Transfinite methods in geometry

A. Beutelspacher *P. J. Cameron*

Dedicated to J. A. Thas on his fiftieth birthday

Abstract

Some very difficult questions in finite geometry (for example, concerning ovoids, spreads, and extensions) have fairly trivial resolutions in the infinite case, by standard transfinite induction. We give some old and new results of this kind, and a number of open questions. The paper is expository in nature, and is intended to illustrate the technique rather than give the most general applications.

1 Introduction

Jef Thas has contributed to almost every part of finite geometry. Among these are the related topics of ovoids, spreads and extensions. To cite just a couple among many important papers, we mention [12], [13]. It is our purpose here to show that, for infinite geometries, Jef's great ingenuity is not needed; constructions are easy, and everyone can have a go.

The basic tool is transfinite induction, which is outlined briefly in Section 2. In the following sections, we construct extensions of Steiner systems and projective planes; blocking sets, ovoids, and parallelisms of projective spaces; and partitions into ovoids for a variety of quadrangle-like structures, leading to flat geometries with diagrams like $C_2.C_2$ and "large sets" of Steiner systems.

The paper also contains a number of open questions.

Received by the editors in February 1994
AMS Mathematics Subject Classification: Primary 51A05, Secondary 05B25
Keywords: transfinite methods, ovoids, spreads, extensions, blocking sets.

2 Transfinite induction

We work in Zermelo–Fraenkel set theory with the Axiom of Choice. This axiom will normally be invoked in the form that

> *every set can be well-ordered (i.e., put into bijective correspondence with an ordinal number).*

Recall that a *cardinal number* is an ordinal which is not bijective with any smaller ordinal; thus the smallest ordinal bijective with any set is a cardinal, and if $|X| = n$, then we can well-order X as $X = (x_i : i < n)$, where the indices are ordinals.

We customarily divide ordinals into three classes: *zero*, *successor ordinals* (of the form $m + 1$ for some ordinal m), and *limit ordinals* (the others). Any infinite cardinal is a limit ordinal.

Transfinite induction is the principle asserting that

> *if P is a proposition about ordinals such that the truth of $P(m)$ for all $m < n$ implies the truth of $P(n)$, then P holds for all ordinals.*

Often the hypothesis of the principle is separated into three parts (corresponding to the division above). This is the form in which we will invoke it.

Theorem 2.1 *Let P be a proposition about ordinal numbers. Then P holds for all ordinals provided that the following conditions are satisfied:*

(a) $P(0)$ is true;

(b) $P(m)$ implies $P(m + 1)$ for any m;

(c) if n is a limit ordinal and $P(m)$ holds for all $m < n$, then $P(n)$ is true.

There is an analogous statement involving constructions rather than proofs (sometimes referred to as *transfinite recursion*). Typically, we start with the empty set; at each successor ordinal, we add a point; and at a limit ordinal, we take the union of the sets previously constructed. The possibility of adding a point at stage i often requires a small argument depending on the fact that the number of points already added is less than the total number of points available. Several examples below will illustrate this.

3 Extensions of Steiner systems

As usual, a *Steiner system $S(t, k, v)$* consists of a v-set of points with a collection of k-subsets called blocks, so that any t points lie in a unique block. In this paper, v will always be an infinite cardinal number, and k may be finite or infinite; but we assume that t is finite. In the finite case, non-degeneracy is forced by requiring that $0 < t < k < v$. However, if v is infinite, there can be non-trivial systems with $k = v$; we require that each block is a proper subset of the point set.

We begin with a simple fact about infinite Steiner systems.

Lemma 3.1 *A $S(t,k,v)$, with $t \geq 2$ and v infinite, has v blocks.*

Proof. There are not more than v blocks, since there are just v t-subsets of the point set. If $k < v$, the assertion follows by cardinal arithmetic; so suppose that $k = v$. Let B be a block, and S a $(t-1)$-set disjoint from B. Then each point of B lies in a block with S, and such blocks meet B in at most $t-1$ points; so there are v of them. □

By an *extension* of a class $\mathcal{K}$ of incidence structures, we mean a structure with the property that, for any point p, the *derived* structure at p (formed by the points different from p and the blocks incident with p) is a member of $\mathcal{K}$. An extension of a particular member X of $\mathcal{K}$ additionally has the property that at least one such substructure is isomorphic to X. An r-fold extension is defined analogously.

