Regular partial conical flocks

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Abstract

This article provides a classification of the partial conical flocks in PG(3,q) of (q-1) conics of a quadratic cone which admit a linear regular automorphism group on the conics.

1 Introduction.

Let C be a quadratic cone in $PG(3,q)$ with vertex v_o. A partial flock of C is a set of t conics, $1 \leq t \leq q$ which are mutually skew and lie within $C - \{v_o\}$. Using results of Johnson [10] and Gevaert and Johnson [5], a partial flock of a quadratic cone is equivalent to a translation net of degree $qt+1$ and order q^2 consisting of t reguli(regulus nets) which mutually share a component(line). The deficiency of the partial flock is $q-t$.

In this article, we consider partial flocks of deficiency 1 on which there is a subgroup of $PGL(4,q)$ that acts regularly on the $(q-1)$ conics of the partial flock. Such partial flocks are said to be **regular partial conical flocks.**

Our main result is as follows:

Theorem 1 Let P be a regular partial conical flock of deficiency one in $PG(3,q)$. Then P may be extended to a flock of a cone $x_ox_1 = x_2^2$ with planes containing the conics given by $x_ot - x_1f(t) + x_2g(t) + x_3 = 0$ for all $t \,\epsilon\, GF(q)$, where the functions f and g are determined as follows:

Type I. $g(t) = \alpha t^{1+k}$, $f(t) = \beta t^{1+2k}$ where α, β are constants in $GF(q)$, and k a positive integer.

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Type II. q **is odd and** $f(t) = (f_1 + f_2 t^{(q-1)/2})t^j$ **where** f_1, f_2 **are constants in** $GF(q)$ **and** j **is a positive integer.**

(a) If $q \equiv -1$ **mod 4 then** $g(t) = (g_1 + g_2 t^{(q-1)/2})t^{(j+1)/2}$ **if** j **is odd and** $g(t) = (g_1 + g_2 t^{(q-1)/2})t^{(q+1)(j+1)/4}$ **if** j **is even where** g_1, g_2 **are constants. Moreover, if** j **is odd then** $(j^2 - 1) \equiv 0$ **mod** $(q-1)$ **and if** j **is even** $2(j^2 - 1) \equiv 0$ **mod** $(q-1)$.

(b) **If** $q \equiv 1$ **mod 4 then** $g(t) = (g_1 + g_2 t^{(q-1)/2})t^{(j+1)/2}$ **for** j **odd as in II(a) or** $g(t) = (g_1 t^{(q-1)/4} + g_2 t^{3(q-1)/4})t^{(j+1)/2}$.

Type III. There is an integer $s\ \epsilon\{12, 24, 60\}$ **where** $s|(q-1)$ **such that** $f(t) = t\ f_1$ **and** $g(t) = t\ g_1$ **where** f_1, g_1 **are constants for all** t **of order divisible by** $(q-1)/s$ **.**

Furthermore, If q **is even, then the case I holds and, in addition, this situation occurs if it is merely assumed that the group is transitive on the partial flock.**

When a partial flock is a flock, there is a corresponding translation net of degree $q^2 + 1$ and thus a corresponding translation plane. There are various examples of regular flocks of various types. For example, when $k = 1$ in type I, there is a flock corresponding to the $Betten/Walker$ translation planes, when $k = 2$ in type I, there are the flocks corresponding to the $Barriga/Cohen$-Ganley translation planes and when $k = (p^s - 1)/2$ in type I, where $q = p^r$ and p is odd, there are flocks corresponding to certain Knuth semifields(see Johnson [8] for these connections *and/or* Gevaert — Johnson [5]).

Our main result shows the pattern that exists within regular partial conical flocks. Furthermore, this result shows that a computer search for flocks of quadratic cones of this type might be quite easy.

2 The group of a partial conical flock of deficiency 1.

In this section, we consider the collineation group in $PL(4, q)$ acting on a partial flock of deficiency one.

Payne and Thas [15] have shown that every partial flock of deficiency 1 may be uniquely extended to a flock of a quadratic cone. We shall use this both in the proof of our main result and also to realize the automorphism group of a partial flock of deficiency 1 as a quotient group of a collineation group of the translation plane corresponding to the flock extending the given partial flock.

Actually, without using the extension result of Payne and Thas, a partial conical flock of deficiency 1 in $PG(3, q)$ is equivalent to a translation plane of order q^2 and kernel $GF(q)$ which admits a Baer collineation group of order q(*see* Johnson [10]). We note this formally within the following proposition.

Proposition 2 **Let** P **be a partial flock of a quadratic cone of deficiency 1 in** $PG(3, q)$**. Let** π^* **denote the associated translation plane admitting a Baer collineation group** E **of order** q**. Then the net of degree** $q + 1$

which contains the subplane which is pointwise fixed by the Baer group is derivable and the translation plane π obtained by the derivation of this net corresponds to the flock which uniquely extends P. Furthermore, E acts an an elation group in π.

Proof: Johnson [10], Payne and Thas [15].

Theorem 3 Let P be a partial flock of a quadratic cone of deficiency 1 in $PG(3,q)$ and let G denote the automorphism group of P in $PGL(4,q)$. Let π^* denote the translation plane corresponding to P admitting a Baer group E of order q and let π denote the translation plane corresponding to the flock extending P. Let K^* denote the subgroup of kernel homologies of order $q-1$ of π or π^*. Let $\hat{G}$ denote the collineation group in $GL(4,q)$ of pi^* which fixes the net N^* of degree $q+1$ which contains the subplane pointwise fixed by E.

