathcal{O}_Y$ at y by $\mathcal{I}\mathcal{O}_{Y,y}$, in stead of the more cumbersome $(f^{-1}(\mathcal{I})\mathcal{O}_Y)_y$ or $\mathcal{I}_x\mathcal{O}_{Y,y}$, where $x = f(y)$.

0.4. Lemma. *Let $X = Sp\,A$ be an affinoid variety and let $\mathfrak{a}$ be an ideal in A. Let U be an admissible open of X, contained in $X \setminus V(\mathfrak{a})$. Then $\mathfrak{a}\mathcal{O}_X(U) = \mathcal{O}_X(U)$.*

Proof. Let $\{U_i\}_i$ be an admissible affinoid covering of U. Note that the $U_i \cap U_j$ are also affinoid since U is separated and [BGR,9.6.1. Proposition 6].

First of all we have, for each i, that $\mathfrak{a}\mathcal{O}_X(U_i) = \mathcal{O}_X(U_i)$. Indeed, suppose the contrary. So there exists a maximal ideal $\mathfrak{m}$ of $\mathcal{O}_X(U_i)$ containing $\mathfrak{a}\mathcal{O}_X(U_i)$. Let $x \in U_i$ be the corresponding point. Then we have that $f(x) = 0$, for all $f \in \mathfrak{a}$, contradicting the fact that $U_i \subset U \subset X \setminus V(\mathfrak{a})$.

Due to the fact that $\mathcal{O}_X$ is a sheaf, we have, by [BGR,9.2.1. Definition 2], the following exact diagram

$$\mathcal{O}_X(U) \xrightarrow{\sigma} \prod_i \mathcal{O}_X(U_i) \underset{\sigma''}{\overset{\sigma'}{\rightrightarrows}} \prod_{i,j} \mathcal{O}_X(U_i \cap U_j), \tag{1}$$

where σ is induced by the restriction maps $\mathcal{O}_X(U) \to \mathcal{O}_X(U_i)$ and where σ' (respectively, σ'') is induced by the restriction maps $\mathcal{O}_X(U_i) \to \mathcal{O}_X(U_i \cap U_j)$ (respectively, $\mathcal{O}_X(U_j) \to \mathcal{O}_X(U_i \cap U_j)$). On the other hand, let $\mathcal{I}$ denote the coherent $\mathcal{O}_X$-ideal corresponding to $\mathfrak{a}$. Our previous remark implies that $\mathcal{I}(U_i) = \mathcal{O}_X(U_i)$, as well as $\mathcal{I}(U_i \cap U_j) = \mathcal{O}_X(U_i \cap U_j)$, since the intersections $U_i \cap U_j$ are also affinoid. Hence the exact diagram expressing that $\mathcal{I}$ is a sheaf is identical with (1), except for the first module, which equals $\mathcal{I}(U)$. By exactness this implies that these two also must agree, so that we have that $\mathfrak{a}\mathcal{O}_X(U) = \mathcal{I}(U) = \mathcal{O}_X(U)$. ■

1 Blowing Up in Rigid Analytic Geometry

1.1 Invertible Ideals

1.1.1. Definition. Let X be a rigid analytic variety. We call a sheaf of $\mathcal{O}_X$-ideals $\mathcal{I}$ *invertible*, if, for each point $x \in X$, we have that $\mathcal{I}_x$ is an invertible ideal in $\mathcal{O}_{X,x}$ in the ring theoretic sense, meaning that $\mathcal{I}_x$ is generated by a regular element of $\mathcal{O}_{X,x}$. It is then an exercise to prove that $\mathcal{I}$ is invertible, if and only if, it is locally (in the Grothendieck topology on X) free of rank one.

1.1.2. Lemma. *Let $(S, \mathfrak{p})$ be a local ring and $\mathfrak{a}$ an ideal of S. If $\mathfrak{a} = (\alpha_1, \dots, \alpha_s)$ is invertible, then there exists an i, such that $\mathfrak{a} = \alpha_i S$ and α_i is a regular element of S.*

Proof. Let e be a regular element of S, such that $\mathfrak{a} = eS$. Hence we can find $r_i, s_i \in S$, for $i = 1, \dots, s$, such that $\alpha_i = r_i e$ and

$$e = \sum_{i=1}^{s} s_i \alpha_i. \tag{1}$$

But then, substituting $\alpha_i = r_i e$ in (1), and using that e is regular, we find that

$$1 = \sum_{i=1}^{s} r_i s_i.$$

Hence one of the $r_i \notin \mathfrak{p}$, say $r_1 \notin \mathfrak{p}$, and hence is a unit in S. Hence we obtain that $\mathfrak{a} = eS = \alpha_1 S$. ■

1.1.3. Lemma. *Let $(S, \mathfrak{p}) \to (T, \mathfrak{q})$ be a faithfully flat local morphism of local rings. Let $\mathfrak{a}$ be an ideal of S. Then $\mathfrak{a}$ is invertible, if and only if, $\mathfrak{a}T$ is invertible.*

Proof. $\Rightarrow$. Let e be a regular element of S, generating $\mathfrak{a}$. Then clearly $\mathfrak{a}T = eT$ and since $S \to T$ is flat, e is regular in T.

$\Leftarrow$. Let $\mathfrak{a} = (\alpha_1, \dots, \alpha_s)$, with the $\alpha_i \in S$. By (1.1.2), we can find an i, say $i = 1$, such that $\mathfrak{a}T = \alpha_1 T$ and α_1 is regular in T. Since $S \to T$ is faithfully flat, we have that

$$\mathfrak{a} = \mathfrak{a}T \cap S = \alpha_1 T \cap S = \alpha_1 S,$$

and clearly, since α_1 is regular in T, it is also regular in S, since $S \to T$ is injective. ■

Remark. Note that for the only if part, we do not need to assume that S nor T are local and we only need that $S \to T$ is (not necessarily faithfully) flat.