Thus, for example, an extension of the class of $S(t,k,v)$ systems is a $S(t+1,k+1,v+1)$ system, while an extension of a particular Steiner system X must have a derived structure isomorphic to X. Our first easy result is well-known, at least for finite k. We give the proof in some detail; similar arguments later will just be outlined.

Theorem 3.1 *Any $S(t,k,v)$ with $t \geq 2$ and v infinite has an extension.*

Proof. In order to extend an incidence structure $X = (P, \mathcal{B})$, we add a point ∞, and decree that the blocks containing ∞ in the extension are all sets $B \cup \{\infty\}$ for $B \in \mathcal{B}$. The non-trivial part is the choice of the set $\mathcal{C}$ of blocks not containing ∞. In this case, we require that $\mathcal{C}$ is a collection of $(k+1)$-subsets of P with the properties that no $(t+1)$-subset of a member of $\mathcal{C}$ lies also in a block of X, but any $(t+1)$-set not in a block of X is contained in a unique member of $\mathcal{C}$. Such a set is constructed by transfinite induction. We well-order the $(t+1)$-subsets of P which are not contained in members of X as $(U_i : i < v)$. At the i^{th} stage in the construction, we must add to $\mathcal{C}$ a $(k+1)$-set containing U_i, if one does not exist already; it must meet the sets of $\mathcal{B}$ and the previously-chosen sets of $\mathcal{C}$ in at most t points. We need a "technical lemma" to guarantee that such a set exists. For the remainder of the proof, we assume that k is infinite; only trivial modifications are required if k is finite.

Lemma 3.2 *Let $\mathcal{B}$ be a set of k-subsets of the v-set P, with $|\mathcal{B}| = v$. Suppose that any $t+1$ points of P lie in at most one member of $\mathcal{B}$. If $\mathcal{B}' \subseteq \mathcal{B}$ with $|\mathcal{B}'| < v$, then there is a point of P lying in no set in $\mathcal{B}'$.*

Proof. If $k < v$, then $|\bigcup \mathcal{B}'| \leq k|\mathcal{B}'| < v$; so assume that $k = v$. Take $B \in \mathcal{B} \setminus \mathcal{B}'$. Any member of $\mathcal{B}'$ contains at most t points of B; so some point lies in no such set. □

Now we proceed to the construction of $\mathcal{C}$. We assume that, at the conclusion of stage i of the induction, we have a set $\mathcal{C}_i$ such that

(a) any two members of $\mathcal{B} \cup \mathcal{C}_i$ meet in at most t points;

(b) for $j < i$, U_j is contained in a (unique) member of $\mathcal{C}_i$.

We start with $\mathcal{C}_0 = \emptyset$.

Suppose that $\mathcal{C}_i$ has been constructed. If U_i lies in a member of $\mathcal{C}_i$, then set $\mathcal{C}_{i+1} = \mathcal{C}_i$. Otherwise, we must enlarge U_i to a k-set S meeting every member of $\mathcal{B} \cup \mathcal{C}_i$ in at most t points. This is also done by transfinite induction; it is enough to show that, if $|S| < k$, then a point can be added to S without violating this condition. Let $|S| = l < k$. There are l t-subsets of S (if l is infinite; the argument is similar if l is finite). Each lies in a unique member of $\mathcal{B}$. Also, there are at most $|i|$ sets in $\mathcal{C}_i$. So fewer than v sets meet S in t points, and by Lemma 3.2 we can choose a point in none of them. Having constructed our k-set S, we put $\mathcal{C}_{i+1} = \mathcal{C}_i \cup \{S\}$.