Then either

(i) π is Desarguesian and the flock extending P is linear or

(ii) $G \cong \hat{G}/EK^*$.

Proof: The partial spread S_P which corresponds to the partial flock P may be obtained as follows: Embed the projective space $PG(3,q)$ in $PG(5,q)$ so that the cone C belongs to the Klein quadric Q. Let $\sum_i$ for $i = 1, 2, \ . \ . \ . \ , q-1$ denote the planes which contain the $q-1$ conics of P. Let $\sum_i^{\perp}$ denote the planes polar to $\sum_i$ with respect to the Klein quadric. Project $\sum_i^{\perp} \cap Q$ onto a set of mutually skew lines of $PG(3,q)$ or mutually disjoint 2-dimensional subspaces of the 4-dimensional vector space V_4 by use of the Klein correspondence. This image set defines the corresponding partial spread of degree q^2-q and consists of $q-1$ reguli (or regulus nets) mutually containing a line(component).

By Ostrom [16], since extensions exists and we have what is called a net of "critical deficiency", there are exactly two extensions of S_P to affine planes, namely π^* and π.

Let G denote the subgroup of $PGL(4,q)$ of the partial flock P and let $\hat{G}$ denote the collineation group of the net S_P obtained from G via the Klein correspondence and the Thas-Walker construction above.

Let $\sigma \ \epsilon \hat{G}$ and consider $\pi \ \sigma$ as the translation plane obtained by defining the points of the plane as points of π and the lines as the σ-images of lines of π so that $\pi \cong \pi \ \sigma$. Since π admits an elation group of order q and contains the partial spread S_P and a translation plane cannot admit both a Baer group and an elation group of order q(*Jha*-Johnson [7]), it follows that $\pi = \pi \ \sigma$.

Similarly, $\pi^* = \pi^* \sigma$.

Thus, $\hat{G}$ is a collineation group of both π and π^* and thus must leave invariant the net D^* of π^* *and/or* the net D of π where D^* is the net defined by the Baer subplane Fix E and D is the derived net.

The Baer subplanes incident with the zero vector of the $q-1$ regulus nets of π^* distinct from N^* correspond to the q^2-1 points of the conics of the partial flock. Considering these Baer subplanes within π, it follows that the group E leaves each of these Baer subplanes invariant. Also, K^* leaves each such subplane invariant.

Let H denote the subgroup of $\hat{G}$ which leaves each such Baer subplane invariant. Clearly H is a normal subgroup of $\hat{G}$ and $\hat{G}/H \cong G$. And, $E\ K^*$ is contained in H.

Assume that π is not Desarguesian. It remains to show that $H \leq E\ K^*$.

We may choose coordinates so that the spread for π has the following form:

$x = 0,\ y = x \begin{bmatrix} u + g(t) & f(t) \\ t & u \end{bmatrix}$ for all $u,\ t \in K \cong GF(q)$ and $g,\ f$ functions on K such that f is 1-1 and $g(0) = f(0) = 0$ (see Gevaert and Johnson [5]). In this representation, the group E has the form $< \begin{bmatrix} 1 & 0 & u & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} |u \in K >$ *and* the net D is represented by the regulus net $x = 0,\ y = x \left[\begin{smallmatrix} u & 0 \\ 0 & u \end{smallmatrix}\right]$ for all $u \in K$.

Suppose that the axis of E acting in π is not invariant under the full collineation group. Then by Hering- Ostrom ([13], [17]), $SL(2, q)$ is generated by the elation groups and π is Desarguesian by Foulser-Johnson-Ostrom [4].

If the axis of E is left invariant, it follows from Gevaert-Johnson-Thas [6] that the reguli defined by the orbits of E (that share the axis of E) are permuted by the full collineation group of π.

$\hat{G}$ is a collineation group of pi^* which leaves the net D invariant. If $\sigma\ \epsilon\hat{G}$ then σ must have the form $\begin{bmatrix} A & 0 \\ 0 & Av_o \end{bmatrix}$ where A is a 2×2 matrix over K and I denotes the 2×2 identity matrix for some $v_o \epsilon\ K$ since σ must leave the net D represented as above invariant. Clearly, $\hat{G}$ normalizes E.

Now assume that $\sigma\ \epsilon\hat{G} - EK^*$, and that σ fixes each regulus net D_t defined by $x = 0,\ y = x \begin{bmatrix} u + g(t) & f(t) \\ t & u \end{bmatrix}$ for all $u \in K$ and t fixed in K.

First assume that $|\sigma| = p$ where $q = p^r$, p a prime. $In < \sigma, E >$, there exists an element σ^* of order p which fixes $y = 0 (and\ x = 0)$. Thus, there must be a Baer p-element $in < \sigma,\ E >$. Thus, $p = 2$ by Foulser [3] since E is an elation group and Baer p-elements and elations are incompatible in translation planes when p is odd.

σ^* fixes each of the q regulus nets D and D_t and fixes two components of D and furthermore, fixes a Baer subplane π_o pointwise. σ^* must fix a third component of the net D and , by assumption, $\hat{G} \leq GL(4, q)$. Since D is derivable, σ^* must leave invariant the subplane π_1 of D which contains the 1-space of π_o incident with $x = 0$. Furthermore, σ^* cannot fix any other Baer subplanes of D incident with the zero vector since the action on $x = 0$ makes σ^* an elation on $x = 0$(thinking of $x = 0$ as a Desarguesian spread). Hence $\pi_o = \pi_1$.