1.1.4. Proposition. *Let $X = Sp(A)$ be an affinoid variety and $\mathfrak{a}$ an ideal of A. Let x be a point of X and $\mathfrak{m}$ the corresponding maximal ideal of x. Then $\mathfrak{a}\mathcal{O}_{X,x}$ is invertible, if and only if, $\mathfrak{a}A_{\mathfrak{m}}$ is.*

Proof. By [BGR,7.3.2. Proposition 3], we know that

$$\widehat{\mathcal{O}}_{X,x} \cong \widehat{A}_{\mathfrak{m}}.$$

Hence the proposition now follows from applying (1.1.3) twice. ∎

1.2 Definition of Blowing Up

1.2.1. Definition. Let X be a rigid analytic variety and let Z be a closed analytic subvariety of X. Let $\mathcal{I}$ be the coherent $\mathcal{O}_X$-ideal defining Z. We call a map of rigid analytic varieties

$$\pi : \tilde{X} \to X,$$

the *blowing up* of X with *center* Z, (or, the blowing up of X with respect to the $\mathcal{O}_X$-ideal $\mathcal{I}$), if the following two conditions hold.

(i) $\pi^{-1}(\mathcal{I})\mathcal{O}_{\tilde{X}}$ is an invertible $\mathcal{O}_{\tilde{X}}$-sheaf.

(ii) If $f : Y \to X$ is a map of rigid analytic varieties, such that $f^{-1}(\mathcal{I})\mathcal{O}_Y$ is invertible, then there exists a unique factorization $g : Y \to \tilde{X}$ of f over π, i.e., the following diagram commutes

$$\begin{array}{ccc} Y & \xrightarrow{\;g\;} & \tilde{X} \\ \Vert & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow[\;f\;]{} & X. \end{array}$$

Sometimes we will call $\tilde{X}$ the blowing up of X, rather than the map π. In these cases it should be clear which map we mean.

Note that since a blowing up is defined by an universal problem, if it exist, it must be unique (up to an isomorphism).

1.2.2. Example. Let X be a rigid analytic variety and take $Z = \emptyset$, which is considered as a closed analytic subvariety of X by the $\mathcal{O}_X$-ideal $\mathcal{O}_X$ itself. Clearly this ideal is already invertible, hence the blowing up of X with empty center is nothing but the identity map on X.

1.2.3. Example. Let X be a rigid analytic variety and take now as center Z any closed analytic subvariety structure on the the whole space X. In other words,

X and Z have the same underlying point set. The defining coherent $\mathcal{O}_X$-ideal of Z is now a nilpotent ideal. Since this can never be invertible, unless there are no points, the blowing up must be the empty rigid analytic variety.

1.2.4. Lemma. *Let X be a rigid analytic variety and $\mathcal{I}$ a coherent $\mathcal{O}_X$-ideal. In order to check that a map of rigid analytic varieties $\pi : \tilde{X} \to X$ for which (1.2.1.(i)) holds, is the blowing up of X with respect to $\mathcal{I}$, it is enough to check condition (1.2.1.(ii)) for Y affinoid.*

Proof. Assume that (1.2.1.(ii)) has been verified for all Y affinoid and let Y be an arbitrary rigid analytic variety. Let $f : Y \to X$ be a map of rigid analytic varieties such that $\mathcal{I}\mathcal{O}_Y$ is invertible. Let $\{Y_i\}_i$ be an admissible affinoid covering of Y and denote by f_i the restriction of f to Y_i.

Because being invertible is a local property, we can find, by our assumption, unique maps of rigid analytic varieties $g_i : Y_i \to \tilde{X}$, making the following diagram commute

$$\begin{array}{ccc} Y_i & \xrightarrow{g_i} & \tilde{X} \\ \downarrow & & \downarrow \pi \\ Y & \xrightarrow[f]{} & X. \end{array} \tag{1}$$

By the uniqueness of these g_i, we must have that g_i and g_j agree on $Y_i \cap Y_j$, for all i and j. Indeed, let $\{U_k\}_k$ be an admissible affinoid covering of $Y_i \cap Y_j$. Then by our assumption, $g_i|_{U_k}$ and $g_j|_{U_k}$ must agree, since they are both solutions to the commutativity of the following diagram

$$\begin{array}{ccc} U_k & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \pi \\ Y & \xrightarrow[f]{} & X. \end{array}$$

Since this holds for all k, our claim follows.

Hence we can paste the g_i together ([BGR,9.3.3. Proposition 1]) in order to obtain a map $g : Y \to \tilde{X}$, which, when restricted on Y_i, equals g_i. Hence this map gives a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{g} & \tilde{X} \\ \| & & \downarrow \pi \\ Y & \xrightarrow[f]{} & X. \end{array} \tag{2}$$

The uniqueness of g follows from the following observation. The restriction to Y_i of any other morphism g which makes diagram (2) commute, is also a solution to the commutativity of (1), and hence must be equal to g_i. ■

1.3 Blowing Up at the Origin

1.3.1. Definition. Let $\mathbb{P}^n$ denote the n-dimensional projective space over K ([BGR,9.3.4., Example 3]). Define the following analytic subset X of $R^n \times \mathbb{P}^{n-1}$, by

$$\tilde{X} \stackrel{\text{def}}{=} \{(x,\xi) \in R^n \times \mathbb{P}^{n-1} \mid \text{ for all } i,j = 1,\dots,n : x_i\xi_j = x_j\xi_i\},$$

where $(x,\xi) = (x_1,\dots,x_n,\xi_1;\dots;\xi_n)$ are coordinates in $R^n \times \mathbb{P}^{n-1}$. We will consider $\tilde{X}$ as a rigid analytic variety by putting the reduced closed subvariety structure on it. Let us denote by $\tilde{\pi}$ the canonical projection $R^n \times \mathbb{P}^{n-1} \to R^n$ and define π as the restriction of $\tilde{\pi}$ to $\tilde{X}$. In other words,

$$\pi : \tilde{X} \to R^n : (x,\xi) \mapsto x.$$

We will prove below that π is the blowing up of R^n with center the origin, where we consider the origin with its reduced closed subvariety structure.