If i is a limit ordinal, we put $\mathcal{C}_i = \bigcup_{j<i} \mathcal{C}_j$.

Now $\mathcal{C} = \mathcal{C}_v$ is the required set of blocks in the extension. □

4 Extensions of projective planes

An r-fold extension of an infinite projective plane is a Steiner system $S(r+2, v, v)$ having the property that no two blocks meet in exactly r points.

Theorem 4.1 *Let $X = (P, \mathcal{B})$ be an r-fold extension of an infinite projective plane, with $r \geq 0$. Then X can be extended to a structure in which any two blocks not containing the added point meet in exactly $r+2$ points.*

In particular,

> *any infinite projective plane has an r-fold extension in which any two blocks meet in $r+1$ points.*

The proof is by transfinite induction as before. We require that a new block must meet all existing blocks in $r+2$ points. The possibility of doing this is guaranteed by a technical lemma as follows.

Lemma 4.2 *Let v and w be infinite cardinal numbers with $w < v$, and t a positive integer. Let X be a set of size v, $\mathcal{B}$ a family of at most v subsets of X, each of size v, with the property that any t points of X are contained in at most w members of $\mathcal{B}$, and any $t+1$ points in at most one member of $\mathcal{B}$. Let Y be a set of size less than v with the property*

$$(\forall B \in \mathcal{B})\ |Y \cap B| \leq t.$$

Then there exists Z with $Y \subseteq Z \subseteq X$ such that

$$(\forall B \in \mathcal{B})\ |Z \cap B| = t.$$

Proof. Enumerate $\mathcal{B} = (B_i : i < v)$. (We assume that $|\mathcal{B}| = v$, though this assumption is not necessary.) Set $Y_0 = Y$, and define Y_i for $i \leq v$ by transfinite induction so that

(i) $|Y_i \cap B| \leq t$ for all $B \in \mathcal{B}$;

(ii) $|Y_i \cap B_j| = t$ for all $j < i$;

(iii) $|Y_i| \leq \max\{|Y|, |i|\}$ if the right-hand side is infinite; otherwise $|Y_i|$ is finite.

These conditions clearly hold for $i = 0$.

Suppose that Y_i is defined so that (i)–(iii) hold. Let $|Y_i \cap B_i| = t - m$. We want to add m points of B_i to Y_i to form Y_{i+1} without violating (i)–(iii). We describe next how to add one point; repeat this procedure m times. (Of course, if $m = 0$, there is nothing to do.)

There are fewer than v t-subsets of Y_i, each lying in at most w members of $\mathcal{B}$; so fewer than v members of $\mathcal{B}$ meet Y_i in t points. Each of these contains at most t points of B_i; so fewer than v points of B_i lie on a block with t points of Y_i. Choose a point $p \in B_i$ different from all of these; we can add it to Y_i without violating (i)–(iii).

If i is a limit ordinal, set $Y_i = \bigcup_{j<i} Y_j$. Clearly (i) and (ii) hold, and $|Y_i| \leq |i| \max\{|Y|, |i|\} = \max\{|Y|, |i|\}$.

Now $Z = Y_v$ is the required set. □

5 Blocking sets; extensions of projective spaces

A construction very similar to the one just given shows the following result.

Theorem 5.1 *Let X be an infinite Steiner system $S(t, v, v)$, and let l be a cardinal number satisfying $t \leq l < v$. Then the point set of X can be partitioned into subsets $(P_i : i < v)$ with the property that each set P_i intersects every block in exactly l points.*

Proof (outline). We proceed by induction as usual. Assume that the points are well-ordered as $(p_j : j < v)$, and that by stage i all points p_j for $j < i$, but not p_i, lie in some set P_t. Assume also that the blocks are well-ordered as $(B_k : k < v)$, and that we have found a set S such that $p_i \in S$, S is disjoint from the earlier sets P_j, and that S meets any block in at most l points, and all B_j for $j < k$ in exactly l points. We have to add at most l points of B_k to S, avoiding all intersections of B_k with earlier sets P_j (at most $|i| \cdot l$ points), and all intersections with blocks B which already meet S in l points (at most $|S|$ of these if S is infinite, since $l \geq t$); since $|B_k| = v$, suitable points can be found one at a time. □

The proposition applies to the lines of an infinite projective or affine plane, or a projective or affine space over an infinite field, and yields blocking sets for these structures. We can block higher dimensional spaces, too. For example, Beutelspacher and Mazzocca [2] showed the following proposition.