Without loss of generality, we may choose coordinates so that

$\sigma^* = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. The functions f and g may actually change under a coordinate change but the general form of the representation of the spread does not and since we have so far not made any use of the specific functions, the choice of σ^* is admissible.

Thus, the image of $y = x\begin{bmatrix} g(t) & f(t) \\ t & 0 \end{bmatrix}$ under σ^* is

$$y = x\left[\begin{smallmatrix}1 & 1\\0 & 1\end{smallmatrix}\right]\begin{bmatrix} g(t) & f(t) \\ t & 0 \end{bmatrix}\left[\begin{smallmatrix}1 & 1\\0 & 1\end{smallmatrix}\right] = \begin{bmatrix} g(t)+t & g(t)+t+f(t) \\ t & t \end{bmatrix}.$$

This latter component belongs to the regulus net D_t so that

$f(t) = g(t) + t + f(t)$ so that $g(t) = t$ for all t.

The planes of the flock corresponding to π containing the conics are:

$x_o t - f(t)x_1 + x_2 t + x_3$. These planes contain a common point

(1,0,1,0). It then follows from Thas [18] that the flock is linear and thus the translation plane is Desarguesian.

Hence, $|\sigma| \neq p$.

More generally, it follows that $|\sigma| \neq p^t$ for any integer t.

Let $|\sigma| = u$ where u is a prime. Then $(u,p) = 1 = (u,q)$ so that σ fixes a component of each of the q regulus nets.

Let σ fix $y = x\begin{bmatrix} u_t + g(t) & f(t) \\ t & u_t \end{bmatrix}$ for all t where $u_o = 0$ and $u_t \in K$ depends on t.

Then let $A = \left[\begin{smallmatrix}a_1, a_2\\a_3, a_4\end{smallmatrix}\right]$. If u does not divide $q-1$ then σ fixes a third component of D which we may assume is $y = x$ so that $v_o = 1$ in the original representation of σ. Then σ fixes each component $y = x\left[\begin{smallmatrix}u & 0\\0 & u\end{smallmatrix}\right]$ for $u \in K$ and fixes $y = x\begin{bmatrix} u_t + g(t) & f(t) \\ t & u_t \end{bmatrix}$ for each t.

Hence, $y = x\begin{bmatrix} u + g(t) & f(t) \\ t & u \end{bmatrix}$ maps to

$$y = x\ A^{-1}\begin{bmatrix} g(t) + u_t + (u-u_t) & f(t) \\ t & u_t + (u-u_t) \end{bmatrix}A =$$

$$A^{-1}\begin{bmatrix} g(t) + u_t & f(t) \\ t & u_t \end{bmatrix}A + \begin{bmatrix} u-u_t & 0 \\ 0 & u-u_t \end{bmatrix} =$$

$$\begin{bmatrix} g(t) + u_t & f(t) \\ t & u_t \end{bmatrix} + \begin{bmatrix} u-u_t & 0 \\ 0 & u-u_t \end{bmatrix} = \begin{bmatrix} g(t) + u & f(t) \\ t & u \end{bmatrix}.$$

Hence σ fixes each component and is thus a kernel homology – contrary to assumption.

Thus, u must divide $q-1$.

First assume that $u = 2$ and q is odd. Then σ is a Baer involution. But, σ must fix the subplanes π_1, π_2 of the net D which contains the 1-spaces fixed pointwise by σ that lie on $x = 0$, $y = 0$ respectively. Since the fixed point space of σ does not lie in D, it follows that σ induces an affine homology of order 2 on each of the subplanes π_1, π_2.

It follows that $\pi_1 \neq \pi_2$ and by appropriate choice of representation

$\sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Then σ maps

$y = x \begin{bmatrix} g(t) & f(t) \\ t & 0 \end{bmatrix}$ onto $y = x \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} g(t) & f(t) \\ t & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$= y = x \begin{bmatrix} -g(t) & f(t) \\ t & 0 \end{bmatrix}$. Hence, $g(t) = -g(t)$ so that $g \equiv 0$. By Thas [18], it follows that the plane must be a semifield plane of Knuth type. However, although σ fixes $(x, x \begin{bmatrix} 0 & f(t) \\ t & 0 \end{bmatrix})$, the action on the 1- spaces is $x \to x \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and thus cannot fix each Baer subplane incident with the zero vector of the nets D_t, for $t \neq 0$.

Thus, $u \neq 2$, and u divides $q - 1$ and hence, fixes at least two Baer subplanes of D(note we are only assuming that σ fixes each Baer subplane incident with the zero vector of the regulus nets D_t for $t \neq 0$).

Without loss of generality, we may assume that

$\sigma = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & av_o & 0 \\ 0 & 0 & 0 & bv_o \end{bmatrix}$. Then

$\begin{bmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{bmatrix} \begin{bmatrix} g(t) + u_t & f(t) \\ t & u_t \end{bmatrix} \begin{bmatrix} av_o & 0 \\ 0 & bv_o \end{bmatrix} =$

$\begin{bmatrix} v_o(g(t) + u_t) & a^{-1}bf(t)v_o \\ b^{-1}av_ot & v_ou_t \end{bmatrix} = \begin{bmatrix} g(t) + u_t & f(t) \\ t & u_t \end{bmatrix}$ only if $v_ou_t = u_t$ and $b^{-1}av_ot = t$ for all $t \in K$. If $v_o = 1$ then $b = a$ and σ becomes a kernel homology – a contradiction. Thus, $u_t = 0$ then $v_og(t) = g(t)$ implies either $g \equiv 0$ which we have seen above implies that the plane is a semifield plane of Knuth type or $v_o = 1$ and σ is a kernel homology.