1.3.2. Proposition. *The in (1.3.1) defined rigid analytic variety $\tilde{X}$, is the blowing up of $X = R^n$ at the origin (with its reduced closed subvariety structure).*

Proof. Let $I = (S_1,\dots,S_n)$ be the maximal ideal in $K\langle S\rangle$ corresponding to the origin, where $S = (S_1,\dots,S_n)$ is a set of variables. For each k, let

$$\tilde{X}_k = \{(s,\xi) \mid \forall i,j : s_i\xi_j = s_j\xi_i \quad \text{and} \quad |\xi_i| \le |\xi_k|\}.$$

Then $\tilde{X}_k$ is an admissible affinoid of $\tilde{X}$. Indeed, letting $t_i = \xi_i/\xi_k$, for all i, we see that the affinoid algebra of $\tilde{X}_k$ is isomorphic with

$$(1) \qquad A_k = \frac{K\langle S,T\rangle}{(S_1 - S_kT_1,\dots,S_n - S_kT_n, T_k - 1)},$$

where $T = (T_1,\dots,T_n)$ is another set of variables. Under this isomorphism, the image of I in A_k is the ideal generated by S_k and the latter element is regular in A_k. (Recall that an element in a ring R is called *regular* if it is not a zero-divisor). This proves that $I\mathcal{O}_{\tilde{X}}$ is invertible.

Therefore, by (1.2.4), we need only to show that every map $f : Y \to X$, with $Y = Sp\,B$ affinoid and such that IB is invertible, can uniquely be factored over $\tilde{X}$. Let

$$\mathfrak{a}_k = ((S_k)B : IB) \stackrel{\text{def}}{=} \{s \in B \mid sIB \subset (S_k)B\},$$

for every k. Let $Y_k = Y \setminus V(\mathfrak{a}_k)$. A point $y \in Y$ belongs to Y_k, if and only if, $I\mathcal{O}_{Y,y} = (S_k)\mathcal{O}_{Y,y}$. Using lemma (1.1.2), we therefore get that the Y_k cover Y, in other words, form an admissible covering of Y. (Here we used that each covering by Zariski open subsets is admissible).

In order to define a factorization $g : Y \to \tilde{X}$ of f, it is enough, by [BGR,9.3.3. Proposition 1], to define factorizations $Y_k \to \tilde{X}$, which agree on the intersections $Y_k \cap Y_l$, for all $k \neq l$. Let us therefore define maps $g_k : Y_k \to \tilde{X}_k$, such that following diagram commutes

$$\begin{array}{ccc} Y_k & \xrightarrow{g_k} & \tilde{X}_k \\ \downarrow & & \downarrow \pi \\ Y & \xrightarrow[f]{} & X. \end{array}$$

Therefore, by [BGR,9.3.3. Corollary 2], in order to define these maps g_k it is enough to give morphisms $\gamma_k : A_k \to B_k = \mathcal{O}_Y(Y_k)$, such that following diagram commutes

$$\begin{array}{ccc} K\langle S\rangle & \xrightarrow{\sigma} & B \\ \downarrow & & \downarrow u_k \\ A_k & \xrightarrow[\gamma_k]{} & B_k. \end{array} \tag{2}$$

where $\sigma : K\langle S\rangle \to B$ denotes the algebra morphism corresponding to the map $f : Y \to X$.

Fix k and let $s_i = \sigma(S_i)$. Consider the s_i as elements of B_k, via the canonical map u_k. By construction and (0.4), we have that $\mathfrak{a}_k B_k = B_k$, or, in other words, that s_k generates IB_k. Since the latter is invertible, because u_k is flat, we must have that s_k is regular in B_k. So, we can find, for all i, unique elements $t_i \in B_k$, with $t_k = 1$, such that

$$s_i = t_i s_k. \tag{3}$$

Define now the γ_k, by sending S_i to s_i and T_i to t_i. It is left to the reader to verify that the thus constructed maps g_k agree on their common intersection and that we obtain a factorization g of the original map f.

To finish the proof, we have to show that this map g is unique. Therefore, it is enough to prove the uniqueness of the γ_k. Hence, let γ be an other map making diagram (2) commute.

From the commutativity of (2), we get that $s_i = \gamma(S_i)$. Hence, from the relation $S_i = S_k T_i$ in A_k, we get that

$$s_i = s_k \gamma(T_i),$$

for all i. Hence, by the uniqueness of t_i in equation (3), we obtain that $t_i = \gamma(T_i)$, what we had to prove. ■

1.4 Flat Base Change and Blowing Up

1.4.1. Proposition. *Let S be a rigid analytic variety. Let X and Y be a rigid analytic varieties over S. Let Z be a closed analytic subvariety of X. Suppose that the blowing up of X with center Z exists, say*

$$\pi : \tilde{X} \to X.$$

If $Y \to S$ is flat, then

$$\pi \times_S 1_Y : \tilde{X} \times_S Y \to X \times_S Y$$

is the blowing up of $X \times_S Y$ with center $Z \times_S Y$.

Proof. Let $\mathcal{I}$ be the coherent $\mathcal{O}_X$-ideal defining Z. Put $\mathcal{J} = \mathcal{I}\mathcal{O}_{X \times_S Y}$. Hence $\mathcal{J}$ is the coherent $\mathcal{O}_{X \times_S Y}$-ideal defining $Z \times_S Y$.

