The Van Der Put Base for C^n-Functions

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1 Introduction

Let K be an algebraic extension of $\mathbb{Q}_p$, the field of p-adic numbers. As usual, we write $\mathbb{Z}_p$ for the ring of p-adic integers and $C(\mathbb{Z}_p \to K)$ for the Banach space of continuous functions from $\mathbb{Z}_p$ to K. We have the following well-known bases for $C(\mathbb{Z}_p \to K)$: on one hand, we have the Mahler base $\begin{pmatrix} x \\ n \end{pmatrix}$ $(n \in \mathbb{N})$, consisting of polynomials of degree n (see [3] p. 149 or [1]) and on the other hand we have the van der Put base $\{e_n \mid n \in \mathbb{N}\}$ (see [3] p. 189 or [4] p. 61) consisting of locally constant functions. e_n is defined as follows: $e_0(x) = 1$ and for $n > 0$, e_n is the characteristic function of the ball $\{\alpha \in \mathbb{Z}_p \mid |\alpha - n| < 1/n\}$. For every $f \in C(\mathbb{Z}_p \to K)$ we have the following uniformly convergent series:

$$f(x) = \sum_{n=0}^{\infty} a_n \begin{pmatrix} x \\ n \end{pmatrix} \quad \text{where} \quad a_n = \sum_{j=0}^{n} (-1)^{n-j} \begin{pmatrix} n \\ j \end{pmatrix} f(j)$$

$$f(x) = \sum_{n=0}^{\infty} b_n e_n(x) \quad \text{where} \quad b_0 = f(0) \text{ and } b_n = f(n) - f(n_-).$$

Here n_- is defined as follows.
For every $n \in \mathbb{N}_0$, we have a Hensel expansion $n = n_0 + n_1 p + \ldots + n_s p^s$ with $n_s \neq 0$. Then $n_- = n_0 + n_1 p + \ldots + n_{s-1} p^{s-1}$. We further put $\gamma_0 = 1$, $\gamma_n = n - n_- = n_s p^s$, $\delta_0 = 1$, $\delta_n = p^s$ and $n_\sim = n - \delta_n$. Remark that $|\delta_n| = |\gamma_n|$. Let $f : \mathbb{Z}_p \to K$. The (first) difference quotient $\phi_1 f : \nabla^2 \mathbb{Z}_p \to K$ is defined by $\phi_1 f(x,y) = \frac{f(y)-f(x)}{y-x}$, where $\nabla^2 \mathbb{Z}_p = \mathbb{Z}_p \times \mathbb{Z}_p \backslash \{(x,x) \mid x \in \mathbb{Z}_p\}$. f is called continuously differentiable (or strictly

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differentiable, or uniformly differentiable) at $a \in \mathbb{Z}_p$ if $\lim_{(x,y)\to(a,a)} \phi_1 f(x,y)$ exists. We will also say that f is C^1 at a.

In a similar way, we may define C^n-functions as follows: for $n \in \mathbb{N}$, we define $\nabla^{n+1}\mathbb{Z}_p = \{(x_1, \ldots, x_{n+1}) \mid x_i \neq x_j \text{ if } i \neq j\}$ and the n-th difference quotient $\phi_n f : \nabla^{n+1}\mathbb{Z}_p \to K$ by

$$\phi_0 f = f \text{ and } \phi_n f(x_1, x_2, \ldots, x_{n+1})$$

$$= \frac{\phi_{n-1} f(x_2, x_3, \ldots, x_{n+1}) - \phi_{n-1} f(x_1, x_3, \ldots, x_{n+1})}{x_2 - x_1}.$$

A function f is called a C^n-function if $\phi_n f$ can be extended to a continuous function $\overline{\phi_n f}$ on $\mathbb{Z}_p^{n+1}$. Recall from [2,3] that $\overline{\phi_n f}(x, x, \ldots, x) = \frac{f^{(n)}(x)}{n!}$, for all $x \in \mathbb{Z}_p$. The set of all C^n-functions from $\mathbb{Z}_p$ to K will be denoted by $C^n(\mathbb{Z}_p \to K)$. For any C^n-function f, we define

$$||f||_n = \max\{||\phi_j f||_s \mid 0 \leq j \leq n\}$$

where $||\,||_s$ is the sup norm. (For $f : X \to K$, $||f||_s = \max_{x\in X} |f(x)|$)
$||\,||_n$ is a norm on C^n, making C^n into a Banach space.