If $g \equiv 0$, then $f(t) = \gamma\, t^\rho$ for some constant γ and $\rho \in$ aut K.

Since f is 1-1, $a^{-1}bv_o = 1$. Hence, $b^{-1}av_o = (b^{-1}a)^2$ and $f((b^{-1}a)^2t) = f(t)$ so that we must have $(b^{-1}a)^2 \in$ Fix ρ.

Let $a^{-1}b = c$ and consider $\sigma\, a^{-1}I_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & v_o & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. The action of this element on $(x,\, x \begin{bmatrix} 0 & f(t) \\ t & 0 \end{bmatrix}) \to$

$(x \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix},\, x \begin{bmatrix} 0 & f(t) \\ t & 0 \end{bmatrix} \begin{bmatrix} c^{-1} & 0 \\ 0 & 1 \end{bmatrix})$ and cannot leave invariant each 1 — dimensional subspace on $y = x \begin{bmatrix} 0 & f(t) \\ t & 0 \end{bmatrix}$ unless $c = 1$ and since $\sigma\, a^{-1}I_2$ must leave each Baer subplane incident with the zero vector on each D_t , for $t \neq 0$, it follows that this element must fix each 1-dimensional subspace of the indicated fixed component. Hence, $c = 1 = a^{-1}b$ so that $a = b$ and the restrictions above show that $v_o = 1$ and hence, σ is a kernel homology.

Thus, we have completed the proof.

3 A preview.

In this section, we shall assume that our main result stated in the introduction is valid. One of the main tools in the proof of this result is the theorem of Hiramine[12] classifying translation planes admitting certain autotopism groups. In the process of listing this theorem, we shall give a variation which we shall use in the proof of the main result.

Assume that P is a regular partial conical flock of deficiency 1 and let G denote a subgroup of $PGL(4, q)$ which acts transitively on the $q-1$ conics of the partial flock. Assume P is not linear. Let G^* denote the collineation group of the translation plane π corresponding to P which admits an elation group E of order q such that G^* leaves invariant the regulus net D corresponding to the extension of P to a flock. Let K^* denote the kernel homology group of π. Then, by Theorem 3, $G \cong G^*/EK^*$ (note that the subgroup G of $PGL(4, q)$ is not considered within G^*/K^* even though this group is isomorphic to a subgroup of a group isomorphic to $PGL(4, q)$).

In [12], Hiramine studies translation planes $\sum$ of order q^2 and kernel containing $K \cong GF(q)$ that admit an autotopism group A which fixes a net N of degree $q + 1$ and acts transitively on the remaining $q^2 - q$ components. It follows that there is a collineation group E of order q which, due to the assumption that it is an autotopism group, acts as a Baer group with Fix E a Baer subplane of the net N. Hiramine shows that N defines a regulus in $PG(3, K)$(alternatively, it is possible to see this using the result of Johnson [10] on Baer groups and partial flocks, and the extension result of Payne and Thas [15]).

Hiramine determines the general structure of the spread of $\sum$. Since $\sum$ is derivable using the net D, there is a corresponding translation plane also with kernel K that admits the collineation group acting transitively on the components of $\sum$ which are not in D and where the subgroup E now acts as an elation group. We shall completely determine the structure shortly.

Theorem 4 (Hiramine [12](part (i) when the group is an autotopism group, for part (ii) see Johnson [8*])*. Let $\sum$ denote a translation plane of order q^2 and kernel containing $K \cong G(q)$ which admits a collineation group G that has an orbit of length $q^2 - q$.

(i) If G fixes two components then the remaining components define a derivable(regulus) net N.

(ii) Derive the plane $\sum$ by replacing N. The constructed translation plane $\sum^*$ admits a collineation group which fixes a regulus net N^*, acts transitively on the components of $\sum^* - N^*$ and contains an elation group of order q. The spread for Σ^* has the following form:

$x=0$, $y=x\begin{bmatrix} u+\alpha t^{1+k} & \beta t^{1+2k} \\ t & u \end{bmatrix}$ **for all** $u,t \in K \cong GF(q)$ **where** α,β **are constants in** K **and** k **is a fixed positive integer.**

Now consider the initial problem:

Determine the regular partial flocks of deficiency 1.

In the corresponding translation plane, using the collineation group acting on the translation plane admitting the elation group then there must be an orbit on the line at infinity of length q^2-q(*see* section 4). If the group fixes two Baer subplanes of the net corresponding to the extension conic of the partial flock then we may apply the above result. It turns out that the group either fixes or interchanges two Baer subplanes or there is a quotient group isomorphic to A_4, S_4, or A_5. We consider these situations in the next section.

4 Transitive groups.

In this section, we give the proof to our main theorem stated in the introduction.

We stated the result of Hiramine [12] in section 3 which we shall make use of to prove our main result. And, if the indicated collineation group is actually an autotopism group then the spread has the form indicated in section 3 and the translation to the flock produces the representation in case I.