Theorem 5.2 *If l, h, n are positive integers with $h \leq n$ and $h + 2 \leq l$, and F an infinite field, then there is an h-blocking set in $\mathrm{PG}(n, F)$ or $\mathrm{AG}(n, F)$, which meets every h-subspace in at most l points and meets some h-subspace in exactly l points.*

A consequence of Proposition 5.1 is the following extension result:

Theorem 5.3 *Let F be an infinite field with $v = |F|$, and n a positive integer (at least 2). Then there exists a $S(3, v, v)$, all of whose derived structures are isomorphic to the point-line structure of $\mathrm{PG}(n, F)$.*

Proof. Take a set S of points in $\mathrm{PG}(n+1,F)$ meeting every line in exactly two points, and let the blocks be the intersections of S with all planes spanned by three of its points. $\square$

This can be extended further. For example, a modification of the proof of Proposition 5.1 shows that, if $n>1$, $r>0$, and F is an infinite field, then there is a set of points of $\mathrm{PG}(n+r,F)$ meeting every r-subspace in a basis (a set of $r+1$ independent points). The points and $(r+1)$-space sections of such a set form an r-fold extension of $\mathrm{PG}(n,F)$.

6 Ovoids; extensions of affine spaces

Proposition 5.1, applied to the set of lines of a projective space $\mathrm{PG}(n,F)$ over an infinite field F, shows that we can choose a set of points meeting every line in exactly k points, for any $k\geq 2$. These sets are analogues of hyperovals in projective planes. With a little care, it is possible to construct similar analogues of ovals.

Theorem 6.1 *Let n,k be integers at least 2, and F an infinite field. Then there is a set S of points of $\mathrm{PG}(n,F)$ such that*

(a) $|S\cap L|=0,1$ or k for any line L;

(b) for any point $p\in S$, the union of the lines L satisfying $S\cap L=\{p\}$ is a hyperplane H_p.

Lemma 6.2 *Let S be a set of fewer than $|F|$ points of $\mathrm{PG}(n,F)$, and U a proper subspace disjoint from S. Then U is contained in a hyperplane disjoint from S.*

Proof. If $U\neq\emptyset$, choose a hyperplane of the quotient space by U which is disjoint from all points $\langle U,p\rangle/U$ (using induction on n). If $U=\emptyset$, choose a point $p\notin S$ and replace U by $\{p\}$. $\square$

Proof of Theorem 6.1. Our strategy is to ensure that $|S\cap L|=k$ for each line L not contained in a previously-constructed "tangent hyperplane" H_p. Accordingly, well-order the lines as $(L_i : i<v)$, where $v=|F|$. At stage i, we will have a set S_i of points, and a hyperplane H_p containing p for each $p\in S_i$, such that

(i) $H_p\cap S_i=\{p\}$;

(ii) $|L\cap S_i|\leq k$ for all lines L;

(iii) for $j<i$, we have $|L_j\cap S_i|=0,1$ or k, and, if $S_i\cap L_j=\{p\}$, then $L_j\subseteq H_p$.

We begin as usual with $S_0=\emptyset$.

Suppose that S_i has been constructed. If L_i is contained in H_p for some $p\in S_i$, then take $S_{i+1}=S_i$. Otherwise, suppose that $|S_i\cap L_i|=k-m$. We have to add m points of L_i to S_i. We must avoid any point lying on a line containing k points of S_i, and the intersection of L_i with H_p for any $p\in S_i$; there are fewer than v such "bad" points. If x is not bad, Lemma 6.2 permits us to find a hyperplane H_x disjoint from

S_i and containing x but not L_i. Add x to S_i. Repeat this m times, and let S_{i+1} be the resulting set.