For C^n-functions the polynomials $\binom{x}{i}$ $(i \in \mathbb{N})$ still remain a base, we only have to add the factor $\gamma_i \gamma_{[i/2]} \ldots \gamma_{[i/n]}$ where $\gamma_i = i - i_-$ and $[\alpha]$ denotes the entire part of α, to obtain the orthonormal base $\gamma_i \gamma_{[i/2]} \ldots \gamma_{[i/n]} \binom{x}{i}$. A similar property does not hold for the van der Put base.
In the case $n = 1$, we have the following property:
$\{\gamma_i e_i(x) \mid i \in \mathbb{N}\} \cup \{(x - i) \cdot e_i(x) \mid i \in \mathbb{N}\}$ is an orthonormal base for $C^1(\mathbb{Z}_p \to K)$. Therefore every continuous differentiable function f can be written under the form $f(x) = \sum a_n e_n(x) + \sum b_n(x - n)e_n(x)$ where $a_0 = f(0), a_n = f(n) - f(n_-) - (n - n_-) \cdot f'(n_-)$, $b_0 = f'(0)$ and $b_n = f'(n) - f'(n_-)$. For details we refer to [3]. The construction uses the antiderivation map $P : C(\mathbb{Z}_p \to K) \to C^1(\mathbb{Z}_p \to K)$, given by $Pf(x) = \sum_{n=0}^{\infty} f(x_n)(x_{n+1} - x_n)$. Here for $x = \sum_{j=-\infty}^{+\infty} a_j p^j$, we write $x_n = \sum_{j=-\infty}^{n-1} a_j p^j$.
The antiderivative P has among others the following properties:
*) $(Pf)' = f$
*) P is a linear isometry of $C(\mathbb{Z}_p \to K)$ into $C^1(\mathbb{Z}_p \to K)$
*) If $f(x) = \sum_{n=0}^{\infty} a_n e_n(x)$ then $Pf(x) = \sum_{n=0}^{\infty} a_n (x - n)e_n(x)$.

In this note, we will construct an orthonormal base for $C^n(\mathbb{Z}_p \to K)$. We will show that $\{\gamma_i^n e_i(x), \gamma_i^{n-1}(x - i)e_i(x), \ldots, (x - i)^n e_i(x) \mid i \in \mathbb{N}\}$ is an orthonormal base for $C^n(\mathbb{Z}_p \to K)$.

To prove this, we will use the C^n-antiderivative $P_n : C^{n-1}(\mathbb{Z}_p \to K) \to C^n(\mathbb{Z}_p \to K)$ defined by

$$P_n f(x) = \sum_{m=0}^{\infty} \sum_{j=0}^{n-1} \frac{f^{(j)}(x_m)}{(j+1)!}(x_{m+1} - x_m)^{j+1}.$$

Notice that P_1 is what we have called P above.
To simplify the notations, we will treat only the case $n = 2$ in full detail. The general case may be handled in a similar way. In section 2 we will show that $||\gamma_i^{n-j}(x-i)^j e_i(x)||_n = 1$. In section 3 we give a necessary and sufficient condition for C^1-functions to be C^2. Finally, in section 4, we prove our main result.

2 Norms

To simplify the computations, we start with the following two lemmas.

Lemma 1. *For $(t_1, \ldots, t_k) \in \nabla^k X = \{(x_1, x_2, \ldots, x_k) \mid x_i \neq x_j \text{ if } i \neq j\}$ with $t_1 = x$, $t_i = y$ and $t_k = z$, we have*

$$\phi_2 f(x, y, z) = \sum_{j=2}^{k-1} \mu_j \phi_2 f(t_{j-1}, t_j, t_{j+1})$$

with

$$\mu_j = \begin{cases} \frac{(t_{j+1}-t_{j-1})(t_j-t_k)}{(z-x)(y-z)} & \text{for } j \geq i \\ \frac{(t_{j+1}-t_{j-1})(t_j-t_1)}{(z-x)(y-x)} & \text{for } j \leq i. \end{cases}$$

Moreover, $\sum_{j=2}^{k-1} \mu_j = 1$.