Let G denote the regular group on the partial flock. Assume that the corresponding translation plane π is not Desarguesian which admits an elation group E and let G^* denote the collineation group of π which fixes the extended regulus net D so that $G \cong G^*/EK^*$. Interpreting the results of section 3, we have the following lemma:

Lemma 5 **(i)** G^* **fixes** D **and** G^*/K^* **acts regularly on the remaining** q^2-q **components.**

(ii) If G^* **fixes two Baer subplanes of the net** D **incident with the zero vector then the representation is as stated in the theorem case I.**

Proof: (i) G^* contains the elation group E which fixes each of the $q-1$ regulus nets corresponding to the partial flock(and, of course, fixes the extended regulus net D corresponding to the extending flock). E acts regularly on the components distinct from the axis of each such regulus net. Since G acts transitively on the conics, and the conics correspond to the regulus nets different from D(actually to the set of Baer subplanes of the nets), it follows that G^* acts transitively on the components not in D. And, since $|G|=(q-1)$ by regularity, $|G^*|=q(q-1)^2$ and G^* contains the kernel homology group K^* of order $q-1$.

(ii). If G^* fixes two Baer subplanes of the net D then, in the derived translation plane, G^* becomes an autopism group of the derived plane which acts transitively on the q^2-q components not in the derived net D^* and we apply Hiramine's results.

Remark 1 **In the following, we shall assume that the spread for the translation plane** π **is** $x=0$, $y=x\begin{bmatrix} u+g(t) & f(t) \\ t & u \end{bmatrix}$ **for all** $t,u \in K \cong GF(q)$

where g, f are functions on K, f $1-1$ and $g(0) = f(0) = 0$. We assume that π is not Desarguesian so that E is normal in G^*. We assume that G^* fixes the net D with components $x = 0$, $y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$ for all $u \in K$ and the axis of E is $x = 0$.

Lemma 6 . Let $G^* = EG^*_{(y=0)}$. Let $G^*_{[y=0]}$ denote the subgroup of $G^*_{(y=0)}$ which fixes $y = 0$ pointwise. Then $EG^*_{[y=0]}$ fixes each Baer subplane of D incident with the zero vector.

Proof: The indicated subgroup consists of central collineations. Any central collineation g acting on a projective plane will fix any subplane which contains the axis and coaxis(if the homology case) and such that if for some point P of the subplane not on the axis or coaxis then $P\ g$ is also a point of the subplane(*e.g.* Lüneburg [14]).

Since the Baer subplanes of D incident with the zero vector contain the axis and coaxis of the central collineations and each central collineation leaves invariant the set of points at infinity of the Baer subplanes, it follows that $EG^*_{[y=0]}$ fixes each such Baer subplane.

Lemma 7 (i) If q is even then assume merely that there is a transitive group G on the partial flock. Then there is a subgroup G^*_1 of order divisible by $q^2 - q$ which fixes two Baer subplanes.

(ii) If $q \equiv -1 \bmod 4$ then G^* fixes or interchanges two Baer subplanes of D incident with the zero vector.

(iii) If $q \equiv 1 \bmod 4$ then either G^* fixes or interchanges two Baer subplanes of D or

$G^*_{(y=0)}/G^*_{[y=0]} \cong A_4, S_4$**, or** A_5.

Proof: $EG^*_{[y=0]}$ fixes each Baer subplane incident with the zero vector of D.

The collineation group of the regulus net D within $GL(4, K)$ is $G_1 * G_2$ (central product) where $G_i \cong GL(2, q)$ for $i = 1, 2$(*see* Foulser [3], or Johnson [9]) and where G_1 acts trivially on the Baer subplanes of D incident with the zero vector and G_2 acts trivially on the components of D. Thus, $EG^*_{[y=0]}K^* \leq G_1$.

We first consider that G^*/EK^* contains p-elements(of course, it does not by assumption if p is odd).

If so, then there is a p-element not in E and since $G^* = EG^*_{(y=0)}$, we may assume that there is a Baer p-element ρ in G^*. Hence, $p = 2$ by Foulser [3]. It follows that the order of G^* is divisible by 2 $q(q^2 - 1)$(assuming that K^* is a subgroup of G^*). Moreover, the Baer involution is in $G^*_{(y=0)}$ and by Jha-Johnson [7], there can be no 2-group of order divisible by 4 q since in this case, there would be a Baer 2-group of order ≥ 4. Hence, $(G^*_{(y=0)}/G^*_{[y=0]})/(K^*G^*_{[y=0]}/G^*_{[y=0]})$ is a subgroup A of $PGL(2, q)$, q is even and the 2-order of the previous group is exactly 2. If there is a nontrivial subgroup M of order divisible by $(q - 1)$, then M is cyclic and leaves exactly two 1-spaces invariant on $y = 0$. Also, $M \leq A \cap PSL(2, q) = A^*$ and A^* is either cyclic or dihedral of order 2 k where $k|(q - 1)$. In any case, we may assume that M is characteristic in A^* which is normal in A. Thus, there is a subgroup of order divisible by $q(q-1)$ of G^* which fixes two Baer subplanes. It remains to show that this subgroup has an orbit of length q^2-q. In the above argument, the only possible group element which does not fix at least two Baer subplanes has order 2. Hence, if a group of order $2q(q-1)^2t$ has an orbit of length $q(q-1)$ then the point stabilizer subgroup has order $2(q-1)t$. If a subgroup U of index 2 is not transitive on this set then the orbits of U are permuted transitively by the full group, so each orbit of U has the same length $z|q(q-1)$ and the point stabilizer subgroups all have order $2q(q-1)^2/z \leq 2(q-1)t$ so that $q(q-1) \leq z$. That is, there must be a subgroup which fixes at least two Baer subplanes and which has an infinite point orbit of $q(q-1)$.