At limit ordinals, take the union as usual.

The set $S = S_v$ is the required set. $\square$

The set just constructed with $k = 2$ is an ovoid in $\mathrm{PG}(n, F)$. If P is an ovoid, and $\mathcal{B}$ the set of sections by planes spanned by three points of P, then $(P, \mathcal{B})$ is an extension of the point-line structure of $\mathrm{AG}(n, F)$ (and all its derived structures are isomorphic): compare Proposition 5.3.

7 Spreads and parallelisms

A *spread* is a family of (pairwise disjoint) blocks which partitions the point set of an incidence structure. A *parallelism* or *resolution* is a partition of the block set into spreads. Beutelspacher [1] showed the following for projective spaces.

Theorem 7.1 *Let F be an infinite field and n, h positive integers. Let X be the incidence structure of points and h-subspaces of* $\mathrm{PG}(n, F)$. *Then the following are equivalent:*

(a) X has a spread;

(b) X has a parallelism;

(c) $n \geq 2h + 1$.

Proof. The proof is by arguments of a type which should now be familiar. Note that the condition $n \geq 2h + 1$ is equivalent to the statement that there exist two disjoint h-subspaces. The appropriate technical lemma generalizes Lemma 6.2 from "points" and "hyperplanes" to "h-spaces" and "$(n - h - 1)$-spaces". We will generalize this further in Section 9. $\square$

It is possible to modify the requirements on the spread or parallelism. To give one example (Cameron [4]):

Theorem 7.1 *Let F be an infinite field, and let S, T be sets of points and planes respectively in* $\mathrm{PG}(3, F)$, *with $|S| + |T| < |F|$. Then there exists a set $\mathcal{L}$ of pairwise disjoint lines with the properties*

(a) a point lies on a line of $\mathcal{L}$ if and only if it is not in S;

(b) a plane contains a line of $\mathcal{L}$ if and only if it is not in T.

For example, take $S = \emptyset$, $T = \{\Pi\}$. In the affine 3-space for which Π is the plane at infinity, $\mathcal{L}$ is a spread containing one line from each parallel class.

8 Ovoids in other structures

This section summarises results which have appeared in Cameron [5]. In structures like generalized polygons, a different definition of ovoids is normally used. So, in this section, an *ovoid* in an incidence structure is a set of points which meets every block in exactly one point. Thus an ovoid is the dual concept to a spread, and the dual concept to a parallelism is a partition of the point set into ovoids.

Theorem 8.1 *Let $X = (P, \mathcal{B})$ be an incidence structure with v points, where v is infinite. Suppose that any point lies on v blocks. Suppose further that, if B is a block and S a set of fewer than v points outside B, then there are v points of B which lie in a block with no point of S. Then the point set of X can be partitioned into ovoids.*

We call an incidence structure satisfying the hypotheses of the theorem *standard*. Many incidence structures are standard: for example, any generalized quadrangle with $s = t$ infinite. (In a *generalized quadrangle*, or GQ, any point p not on a block B is collinear with exactly one point of B; so, with B and S as in the theorem, at most $|S|$ points of B are collinear with a point of S.)

We note one contrast with the finite case here. Let X be a symplectic GQ over F, embedded in $\mathrm{PG}(3, F)$ in the usual way (as the points and totally isotropic lines for an alternating form). If F is finite, then any ovoid of X is an ovoid (in a different sense!) of $\mathrm{PG}(3, F)$ (see Cameron [4]). However, if F is infinite, we can start the inductive construction with more than two (and less than $|F|$) points of a hyperbolic line, to obtain an ovoid of X which is not an ovoid of the projective space.