Proof. (Using induction on k)

$$\begin{aligned} \phi_2 f(x, y, z) &= \phi_2 f(x, y, t)\frac{x-t}{x-z} + \phi_2 f(t, y, z)\frac{t-z}{x-z} \\ &= \phi_2 f(x, y, t)\frac{x-t}{x-z} + \phi_2 f(y, t, z)\frac{t-z}{x-z}. \end{aligned}$$

Let $t = t_\ell$ with $i < \ell < k$ (in case $1 < \ell < i$, the proof is similar).
Using the induction hypothesis, we can write $\phi_2 f(x, y, z)$ as

$$\frac{x-t}{x-z}\sum_{j=2}^{\ell-1} \mu_{j,1} \cdot \phi_2 f(t_{j-1}, t_j, t_{j+1}) + \frac{t-z}{x-z}\sum_{j=i+1}^{k-1} \mu_{j,2} \cdot \phi_2 f(t_{j-1}, t_j, t_{j+1})$$

$$\begin{aligned} = \; & \frac{x-t}{x-z}\sum_{j=2}^{i} \mu_{j,1} \cdot \phi_2 f(t_{j-1}, t_j, t_{j+1}) + \frac{x-t}{x-z}\sum_{j=i+1}^{\ell-1} \mu_{j,1} \cdot \phi_2 f(t_{j-1}, t_j, t_{j+1}) \\ & + \frac{t-z}{x-z}\sum_{j=i+1}^{\ell-1} \mu_{j,2} \cdot \phi_2 f(t_{j-1}, t_j, t_{j+1}) + \frac{t-z}{x-z}\sum_{j=\ell}^{k-1} \mu_{j,2} \cdot \phi_2 f(t_{j-1}, t_j, t_{j+1}) \end{aligned}$$

where

$$\begin{aligned} \mu_{j,1} &= \frac{(t_{j+1}-t_{j-1})(t_j-t_\ell)}{(t-x)(y-t)} \quad \text{for } j \geq i \\ &= \frac{(t_{j+1}-t_{j-1})(t_j-t_1)}{(t-x)(y-x)} \quad \text{for } j \leq i \end{aligned}$$

and

$$\begin{aligned} \mu_{j,2} &= \frac{(t_{j+1}-t_{j-1})(t_j-t_k)}{(z-y)(t-z)} \quad \text{for } j \geq \ell \\ &= \frac{(t_{j+1}-t_{j-1})(t_j-t_i)}{(z-y)(t-y)} \quad \text{for } j \leq \ell. \end{aligned}$$

Now for $j \leq i$,

$$\frac{x-t}{x-z}\mu_{j,1} = \frac{t_1-t_\ell}{t_1-t_k}\frac{(t_{j+1}-t_{j-1})(t_j-t_1)}{(t_\ell-t_1)(t_i-t_1)} = \frac{(t_{j+1}-t_{j-1})(t_j-t_1)}{(t_k-t_1)(t_i-t_1)} = \mu_j.$$

For $i+1 \leq j \leq \ell-1$, we have
$\frac{x-t}{x-z}\mu_{j,1} + \frac{z-t}{z-x}\mu_{j,2}$

$$\begin{aligned} &= \frac{t_1-t_\ell}{t_1-t_k}\frac{(t_{j+1}-t_{j-1})(t_j-t_\ell)}{(t_\ell-t_1)(t_i-t_\ell)} + \frac{t_k-t_\ell}{t_k-t_1}\frac{(t_{j+1}-t_{j-1})(t_j-t_i)}{(t_k-t_i)(t_\ell-t_i)} \\ &= \frac{(t_{j+1}-t_{j-1})(t_j-t_k)}{(t_k-t_1)(t_i-t_k)} \\ &= \mu_j \end{aligned}$$

and for $j \geq \ell$,

$$\frac{z-t}{z-x}\mu_{j,2} = \frac{t_k-t_\ell}{t_k-t_1}\frac{(t_{j+1}-t_{j-1})(t_j-t_k)}{(t_k-t_i)(t_\ell-t_k)} = \frac{(t_{j+1}-t_{j-1})(t_j-t_i)}{(t_k-t_1)(t_i-t_k)} = \mu_j.$$

Thus $\phi_2 f(x,y,z) = \sum_{j=2}^{k-1} \mu_j \phi_2 f(t_{j-1}, t_j, t_{j+1})$.