Thus, the quotient group listed above cannot have a subgroup of order $|(q-1)$ and hence, $|G^*_{[y=0]}K^*| = (q-1)^2 s$ for s odd. Thus, there is an affine homology group of order $(q-1)$ which leaves a regulus net invariant.

By Johnson [11], there is a corresponding flock of a hyperbolic quadric in $PG(3,q)$. The flocks of hyperbolic quadrics have been completely determined(Bader, Lunardon [1], and Thas [19], [20]) and the associated translation plane is always a nearfield plane. But, since there is an elation group of order q, it follows that the plane must be Desarguesian.

So, by replacing the group G^* by a possibly index two subgroup, we may assume that there are no p-elements in G^*/EK^*. The above argument on the classification of hyperbolic quadrics shows that we may assume that the homology group $G^*_{[y=0]}$ does not have order $q-1$.

Hence, the group induced on $y=0$ in $PGL(2,q)$ is either a subgroup of a dihedral group of order $2(q-1)$ or $2(q+1)$ or is isomorphic to A_4. S_4, or A_5.

We see that we have the following cases:

I. $G^*_{[y=0]}$ **has order** $(q-1)/2$.

II. $G^*_{[y=0]}$ **has order** $<(q-1)/2$ **and the group induced on** $y=0$ **in** $PGL(2,q)$ **is a subgroup of a dihedral group of order** $2(q-1)$.

III. $G^*_{[y=0]}$ **has order** $(q-1)/t$, **where** $t \geq 3$ **and the group induced on** $y=0$ **in** $PGL(2,q)$ **is isomorphic to** A_4**,** S_4**, or** A_5 **where** $t=12$**, 24, or 60.**

We first consider the problem when there is a homology group of order $(q-1)/2$.

If we have an affine homology group of order $(q-1)/2$ and the group is regular, then there is an element $\sigma \, \epsilon G^*$ which does not fix two Baer subplanes of D but σ^2 fixes all Baer subplanes. Hence, σ interchanges pairs of Baer subplanes. We may choose coordinates so that $\{(x_1,0,y_1,0)\}$ and $\{(0,x_2,0,y_2)\}$ are interchanged. Furthermore, σ

$\sigma = \begin{bmatrix} A & 0 \\ 0 & Av_oI \end{bmatrix}$ so that $(x_1,0)A = (0,x_2)$ for some x_2 depending on x_1. Hence,

$A = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$. Note that $\sigma^2 = ab \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & v_o & 0 \\ 0 & 0 & 0 & v_o \end{bmatrix}$.

Since $EG^*_{[y=0]}$ is a semiregular subgroup of order $q(q-1)/2$, the spread for the translation plane has the following form: $x = 0$, $y = x\begin{bmatrix} g_t t+u & f_t t \\ t & u \end{bmatrix}$ where $(g_t, f_t) = (g_1, f_1)$ or (g_2, f_2) if and only if t is nonzero square or nonsquare respectively.

It follows that there are $1+(q-1)/2$ regulus nets which belong to a Desarguesian spread. Hence, there is a flock of a quadratic cone which contains $(q+1)/2$ conics of a linear flock. By Thas [18](1.5.1), if two flocks of a quadratic cone share $> (q-1)/2$ conics, then they are equal. Hence, the flock is linear and thus the translation plane is Desarguesian.

This proves the lemma since if 4 does not divide $(q-1)$ then the A_4, S_4, A_5 cases cannot occur.

Hence, our main result is proved for q even since we can now apply the results of Hiramine.

If the quotient group involves A_4, S_4, or A_5, there is a homology group of order $(q-1)/s$ where $s = 12$, 24, or 60. Since this homology group must fix N, there must be a homology group :

$$\{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{bmatrix} \mid |u| \mid (q-1)/s\}.$$

This implies that $f(ut) = uf(t)$ and $g(ut) = ug(t)$ for all t and for all such u in K^*.

This shows that we must have case III.

In order to complete the proof when the quotient group does not involve A_4, S_4, or A_5, we need to consider the possible case when the group G^* leaves invariant a set of two Baer subplanes incident with the zero vector of D.

By looking at the proof of the previous lemma, we see that the subgroup H^* which fixes the two subplanes has index 2.

Lemma 8 Assume that $q \equiv -1$ mod 4. Then the functions f and g satisfy situations I or II of the main theorem.

Proof:

Since $q \equiv -1$ mod 4, there are two Baer subplanes of the net D which are either fixed or interchanged by the full group G^*. Thus assume that two Baer subplanes are actually interchanged. We choose the two Baer subplanes which are fixed or interchanged by G^* as

$\{(x_1, 0, y_1, 0) | x_i, y_1 \epsilon\ K\}$ and $(0, x_2, 0, y_2) | x_2, y_2 \epsilon\ K\}$.

Hence, there is an element $\sigma = \begin{bmatrix} 0 & a & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & av_o \\ 0 & 0 & bv_o & 0 \end{bmatrix}$ which interchanges the two subplanes and a group H^* of order $(q-1)^2/2$ containing the kernel homology group

K^* which fixes both subplanes. The elements of H^* thus have the form $\tau_{c,d,w} = \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & cw & 0 \\ 0 & 0 & 0 & dw \end{bmatrix}$ for various elements $c, d, w \in K^*$.

Note that $\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}^{-1} \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} = \begin{bmatrix} d & 0 \\ 0 & c \end{bmatrix}$.