Let z be a point of $\tilde{X} \times_S Y$, with respective projections $\tilde{x} \in \tilde{X}$ and $y \in Y$ and let $s \in S$ be the image of z under the structure map $\tilde{X} \times_S Y \to S$. Note that we have the following isomorphism of local rings

$$\widehat{\mathcal{O}}_{\tilde{X} \times_S Y, z} \cong \widehat{\mathcal{O}}_{\tilde{X},\tilde{x}} \widehat{\otimes}_{\mathcal{O}_{S,s}} \widehat{\mathcal{O}}_{Y,y}. \tag{1}$$

By base change and taking completion, our hypothesis implies that the latter is a faithfully flat $\mathcal{O}_{\tilde{X},\tilde{x}}$-algebra. Hence, by using (1.1.3) twice and the fact that $\mathcal{I}\mathcal{O}_{\tilde{X},\tilde{x}}$ is invertible, we conclude that $\mathcal{I}\mathcal{O}_{\tilde{X} \times_S Y} = \mathcal{J}\mathcal{O}_{\tilde{X} \times_S Y}$ is invertible. This establish condition (i) of the definition (1.2.1) of a blowing up.

Let T be an arbitrary rigid analytic variety and $f : T \to X \times_S Y$ a map of rigid analytic varieties, such that $\mathcal{J}\mathcal{O}_T$ is invertible. But then, by the universal property of blowing up, applied to the composite map

$$T \xrightarrow{f} X \times_S Y \to X,$$

we find a factorization $g : T \to \tilde{X}$, making following diagram commute,

$$\begin{array}{ccc} T & \xrightarrow{g} & \tilde{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi} \\ X \times_S Y & \longrightarrow & X. \end{array}$$

Using the composition $T \xrightarrow{f} X \times_S Y \to Y$ and using the universality of the fiber product, we then obtain a map $g' : T \to \tilde{X} \times_S Y$, making following diagram commute

$$\begin{array}{ccccc} T & \xrightarrow{g'} & \tilde{X} \times_S Y & \xrightarrow{\pi \times_S 1} & X \times_S Y \\ \| & & \downarrow & & \downarrow \\ T & \xrightarrow[g]{} & \tilde{X} & \xrightarrow[\pi]{} & X. \end{array}$$

From this diagram, one obtains that $f = (\pi \times_S 1)g'$, since both maps are the same when composed with the two canonical projections $X \times_S Y \to X$ and $X \times_S Y \to Y$, hence, by universality of the fiber product, they have to be equal. This proves that f factors through g'. The uniqueness of g' follows from the corresponding uniqueness in the fiber product and the blowing up. Hence we proved our claim. ■

Remark. The condition on $Y \to S$ to be flat, can not be left out, as should be clear from proposition (2.2.1) infra.

1.4.2. Corollary. *Let X be a rigid analytic variety and Z a closed analytic subvariety of X. Suppose that*

$$\pi : \tilde{X} \to X$$

is the blowing up of X with center Z. Let Y be a rigid analytic variety. Then

$$\pi \times 1_Y : \tilde{X} \times Y \to X \times Y$$

is the blowing up of $X \times Y$ with center $Z \times Y$.

Proof. The canonical map $Y \to Sp\, K$ is flat and a direct product is nothing but a fiber product over $Sp\, K$. Hence we can apply proposition (1.4.1) to this situation. ■

1.4.3. Corollary. *The blowing up of $R^n \times R^k$ with center R^k exists.*

Remark. As our notation suggests, we consider R^k as a reduced closed subvariety of $R^{n+k} = R^n \times R^k$, where we identify R^k with $O \times R^k$ and where O is the origin of R^n.

Proof. Let

$$\pi : \tilde{X} \to R^n$$

be the blowing up of R^n with center the origin O (considered as a reduced closed analytic subvariety), as given by (1.3.2). We are now done by the previous corollary (1.4.2). ■

1.4.4. Proposition. *Let X and $\tilde{X}$ be rigid analytic varieties and Z a closed analytic subvariety of X. Let $\pi : \tilde{X} \to X$ be a map of rigid analytic varieties. Let $\{X_i\}_i$ be an admissible covering of X and let $\tilde{X}_i = \pi^{-1}(X_i)$. Then the following are equivalent.*

(i) $\pi : \tilde{X} \to X$ *is the blowing up of X with center Z.*

(ii) $\pi|_{\tilde{X}_i} : \tilde{X}_i \to X_i$ *is the blowing up of X_i with center $Z \cap X_i$, for all i.*

Proof. $(i) \Rightarrow (ii)$. By [BGR,7.3.2. Corollary 6] we obtain that the open immersion $X_i \subset X$ is flat. Hence we are able to apply (1.4.1) with $Y = X_i$ and $S = X$. From [BGR,9.3.5. Lemma 3] we get that

$$\tilde{X} \times_X X_i \cong \pi^{-1}(X_i)$$

and that $Z \times_X X_i = X_i \cap Z$.

$(ii) \Rightarrow (i)$. Let $\mathcal{I}$ be the ideal defining Z. It is easy to see, since being invertible is local, that $\mathcal{I}\mathcal{O}_{\tilde{X}}$ is invertible. Let Y be a rigid analytic variety and $f : Y \to X$ be a map of rigid analytic varieties, such that $\mathcal{I}\mathcal{O}_Y$ is invertible. We need to show that f uniquely factorizes over $\tilde{X}$.