To show $\sum_{j=2}^{k-1} \mu_j = 1$, take $f(x) = x^2$. Then $\phi_2 f(x,y,z) = 1$ for all $(x,y,z) \in \mathbb{Z}_p^3$, and the property follows immediately.

In the sequel, we will use the following notation, for $m, x \in \mathbb{Q}_p$: $m \triangleleft x$ if $m = x_i$ for some $i \in \mathbb{Z}$. We sometimes refer to the relation $\triangleleft$ between m and x as "m is an initial part of x" or "x starts with m".

Lemma 2. *Let S be a ball in K and $f \in C(\mathbb{Z}_p \to K)$.*
Suppose that $\phi_2 f(n, n-\delta_n, n+p^k\delta_n) \in S$ for all $n \in \mathbb{N}_0$, $k \in \mathbb{N}$, then $\phi_2 f(x,y,z) \in S$ for all $x, y, z \in \mathbb{Z}_p$, $x \neq y, x \neq z, y \neq z$.

Proof. It suffices to prove the statement for $x, y, z \in \mathbb{N}$, since $\mathbb{N}$ is dense in $\mathbb{Z}_p$, f is continuous and S is closed in K.
S is "convex" in the following sense: if $x_1, x_2, \ldots, x_n \in S$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in K$ with $|\lambda_i| \leq 1$ for all i and $\sum \lambda_i = 1$ then $\sum \lambda_i x_i \in S$.
Let t be the common initial part of x and y, i.e.

$$\begin{aligned} t &= t_0 + t_1 p + \ldots + t_s p^s \\ x &= t_0 + t_1 p + \ldots + t_s p^s + x_{s+1} p^{s+1} + \ldots + x_{s_1} p^{s_1} \\ y &= t_0 + t_1 p + \ldots + t_s p^s + y_{s+1} p^{s+1} + \ldots + y_{s_2} p^{s_2} \text{ with } x_{s+1} \neq y_{s+1}. \end{aligned}$$

Now $\phi_2 f(x,y,z) = \phi_2 f(x,t,z)\dfrac{x-t}{x-y} + \phi_2 f(t,y,z)\dfrac{t-y}{x-y} \in S$ as soon as $\phi_2 f(x,t,z)$ and $\phi_2 f(t,y,z) \in S$ since $|\dfrac{y-t}{y-x}| \leq 1$ and $|\dfrac{t-x}{y-x}| \leq 1$.
Therefore it suffices to show that $\phi_2 f(x,y,z) \in S$ if $y \lhd x$.
Let τ be the common initial part of x and z, i.e.

$$\begin{aligned} \tau &= \tau_0 + \tau_1 p + \ldots + \tau_\sigma p^\sigma \\ x &= \tau_0 + \tau_1 p + \ldots + \tau_\sigma p^\sigma + x_{\sigma+1} p^{\sigma+1} + \ldots + x_{\sigma_1} p^{\sigma_1} \\ z &= \tau_0 + \tau_1 p + \ldots + \tau_\sigma p^\sigma + z_{\sigma+1} p^{\sigma+1} + \ldots + z_{\sigma_2} p^{\sigma_2} \text{ with } x_{\sigma+1} \neq z_{\sigma+1}. \end{aligned}$$