Hence, whenever $\tau_{c,d,w} \in H^*$ then $\sigma^{-1}\tau_{c,d,w}\sigma = \tau_{c,d,w} \in H^*$. Hence, $\tau_{c,d,w}\tau_{d,c,w}$ $= \tau_{cd,cd,w^2} \in H^*$ and since the kernel homology cd $I_4 \in H^*$, it follows that the affine homology $\tau_{1,1,w^2} \in H^*$.

Also, $\tau_{c,d,w}^{-1} \begin{bmatrix} g(t) & f(t) \\ t & 0 \end{bmatrix} \tau_{c,d,w} = \begin{bmatrix} g(t)w & c^{-1}dwf(t) \\ d^{-1}cwt & 0 \end{bmatrix}$. Since, there are two orbits of regulus nets under H^* both of length $(q-1)/2$, it follows that $\{d^{-1}cw\}$ and $\{c^{-1}dw\}$ take on exactly $(q-1)/2$ distinct nonzero elements of K^* as the element $\tau_{c,d,w}$ varies over H^*.

Clearly the indicated sets are both subgroups of K^* since they originate from $\tau'_{c,d,w}s$. Hence, for any generator u of K^*, there exist $\tau_{c,d,w}$ such that $d^{-1}cw = u^2$.

Hence, $c^{-1}d\ w = u^{2j}$. Now since $f(u^2t) = u^{2j}f(t)$ for all $t \in K^*$, and f is 1-1, it follows that $(j, (q-1)/2) = 1$(*i.e.* $u^2 = u^{2j}$ if and only if $u^2 = 1$).

With $d^{-1}cw = u^2$ and $c^{-1}dw = u^{2j}$ then $w^2 = u^{2(1+j)}$, $(d^{-1}c)^2 = u^{2(1-j)}$, and $(c^{-1}d)^2 = u^{2(j-1)}$.

Consider $(\tau_{c,d,w}(c^{-1}I_4))^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & u^{2(j-1)} & 0 & 0 \\ 0 & 0 & u^{2(j+1)} & 0 \\ 0 & 0 & 0 & u^{4j} \end{bmatrix} = \rho_{u^2}$. Since $(q-1)/2$ is odd,

$< u^{4j} >=< u^{2j} >=< u^2 >$.

Hence, there is a subgroup $T =< \rho_{u^2} > of$ order $(q-1)/2$ which acts semiregularly on the $(q-1)$ regulus nets different from D.

$\rho_{u^2}^{-1} \begin{bmatrix} g(t) & f(t) \\ t & 0 \end{bmatrix} \rho_{u^2} = \begin{bmatrix} g(t)u^{2(j+1)} & f(t)u^4 \\ u^{4j}t & 0 \end{bmatrix}$.

Thus, $g(u^4t) = u^{2(1+j)}g(t)$, $f(u^4t) = f(t)u^{4j}$ for all $u, t \in K^*$.

Since $u^{4(q+1)/4} = u^2$, we obtain:

$g(u^4t) = u^{4(1+j)(q+1)/4}g(t)$, and $f(u^4t) = u^{4j}f(t)$.

Hence, we obtain:

$g(v) = v^{(1+j)(q+1)/4}g_t$ and $f(v) = v^jf_t$ where $(j, (q-1)/2) = 1$.

Also, the element σ applied to the components shows that

$g(a^{-1}bu_of(t)) = -u_og(t)$ and $f(a^{-1}bu_of(t)) = a\ b^{-1}u_ot$ for all $t \in K^*$

$(\begin{bmatrix} 0 & b^{-1} \\ a^{-1} & 0 \end{bmatrix} \begin{bmatrix} g(t) & f(t) \\ t & 0 \end{bmatrix} \begin{bmatrix} 0 & au_o \\ bu_o & 0 \end{bmatrix} = \begin{bmatrix} 0 & ab^{-1}u_o \\ a^{-1}bu_of(t) & u_og(t) \end{bmatrix})$.

Since $q \equiv -1 \bmod 4$, and $f(u^4t) = u^{4j}f(t)$, it follows that $f(t) = t^jf_t$ where f_t takes on at most two nonzero values ($f_1 = f_t$ for t nonzero square and $f_2 = f_t$ for t

a nonsquare).

First assume that j is $odd = 1 + 2m$.

Then $g(u^4t) = u^{2(1+j)} = u^{4(1+m)}g(t)$. Hence, $g(t) = t^{(1+m)}g_t = t^{(j+1)/2}g_t$ where g_t takes on at most two nonzero values ($g_1 = g_t$ for t a nonzero square and $g_2 = g_t$ for t a nonsquare).

$g(ct^j f_t)$ (for $c = a^{-1}bu_o$) $= dt^{(j+1)/2}g_t(for\ d = -u_o)$ if and only if

$c^{(j+1)/2}t^{j(j+1)/2}f_t^{(j+1)/2}g_e = dt^{(j+1)/2}g_t($ where $e = ct^j f_t)$ so that

$t^{(j+1)(j-1)/2}$ takes on at most two nonzero constants or g_t and/or g_e is zero for certain values of t.

In the former case, $t^{(j^2-1)/2} = w_1$ when t is a nonzero square $and = w_2$ when t is a nonsquare. Hence, we may evaluate w_i by putting 1 and -1 in the above equation. Since 4 divides $j^2 - 1$, it follows that $w_1 = w_2 = 1$ and hence, $t^{(j^2-1)/2} = 1$ for all nonzero $t \epsilon K$.