Let $Y_i = f^{-1}(X_i)$ and let f_i denote the restriction of f to Y_i. Clearly $\mathcal{I}\mathcal{O}_{Y_i}$ is invertible, for each i, so that by our hypothesis, there exists a unique map $g_i : Y_i \to \tilde{X}_i$, making following diagram commute

(1)
$$\begin{CD} Y_i @>{g_i}>> \tilde{X}_i \\ @| @VV{\pi|_{\tilde{X}_i}}V \\ Y_i @>>{f_i}> X_i. \end{CD}$$

By the universal property of blowing up, we get that g_i and g_j agree on $Y_i \cap Y_j$, for all $i \neq j$. Hence, we can paste the g_i together, in order to obtain a map $g : Y \to \tilde{X}$, such that g_i equals the restriction of g to Y_i, for each i. Therefore, from (1), we get a commutative diagram

(2)
$$\begin{CD} Y @>{g}>> \tilde{X} \\ @| @VV{\pi}V \\ Y @>>{f}> X. \end{CD}$$

The uniqueness of g follows from the fact that any morphism g which renders (2) commutative, when restricted to Y_i, is a solution of the commutativity of (1), and hence must coincide with g_i. ∎

1.4.5. Corollary. *Let X be a rigid analytic variety and Z a closed analytic subvariety of X. Suppose that*

$$\pi : \tilde{X} \to X$$

is the blowing up of X with center Z. Then the restriction

$$\pi|_{\tilde{X} \setminus \pi^{-1}(Z)} : \tilde{X} \setminus \pi^{-1}(Z) \to X \setminus Z$$

is an isomorphism.

Proof. Let $U = X \setminus Z$. Hence U is an admissible open of X. By (1.4.4), we get that

(1)
$$\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$$

is the blowing up of U with center $U \cap Z = \emptyset$. But by example (1.2.2), we know that the identity on U must be the blowing up of U with empty center. Therefore, by universality, (1) must be an isomorphism. Clearly $\pi^{-1}(U) = \tilde{X} \setminus \pi^{-1}(Z)$. ∎

2 Existence of Blowing Up

2.1 Minimality Condition

2.1.1. Definition. Let X be a rigid analytic variety and W a closed analytic subvariety. We say that the pair (X, W) satisfies the *minimality condition*, if, for every analytic subset V of X, the equality

$$V \setminus W = X \setminus W$$

implies $V = X$.

Remark. Note that this condition only depends on the underlying set of W, in other words, on the analytic subset determined by W and not on its structure of an analytic variety.

2.1.2. Lemma. *Let X be a rigid analytic variety and W a closed analytic subvariety of X. Let $\{U_i\}_i$ be an arbitrary covering of X by admissible affinoid opens U_i. Suppose that, for each i, the pair $(U_i, U_i \cap W)$ satisfies the minimality condition. Then the pair (X, W) also satisfies the minimality condition.*

Proof. Straightforward. ■

2.1.3. Proposition. *Let $X = Sp\,A$ be an affinoid variety and W a closed analytic subvariety of X defined by the ideal $\mathfrak{a}$ of A. If $Ann_A(\mathfrak{a}) = 0$, then the pair (X, W) satisfies the minimality condition.*

Proof. Let V be an analytic subset of X, such that

$$V \setminus W = X \setminus W. \tag{1}$$

By [BGR,9.5.2. Corollary 8] we know that V is affinoid. Let $\mathfrak{N}$ be the (radical) ideal in A defining V, i.e. $V = V(\mathfrak{N})$. From (1) one gets that $X = V \cup W$ and therefore that $rad(\mathfrak{a}\mathfrak{N}) = rad(0)$. Hence some power $(\mathfrak{a}\mathfrak{N})^k = 0$. By our hypothesis, we therefore get that $\mathfrak{N}^k = 0$, and thus $X = V$. ■

2.1.4. Corollary. *Let X be a rigid analytic variety and W a closed analytic subvariety with associated $\mathcal{O}_X$-ideal $\mathcal{I}$. If, for every point $x \in X$, we have that $Ann_{\mathcal{O}_{X,x}}(\mathcal{I}_x) = 0$, then the pair (X, W) satisfies the minimality condition.*

Proof. For each point $x \in X$, we can find an admissible affinoid $U = Sp\,C$ in X, containing x, such that $\mathrm{Ann}_{\mathcal{O}_X(U)}(\mathcal{I}(U)) = 0$. Hence by (2.1.3) we get that the pair $(U, W \cap U)$ satisfies the minimality condition. We can now finish the proof by (2.1.2). ■

2.2 Existence of Blowing Up

2.2.1. Proposition. *Let X be a rigid analytic variety and Z a closed analytic subvariety of X. Suppose that the blowing up*

$$\pi : \tilde{X} \to X$$

of X with center Z exists. Let W be a closed analytic subvariety of X. Then there exists a closed analytic subvariety $\tilde{W}$ of $\pi^{-1}(W)$ with the following properties.

(i) The pair $(\tilde{W}, \pi^{-1}(Z) \cap \tilde{W})$ satisfies the minimality condition.

(ii) $\tilde{W} \setminus \pi^{-1}(Z) = \pi^{-1}(W) \setminus \pi^{-1}(Z)$.

(iii) The restriction

$$\pi|_{\tilde{W}} : \tilde{W} \to W$$

is the blowing up of W with center $W \cap Z$.

Remark. We call the closed analytic subvariety $\tilde{W}$ of $\tilde{X}$ the *strict transform* of W under the blowing up map π. By property (iii), this strict transform is unique.

Proof. Let $\mathcal{I}$ be the coherent $\mathcal{O}_X$-ideal defining Z. Let $T = \pi^{-1}(W) = \tilde{X} \times_X W$.