Finally, we give a brief application of Theorem 8.1 to diagram geometries (Buekenhout [3]). Suppose that X_1 and X_2 are two rank 2 geometries (i.e., incidence structures), each having a partition of the point set into ovoids. Suppose that the numbers of ovoids in the partitions are the same for the two geometries. Then there is a rank 3 geometry with string diagram having residues X_1^* and X_2. Moreover, this geometry is *flat*: if the types are 0, 1, 2, then any two varieties of types 0 and 2 are incident. Taking X_1 and X_2 to be GQs with $s = t$ infinite, we obtain a huge number of geometries with the diagram $C_2.C_2$.

The construction proceeds as follows. Type 0 varieties are the blocks of X_1, and type 2 varieties are the blocks of X_2; and we declare any varieties of types 0 and 2 incident. We choose a bijection θ from the ovoids in the partition of X_1 to those in the partition of X_2. Now a type 1 variety is defined to be a pair (p_1, p_2), where p_1 is a point of X_1, p_2 a point of X_2, and the ovoids containing these points correspond under θ. A block B_1 of X_1 is incident with the pair (p_1, p_2) if and only if it is incident with p_1, and similarly for X_2. The diagram properties are easily checked. In fact, we can "glue together" any number of rank 2 geometries all of which have partitions into the same number of ovoids. (This construction is due to Kantor [10].)

Any $C_2.C_2$ geometry has the property that its universal 2-cover is a building (Tits [14]). So we obtain many strange $C_2.C_2$ buildings.

9 Large sets of Steiner systems

A *large set* of Steiner systems $S(t,k,v)$ is a partition of the set of all k-subsets of a v-set into Steiner systems. In this section, as an application of Theorem 8.1, we show the existence of large sets for t,k finite and v infinite. We also give a vector space analogue, which extends a result of Ceccherini and Tallini [6].

Let X be an infinite set, with $|X| = v$. Suppose that t and k are finite with $t < k$. Let $I(t,k,v)$ be the incidence structure whose points and blocks are the k-subsets and t-subsets of X, with incidence being reversed inclusion.

Theorem 9.1 *$I(t,k,v)$ is standard if $t < k$ are finite and v is infinite.*

Proof. The number of blocks (t-subsets of X) is equal to v; and the number of points on a block T is the number of $(k-t)$-subsets of $X \setminus T$, which is also v.

Given a t-set T, and a set $\{K_j : j \in J\}$ of fewer than v k-sets not containing t, the union of all these sets has cardinality less than v, so there are v elements of X not contained in T or any K_j. Now any k-set containing T and $k-t$ of these elements will have the required property. (Two k-sets are collinear if and only if they meet in at least t points.) □

Now an ovoid in $I(t,k,v)$ is an $S(t,k,v)$ by definition, and a partition into ovoids is a large set of such Steiner systems.

We can play the same game with the incidence structure $I = I(t,k,v,F)$ of k-subspaces and t-subspaces of V, where $V = V(v,F)$. Details will not be given here.

Theorem 9.2 *For $t < k$ finite, $I(t,k,v,F)$ is standard if $v \geq 2k-t+1$ and either v or F is infinite.*

Remark. The bound $v \geq 2k-t+1$ is clearly necessary for the existence of a "vector space Steiner system", since such a system must contain two k-spaces intersecting in a $(t-1)$-space. Note that we recover Theorem 7.1 by taking $t = 1$. In the case where $v = 2k-t+1$, any two k-spaces in the same system must intersect in a $(t-1)$-space. In particular, if $t = 2$ and $v = 2k-1$, each system is a projective plane.

10 Miscellanea

Shult and Thas [11] described the universal projective embeddings of dual orthogonal spaces over a field F, under the assumption that the symplectic GQ over F has no ovoids. They showed that the finite fields with this property are precisely those of odd order. Theorem 8.1 implies that no infinite field has this property.

We have not touched here on the literature on free constructions of combinatorial objects such as projective planes.

11 Problems

QUESTION 11.1. Which pairs of structures have a common extension (i.e., are both isomorphic to derived structures of the same structure)? A necessary condition is that the two structures should have isomorphic derived structures. Note that any two projective planes of the same order satisfy this condition.