Thus, $(j^2 - 1)/2 \equiv 0 \bmod (q - 1)$. But, 4 does not divide $(q - 1)$ so that $(j^2 - 1)/2^{a-1} \equiv 0 \bmod (q - 1)$ where $2^{a-1}|(j^2 - 1)$ and 2^a does not divide $(j^2 - 1)$.

Now assume that j is even The exact same argument as above involving the function f shows that $2(j^2 - 1) \equiv 0 \bmod (q - 1)$.

Since we may consider f and g as polynomials of degree $\leq q - 1$, it follows that for j odd $f(t) = (f_1 + f_2t^{(q-1)/2})t^j$ and $g(t) = (g_1 + g_2t^{(q-1)/2})t^{(j+1)/2}$ and for j even, $g(t) = (g_1 + g_1t^{(q-1)/2})t^{(q+1)(j+1)/4}$.

Now assume that $q \equiv 1 \bmod 4$ and assume that there are two Baer subplanes which are interchanged by the full collineation group.

The previous arguments apply to show that $f(u^2t) = u^{2j}f(t)$ and $g(u^2t) = g(t)w$ where $c^{-1}dw = u^{2j}$, $d^{-1}cw = u^2$ so that $w^2 = u^{2(j+1)}$ so that for each u, $w = \pm u^{(j+1)}$.

It follows that the function $f(t)$ is as above in the case $q \equiv -1 \bmod 4$ except that since $4\ |(q - 1)$, this forces j to be odd as f is 1-1.

Let $g(t) = \sum_1^{q-1} h_it^i$ (recall $g(0) = 0$). Thus, $h_iu^{2i} = h_iw$. If $h_i \neq 0$ then

$u^{4i} = w^2 = u^{2(j+1)}$ so that $i \equiv (j + 1)/2 \bmod (q - 1)/4$.

Hence,

$g(t) = g_1t^{(j+1)/2} + g_2t^{(j+1)/2+(q-1)/4} + g_3t^{(j+1)/2+(q-1)/2} + g_4t^{(j+1)/2+3(q-1)/4}$ for all $t \epsilon K$ where g_i are constants in K for $i = 1, 2, 3, 4$.

$g(u^2t) = g(t)w$ and $w = \pm u^{2(j+1)}$ show that either $g_1 = g_3 = 0$ or $g_2 = g_4 = 0$.

This completes the proof of our main result stated in the introduction.

References

1 Bade, L. Lunardon, *G*. On the flocks of $Q^+(3,q)$. Geom. Dedicata 29 (1989), 177-183.

2 Foulser, D.A. Subplanes of partial spreads in translation planes. Bull. London. Math. Soc. 4 (1972), 32-38.

3 Foulser, D.A. Baer p-elements in translation planes. *J*. Alg. 31(1974), 354-366.

4 Foulser, D. A. Johnson, *N.L.* Ostrom, *T.G.* Characterization of the Desarguesian planes of order q^2 by $SL(2,q)$. Internat. J. Math. and Math. Sci. 6(1983), 605-608.

5 Gevaert, H., Johnson, *N.L.* Flocks of quadratic cones, generalized quadrangles, and translation planes. Geom. Ded. 27, no. 3(1988), 301-317.

6 Gevaert, H., Johnson, N.L., Thas, J.A. Spreads covered by reguli. Simon Stevin. Vol. 62(1988), 51-62.

7 Jha, V., Johnson, *N.L.* Coexistence of elations and large Baer groups in translation planes. J. London Math. Soc. (2) 32(1985), 297-304.

8 Johnson, *N.L.* Derivation of partial flocks of quadratic cones. Rendiconti di Mat. VII, vol. 12 (1992), 817-848.

9 Johnson, *N.L.* A group theoretic characterization of finite derivable nets. J. Geom. Vol. 40 (1991), 95-104.

10 Johnson, *N.L.* Translation planes admitting Baer groups and partial flocks of quadric sets. Simon Stevin, vol. 63. no. 3(1989), 281-285.

11 Johnson, *N.L.* Flocks of hyperbolic quadrics and translation planes admitting affine homologies. J. Geom. vol. 34(1989), 50-73.

12 Hiramine, *Y*. On translation planes of order q^2 which admit an autotopism group having an orbit of length q^2-q. Osaka J. Math. 22 (1985), no. 3, 411-431.

13 Hering, Ch. On shears of translation planes. Abh. Math. Seminar Univ. Hamburg, 37 3/4(1972), 258-268.

14 Lüneburg, *H*. **Translation planes.** Springer-Verlag, Berlin, Heidelberg, New York, l980.

15 Payne, S.E., Thas, J.A. Conical flocks, partial flocks, derivation, and generalized quadrangles. Geom. Dedicata 38(1991), 229-243.

16 Ostrom, *T.G.* Nets with critical deficiency. Pacific J. Math. 14 (1964),1381-1387.

17 Ostrom, *T.G.* Linear transformations and collineations of translation planes. J. Alg. 14 (1970), 406-416.

18 Thas, J.A. Generalized quadrangles and flocks of cones. Europ. J. Comb. 8(1987), 441-452.

19 Thas, J.A. Flocks of nonsingular ruled quadrics in $PG(3, q)$. Rend. Acad. Naz. Lincei, 59(1975), 83-85.

20 Thas, J.A. Recent results on flocks, maximal exterior sets and inversive planes. Combinatorics '88, pp. 95-108, Mediterranean Press, Rende (Cosenza), 199.

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