Now $\phi_2 f(x,y,z) = \phi_2 f(x,y,\tau)\dfrac{x-\tau}{x-z} + \phi_2 f(\tau,y,z)\dfrac{\tau-z}{x-z} \in S$ as soon as $\phi_2 f(x,y,\tau)$ and $\phi_2 f(\tau,y,z) \in S$ since $|\dfrac{z-\tau}{z-x}| \leq 1$ and $|\dfrac{\tau-x}{z-x}| \leq 1$.
Therefore it suffices to show that $\phi_2 f(x,y,z) \in S$ if $y \lhd x$ and $x \lhd z$. There exist distinct $u_1, u_2, \ldots, u_i, u_{i+1}, \ldots, u_k$ such that $u_1 = y$, $u_i = x$, $u_k = z$ and $(u_j)_\sim = u_{j-1}$ for $j \leq i$ and $u_j = u_{j-1} + |u_{j-1} - z|^{-1}$ for $j > i$. Now $\phi_2 f(x,y,z) = \phi_2 f(y,x,z) = \sum_{j=2}^{k-1} \lambda_j \phi_2 f(u_{j-1}, u_j, u_{j+1})$ with

$$\begin{aligned} \lambda_j &= \frac{(u_{j+1} - u_{j-1})(u_j - u_k)}{(z-y)(x-z)} \text{ for } j \geq i \\ &= \frac{(u_{j+1} - u_{j-1})(u_j - u_1)}{(z-y)(x-y)} \text{ for } j \leq i. \end{aligned}$$

This finishes the proof since $|\lambda_j| \leq 1$ for all j, $\sum \lambda_j = 1$ and u_j, u_{j-1}, u_{j+1} is of the form $n, n-\delta_n, n+p^\ell\delta_n$ so that $\phi_2 f(u_{j-1}, u_j, u_{j+1}) = \phi_2 f(u_j, u_{j-1}, u_{j+1}) \in S$ by assumption.

Theorem 3. $||e_n||_2 = |\delta_n|^{-2} = |\gamma_n|^{-2}$.

Proof.

$$\phi_2 e_n(m, m-\delta_m, m+p^k\delta_m) = \frac{\frac{e_n(m)-e_n(m-\delta_m)}{\delta_m} + \frac{e_n(m)-e_n(m+p^k\delta_m)}{p^k\delta_m}}{-(p^k+1)\delta_m}.$$

Let $m = m_0 + m_1 p + \ldots + m_{s-1}p^{s-1} + m_s p^s$.
Then $m - \delta_m = m_0 + m_1 p + \ldots + m_{s-1}p^{s-1} + (m_s - 1)p^s$
and $m + p^k\delta_m = m_0 + m_1 p + \ldots + m_{s-1}p^{s-1} + m_s p^s + p^{k+s}$.
We have to consider the following cases.

1) $n \lhd m - \delta_m$

1.1) $n \lhd m$
Then also $n \lhd m + p^k\delta_m$ and thus $\phi_2 e_n(m, m-\delta_m, m+p^k\delta_m) = 0$

1.2) $n \ntriangleleft m$
Then $n \ntriangleleft m + p^k\delta_m$ for all k, so $\phi_2 e_n(m, m-\delta_m, m+p^k\delta_m) = \frac{1}{\delta_m(p^k+1)\delta_m}$ and
$|\phi_2 e_n| = |\delta_m|^{-2} = |\delta_n|^{-2}$ for $p \neq 2$
since $n \lhd m - \delta_m$ and $n \ntriangleleft m$ imply $n = m - \delta_m$ and $m_s - 1 \neq 0$
thus $\delta_n = p^s = \delta_m$.
(For $p = 2$, this case does not arise).

2) $n \ntriangleleft m - \delta_m$

2.1) $n \lhd m$
Then also $n \lhd m + p^k\delta_m$ and thus $\phi_2 e_n(m, m-\delta_m, m+p^k\delta_m) = \frac{-1}{\delta_m(p^k+1)\delta_m}$
and $|\phi_2 e_n| = |\delta_m|^{-2} = |\delta_n|^{-2}$ if $k \neq 0$
since $n \ntriangleleft m - \delta_m$ and $n \lhd m \Rightarrow n = m$.
If $k = 0$ then $n \ntriangleleft m + p^k\delta_m$ and then $\phi_2 e_n(m, m-\delta_m, m+p^k\delta_m) = \frac{-1}{\delta_m^2} = \frac{-1}{\delta_n^2}$.