For an arbitrary $\mathcal{O}_T$-module $\mathcal{F}$, let $\mathcal{A}\mathfrak{nn}_{\mathcal{O}_T}(\mathcal{F})$ denote the sheafification (see [BGR,9.2.2]) of the presheaf

$$U \mapsto \mathrm{Ann}_{\mathcal{O}_T(U)}(\mathcal{F}(\mathcal{U})), \tag{1}$$

where U runs over all admissible opens of T. One obtains that the following sequence of $\mathcal{O}_T$-modules is exact, by checking this on the stalks,

$$0 \to \mathcal{A}\mathfrak{nn}_{\mathcal{O}_T}(\mathcal{F}) \to \mathcal{O}_T \to \mathcal{H}\mathfrak{om}_{\mathcal{O}_T}(\mathcal{F}, \mathcal{F}).$$

This proves that if $\mathcal{F}$ is coherent, then also $\mathcal{A}\mathfrak{nn}_{\mathcal{O}_T}(\mathcal{F})$ is. Let $\mathcal{H}$ be the union

$$\mathcal{H} = \sum_{m=1}^{\infty} \mathcal{A}\mathfrak{nn}_{\mathcal{O}_T}(\mathcal{I}^m \mathcal{O}_T).$$

One verifies that this, being a union of coherent $\mathcal{O}_T$-ideals, is again coherent. By [BGR,9.2.2. Corollary 7], we have, for $t \in T$, that

$$\mathcal{H}_t = \sum_m \mathrm{Ann}_{\mathcal{O}_{T,t}}(\mathcal{I}^m \mathcal{O}_{T,t}). \tag{2}$$

Let $\tilde{W}$ be the closed analytic subvariety of T corresponding to $\mathcal{H}$. In other words, as a set,

$$\tilde{W} = \{t \in T \mid \mathcal{H}_t \neq \mathcal{O}_{T,t}\}, \tag{3}$$

whereas the structure sheaf on $\tilde{W}$ is given by $\mathcal{O}_{\tilde{W}} = \mathcal{O}_T/\mathcal{H}$. We claim that this subvariety satisfies the conditions of the theorem.

Let $t \in T \setminus \pi^{-1}(Z)$. Hence $\pi(t) \notin Z$, so that by definition $\mathcal{I}_{\pi(t)} = \mathcal{O}_{X,\pi(t)}$. Therefore, by (2) we get that $\mathcal{H}_t = 0$, implying that $t \in \tilde{W}$ by (3). This proves (ii).

The sheaf $\mathcal{I}\mathcal{O}_{\tilde{W}}$ is the coherent $\mathcal{O}_{\tilde{W}}$-ideal defining $\tilde{W} \cap \pi^{-1}(Z)$. Let $t \in \tilde{W}$. Using that $\mathcal{O}_{\tilde{W},t} \cong \mathcal{O}_{T,t}/\mathcal{H}_t$, one easily verifies from (2) that

$$\mathrm{Ann}_{\mathcal{O}_{\tilde{W},t}}(\mathcal{I}\mathcal{O}_{\tilde{W},t}) = 0. \tag{4}$$

Therefore, by (2.1.4), we obtain (i).

Let us finally prove (iii). Since $\mathcal{I}\mathcal{O}_{\tilde{X}}$ is invertible, we get, for $t \in \tilde{W}$, by the canonical local surjection $\mathcal{O}_{\tilde{X},t} \twoheadrightarrow \mathcal{O}_{\tilde{W},t}$ that

$$\mathcal{I}\mathcal{O}_{\tilde{W},t} = e\mathcal{O}_{\tilde{W},t}$$

where $e \in \mathcal{O}_{\tilde{X},t}$ is a regular element of $\mathcal{O}_{\tilde{X},t}$. But using (4), we get that e is also a non-zero divisor in $\mathcal{O}_{\tilde{W},t}$. Hence $\mathcal{I}\mathcal{O}_{\tilde{W}}$ is an invertible sheaf on $\tilde{W}$, proving (i) in the definition (1.2.1) of a blowing up.

Let Y be a rigid analytic variety and $f : Y \to W$ a map of rigid analytic varieties, such that $\mathcal{I}\mathcal{O}_Y$ is invertible. Hence by definition of the blowing up π, there exists a unique morphism $g : Y \to \tilde{X}$, making the following diagram commutative

$$\begin{array}{ccc} Y & \xrightarrow{\ g\ } & \tilde{X} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ W & \longrightarrow & X. \end{array}$$

Hence $\pi(g(Y)) \subset W$, meaning that $g(Y) \subset T = \pi^{-1}(W)$. Let $y \in Y$ and put $\tilde{x} = g(y)$. Therefore, $\tilde{x} \in T$ and we have a local morphism

$$\mathcal{O}_{T,\tilde{x}} \to \mathcal{O}_{Y,y}.$$

Since $\mathcal{I}\mathcal{O}_{Y,y}$ is invertible, we get that

$$\mathcal{H}\mathcal{O}_{Y,y} = 0. \tag{5}$$

But this implies that necessarily $\mathcal{H}_{\tilde{x}} \neq \mathcal{O}_{T,\tilde{x}}$. By (3) therefore, we get that $\tilde{x} \in \tilde{W}$, proving, together with (5), that $g : Y \to \tilde{W}$. This finishes the proof of our proposition. ■

2.2.2. Theorem. *Let X be a rigid analytic variety and Z a closed analytic subvariety of X. Then the blowing up of X with center Z exists.*