QUESTION 11.2. For which structures X is there a structure Y, all of whose derived structures are isomorphic to X? When is there such a structure with a transitive automorphism group?

QUESTION 11.3. For which t, k, v is there a Steiner system $S(t, k, v)$ with a t-transitive automorphism group? Note that the techniques of Cohen [7], [8] show that, for any finite $k > t$, there exist (countable) Steiner systems with $(t-1)$-transitive groups. Moreover, the groups Cohen obtains are free of countable rank, and the generators act as cyclic permutations. Perhaps more model-theoretic techniques like those of Hrushovski [9] will settle this question.

QUESTION 11.4. When is there a set of points in $\mathrm{PG}(n, F)$ or $\mathrm{AG}(n, F)$ meeting every h-space in exactly l points?

QUESTION 11.5. Can the point set of $\mathrm{PG}(n, F)$ be partitioned into ovoids?

QUESTION 11.6. Can every infinite affine plane be extended (to an inversive plane)? (See Thas [13] for a finite analogue.) What about further extensions?

QUESTION 11.7. Which GQs have extensions? (For results on the finite version of this question, see Thas [12].)

QUESTION 11.8. For $n > 2$, does $\mathrm{AG}(n, F)$ have a parallelism in which each parallel class has just one line in common with each class of the "standard" parallelism? (Cf. Proposition 7.1)

References

[1] **A. Beutelspacher**. Parallelismen in unendlichen projektiven Räumen endlicher Dimension. *Geom. Dedicata*, 7, pp. 499–506, 1978.

[2] **A. Beutelspacher and F. Mazzocca**. Blocking-sets in infinite projective and affine spaces. *J. Geometry*, 28, pp. 111–116, 1987.

[3] **F. Buekenhout**. Diagrams for geometries and groups. *J. Combin. Theory Ser. A*, 27, pp. 121–151, 1979.

[4] **P. J. Cameron**. *Projective and Polar Spaces*, volume 13 of *QMW Maths Notes*. Queen Mary and Westfield College, University of London, 1992.

[5] **P. J. Cameron**. Ovoids in infinite incidence structures. *Archiv Math.*, 62, pp. 189–192, 1994.

[6] **P. V. Ceccherini and G. Tallini**. A new class of planar pi-spaces and some related topics. *J. Geometry*, 27, pp. 69–86, 1986.

[7] **D. A. Cohen**. An infinite perfect code with a just k-transitive automorphism group, 1993. Preprint.

[8] **D. A. Cohen**. On the ubiquity of infinite perfect codes and infinite steiner systems, 1993. Preprint.

[9] **E. Hrushovski**. A new strongly minimal set. *Ann. Pure Appl. Logic*, 62, pp. 147–166, 1993.

[10] **W. M. Kantor**. Generalized polygons, SCABs and GABs. In L. A. Rosati, editor, *Buildings and the Geometry of Diagrams, Proceedings Como 1984*, volume 1181 of *Lecture Notes in Math.*, pages 79–158. Springer Verlag, 1986.

[11] **E. E. Shult and J. A. Thas**. Hyperplanes of dual polar spaces and the spin module. *Arch. Math.*, 59, pp. 610–623, 1992.

[12] **J. A. Thas**. Extensions of finite generalized quadrangles. *Symp. Math.*, 28, pp. 127–143, 1986.

[13] **J. A. Thas**. Solution of a classical problem on finite inversive planes. In W. M. Kantor et al., editors, *Finite Geometries, Buildings and Related Topics*, pages 145–159. Oxford University Press, 1990.

[14] **J. Tits**. A local approach to buildings. In *The Geometric Vein (Coxeter Festschrift)*, pages 519–547. Springer Verlag, 1981.

A. Beutelspacher
Mathematisches Institut
Justus Liebig Universität
Arndtstr. 2
D-35392 Giessen
Germany

P. J. Cameron
School of Mathematical Sciences
Queen Mary and Westfield College
Mile End Road
London E1 4NS
U.K.