Proof. *Case 1.* Let us assume that $X = Sp\,A$ is affinoid. Let $\mathfrak{a} = (\alpha_1, \dots, \alpha_n)$ be the ideal in A defining Z. By definition of an affinoid algebra there exists a set of variables $U = (U_1, \dots, U_k)$ and an ideal I in $K\langle U\rangle$, such that $A = K\langle U\rangle/I$. Define the following morphism of affinoid algebras

$$u^* : K\langle T, U\rangle \to A \quad : \quad \begin{array}{l} T_i \mapsto \alpha_i \\ U_i \mapsto U_i \bmod I, \end{array}$$

where $T = (T_1, \dots, T_n)$ is another set of variables. Let $u : X \to R^n \times R^k$ be the corresponding map of affinoid varieties. Since u^* is clearly surjective, we have that u is a closed immersion. Hence, from now on we consider X as a closed analytic subvariety of $R^n \times R^k$ through this closed immersion u. Moreover, since $u^*(T) = \mathfrak{a}$, we have, considered as closed analytic subvarieties, that

$$u^{-1}(O \times R^k) = Z, \tag{1}$$

where O is the origin of R^n with its reduced closed subvariety structure. In other words, we can identify Z with $O \times R^k$. Let

$$\pi_1 : X_1 \to R^n \times R^k$$

be the blowing up of $R^n \times R^k$ with center $O \times R^k$, which exists by (1.4.3). Let $\tilde{X}$ be the strict transform of X under the blowing up map π_1, as given by proposition (2.2.1) and let π denote the restriction of π_1 to $\tilde{X}$. From (2.2.1), we know that π is the required blowing up map. Note that this necessarily has to be independent of the chosen closed immersion u.

Case 2. Let X be an arbitrary rigid analytic variety and let $\{X_i\}_i$ be an admissible affinoid covering of X. By Case 1 we can find blowing up maps

$$\pi_i : \tilde{X}_i \to X_i$$

with center $Z \cap X_i$. By (1.4.4), we have, for each $i \neq j$, that

$$\pi_i : {\pi_i}^{-1}(X_i \cap X_j) \to X_i \cap X_j$$

is the blowing up of $X_i \cap X_j$ with center $Z \cap X_i \cap X_j$. Interchanging i and j and using the uniqueness of the blowing up, we obtain that there is an unique isomorphism

$$(3) \qquad {\pi_i}^{-1}(X_i \cap X_j) \cong {\pi_j}^{-1}(X_i \cap X_j),$$

such that, after identifying these two rigid analytic varieties through the unique isomorphism (3), the maps π_i and π_j agree. Hence, by [BGR,9.3.2. and 9.3.3] we can paste the $\tilde{X}_i$ together, along these 'common' opens in order to get a rigid analytic variety $\tilde{X}$ and we can paste the π_i together, in order to get a map

$$\pi : \tilde{X} \to X$$

of rigid analytic varieties, such that

$$\pi^{-1}(X_i) \cong \tilde{X}_i,$$

and $\pi|_{\tilde{X}_i} = \pi_i$. The proof is now finished by (1.4.4). ■

3 Properties of Blowing Up

3.1 Strict Transform

3.1.1. Definition. Let X and Y be rigid analytic varieties and let $f : Y \to X$ be a map of rigid analytic varieties. Let Z be a closed analytic subvariety of X. Let $\pi : \tilde{X} \to X$ be the blowing up of X with center Z and let $\theta : \tilde{Y} \to Y$ be the blowing up of Y with center $f^{-1}(Z)$. We call $\tilde{Y}$ the *strict transform* of Y under π. Moreover, there exists a unique map $\tilde{f} : \tilde{Y} \to \tilde{X}$ making the following diagram commute

$$(1) \qquad \begin{array}{ccc} \tilde{Y} & \xrightarrow{\theta} & Y \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \tilde{X} & \xrightarrow[\pi]{} & X. \end{array}$$

This follows immediately from the definition of blowing up. We call $\tilde{f}$ the *strict transform* of f under the blowing up π.

Note that this definition is compatible with our previous definition of the strict transform of a closed subvariety in (2.2.1). From this it follows that if f is a closed immersion, then also $\tilde{f}$ is.

3.1.2. Proposition. *Let X and Y be rigid analytic varieties and let $f : Y \to X$ be a map of rigid analytic varieties. Let $\pi : \tilde{X} \to X$ be the blowing up of X with center Z, where Z is a closed analytic subvariety of X. Let $\theta : \tilde{Y} \to Y$ be the strict transform of Y under π. Then there exists a map $i : \tilde{Y} \to \tilde{X} \times_X Y$, such that following diagram commutes*

$$\begin{array}{ccccc} & & \tilde{Y} & \xrightarrow{\theta} & Y \\ & & \downarrow{\scriptstyle i} & & \| \\ \tilde{Y} & \xrightarrow{i} & \tilde{X} \times_X Y & \xrightarrow{q} & Y \\ \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f} \\ \tilde{X} & = & \tilde{X} & \xrightarrow{\pi} & X, \end{array}$$

where p and q are the canonical projections onto the first and second factor respectively. Moreover, i is a closed immersion.

Proof. The existence of i follows from the definition of a fiber product. Let

$$\Gamma = \Gamma(f) = X \times_X Y$$

be the graph of f. Hence, $\Gamma \cong Y$. Let us identify Y with Γ under this isomorphism. By using [BGR,6.1.1. Proposition 10], we get a closed immersion $j : Y = \Gamma = X \times_X Y \hookrightarrow X \times Y$, so that we can consider Y as a closed analytic subvariety of $X \times Y$. Moreover, one checks that under these identifications, we have that

$$Y \cap (Z \times Y) = Z \times_X Y = f^{-1}(Z).$$

In other words, $\tilde{Y}$ is the blowing up of Y with center $Y \cap (Z \times Y)$.

But by (1.4.2) we know that $\pi \times 1_Y : \tilde{X} \times Y \to X \times Y$ is the blowing up of $X \times Y$ with center $Z \times Y$. From (2.2.1), since Y is a closed analytic subvariety of $X \times Y$, we know that we can realize the blowing up $\tilde{Y}$ of Y with center $Y \cap (Z \times Y)$ as a closed analytic subvariety of $\tilde{X} \times Y$. Hence we have a commutative diagram

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{i} & \tilde{X} \times_X Y \\ \| & & \downarrow{\scriptstyle u} \\ \tilde{Y} & \xrightarrow{\alpha} & \tilde{X} \times Y, \end{array}$$

where α is a closed immersion. Using [BGR,6.1.1. Proposition 10] again, we have that u is a closed immersion. It is an exercise to conclude that then also i has to be a closed immersion. ■

3.2 Properness of Blowing Up

3.2.1. Theorem. *Let X be a rigid analytic variety and Z a closed analytic subvariety of X. Let $\pi : \tilde{X} \to X$ be the blowing up of X with center Z. Then the map π is proper.*

Proof. We will prove this by splitting up in four cases, according to the way we proved the existence of a blowing up map in (2.2.2).

Case 1. Let $X = R^n$ and let Z be the (reduced) one point set consisting of the origin O. By construction, $\tilde{X}$ is a closed analytic subvariety of $R^n \times \mathbb{P}^{n-1}$ and π is the restriction of $p : R^n \times \mathbb{P}^{n-1} \to R^n$ to $\tilde{X}$. Since p is proper (as a base change of the proper map $\mathbb{P}^{n-1} \to O$), we have that π is also proper. This settles this case.

Case 2. Let $X = R^n \times R^k$ and let $Z = \{O\} \times R^k = R^k$, where O is the origin of R^n. The map π is just the base change of the analogous map in Case 1, and therefore is also proper.

Case 3. Let X be affinoid and consider the closed immersion

$$u : X \to R^n \times R^k$$

as given in the proof of (2.2.2), such that

$$u^{-1}(O \times R^k) = Z.$$

Let $\pi_1 : X_1 \to R^n \times R^k$ be the blowing up map of $R^n \times R^k$ with center R^k. Hence from (2.2.2), we know that $\tilde{X}$ is a closed analytic subvariety of X_1. From the commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\pi} & X \\ \downarrow & & \downarrow u \\ X_1 & \xrightarrow[\pi_1]{} & R^n \times R^k, \end{array}$$

we get that the composite map $u\pi$ is proper, since π_1 is, by Case 2. Therefore we are done by [BGR,9.6.2. Proposition 4].

Case 4. Let finally X be an arbitrary rigid analytic variety. Since properness can be checked on an admissible affinoid covering, we are done by the previous case.
■

3.2.2. Corollary. *Let X be a rigid analytic variety and Z a closed analytic subvariety of X. Let $\pi : \tilde{X} \to X$ be the blowing up of X with center Z. If Z is nowhere dense in X, then $\pi^{-1}(Z)$ is nowhere dense in $\tilde{X}$.*

Remark. Here we say that an analytic subset Z is *nowhere dense* in a rigid analytic variety X, if the difference set $X \setminus Z$ is dense in the Zariski-topology on X, in other words, every non-empty Zariski-open subset of X has a non-empty intersection with $X \setminus Z$.

Proof. Let $U = X \setminus Z$. Let $\tilde{W}$ be an arbitrary Zariski-open in $\tilde{X}$. By (3.2.1) the map π is proper, and hence by [BGR,9.6.3. Proposition 3], $\pi(\tilde{X} \setminus \tilde{W})$ is an analytic subset of X. Let $W = X \setminus \pi(\tilde{X} \setminus \tilde{W})$. By our assumption we have that $U \cap W \neq \emptyset$. Therefore, let u be a point of $U \cap W$. By (1.4.5) there exists a (unique) $\tilde{u} \in \tilde{X} \setminus \pi^{-1}(Z)$, such that $\pi(\tilde{u}) = u$. Hence $\tilde{u} \in \tilde{W}$, since otherwise $u = \pi(\tilde{u}) \in \pi(\tilde{X} \setminus \tilde{W})$. ■

3.2.3. Corollary. *Let X be an irreducible rigid analytic variety and let Z be a closed analytic subvariety of X. Let $\pi : \tilde{X} \to X$ be the blowing up of X with center Z. Unless we are in the extreme case that the underlying point set of Z is equal to the whole space X, we have that π is surjective and $\tilde{X}$ is also irreducible.*

Proof. Let us denote the underlying analytic subset of Z still by Z. From (1.4.5) we get that

$$X \setminus Z \subset \pi(\tilde{X}). \tag{1}$$

From (3.2.1) we have that π is proper. Hence from [BGR,9.6.3. Proposition 3] we have that $\pi(\tilde{X})$ is an analytic subset of X. Combining this with (1), we have that $X = Z \cup \pi(\tilde{X})$. Since X is irreducible and $Z \subsetneq X$ (as sets), we must have that $X = \pi(\tilde{X})$. This proves the first part.

For the irreducibility of $\tilde{X}$, suppose that

$$\tilde{X} = \tilde{X}_1 \cup \tilde{X}_2, \tag{2}$$

with each $\tilde{X}_k$ an analytic subset of $\tilde{X}$. By (3.2.1) and above π is proper and surjective. Hence taking the image of (2) under π, we get that

$$X = \pi(\tilde{X}_1) \cup \pi(\tilde{X}_2),$$

where by the properness of π, both $\pi(\tilde{X}_k)$ are analytic subsets of X. Since the latter is irreducible, this implies that, say, $X = \pi(\tilde{X}_1)$. Since by (1.4.5) the restriction of π to $\tilde{X} \setminus \pi^{-1}(Z)$ induces an isomorphism with $X \setminus Z$, we must have an inclusion

$$\tilde{X} \setminus \pi^{-1}(Z) \subset \tilde{X}_1. \tag{3}$$

But Z is nowhere dense in X, hence, by (3.2.2), the same holds for $\pi^{-1}(Z)$ in $\tilde{X}$. In other words $\tilde{X} \setminus \pi^{-1}(Z)$ is dense in $\tilde{X}$, which together with (3) implies that $\tilde{X} = \tilde{X}_1$, as we needed to show. ■

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