2.2) $n \ntriangleleft m$

2.2.1) $n \ntriangleleft m + p^k\delta_m$ then $\phi_2 e_n(m, m-\delta_m, m+p^k\delta_m) = 0$

2.2.2) $n \lhd m + p^k\delta_m$ then $\phi_2 e_n(m, m-\delta_m, m+p^k\delta_m) = \frac{1}{p^k\delta_m(p^k+1)\delta_m}$
and $|\phi_2 e_n| = p^k|\delta_m|^{-2} = p^{k+2s} \leq |\delta_n|^{-2}$
since $n \ntriangleleft m - \delta_m$, $n \ntriangleleft m$ and $n \lhd m + p^k\delta_m \Rightarrow n = m + p^k\delta_m$.
So $\delta_n = p^k\delta_m$ and $|\delta_n| = p^{-k}|\delta_m| = p^{-s-k}$ for $k \neq 0$.
In case $k = 0$, we have $\phi_2 e_n(m, m-\delta_m, m+p^k\delta_m) = \frac{1}{2\delta_m^2}$
so that $|\phi_2 e_n| = |\delta_m|^{-2} \leq |\delta_n|^{-2}$ in case $p \neq 2$ since then we have $\delta_n = \delta_m$ or $\delta_n = p\delta_m$.
For $p = 2$, $|\phi_2 e_n| = 2 \cdot |\delta_m|^{-2} \leq |\delta_n|^{-2}$ since $\delta_n = 2 \cdot \delta_m$.

$$\begin{aligned}\text{So } ||e_n||_2 &= \max(||e_n||_s, ||\phi_1 e_n||_s, ||\phi_2 e_n||_s) \\ &= \max(1, |\delta_n|^{-1}, |\delta_n|^{-2}) \\ &= |\delta_n|^{-2} = |\gamma_n|^{-2}.\end{aligned}$$

Theorem 4. $||(x-n)e_n(x)||_2 = |\delta_n|^{-1} = |\gamma_n|^{-1}$.

Proof. $||\phi_2(x-i)e_i(x)||_s$

$$= \sup_{n\in\mathbb{N}} \left| \frac{\frac{(n-i)e_i(n)-(n-\delta_n-i)e_i(n-\delta_n)}{\delta_n} + \frac{(n-i)e_i(n)-(n+p^k\delta_n-i)e_i(n+p^k\delta_n)}{p^k\delta_n}}{-(p^k+1)\delta_n} \right|$$

1) $i \lhd n-\delta_n$

1.1) $i \lhd n$ then $i \lhd n+p^k\delta_n$ and therefore

$$\left| \frac{\frac{(n-i)-(n-\delta_n-i)}{\delta_n} + \frac{(n-i)-(n+p^k\delta_n-i)}{p^k\delta_n}}{-(p^k+1)\delta_n} \right| = 0$$

1.2) $i \ntriangleleft n$ then $i \ntriangleleft n+p^k\delta_n$ $(i = n-\delta_n)$ and $\left|\frac{n-\delta_n-i}{(p^k+1)\delta_n^2}\right| = 0$

2) $i \ntriangleleft n-\delta_n$

2.1) $i \lhd n$ then $i \lhd n+p^k\delta_n$ for $k \neq 0$ $(i = n)$ and therefore

$$\left| \frac{\frac{n-i}{\delta_n} + \frac{n-i-(n+p^k\delta_n-i)}{p^k\delta_n}}{-(p^k+1)\delta_n} \right| = \left|\frac{1}{(p^k+1)\delta_n}\right| = \left|\frac{1}{\delta_n}\right| = \left|\frac{1}{\delta_i}\right|.$$

For $k = 0$, $i \ntriangleleft n+\delta_n$ and the valuation is 0.

2.2) $i \ntriangleleft n$

2.2.1) $i \ntriangleleft n+p^k\delta_n$ then the valuation is 0.

2.2.2) $i \lhd n+p^k\delta_n$ then $i = n+p^k\delta_n$ and $\left|\frac{n+p^k\delta_n-i}{p^k(p^k+1)\delta_n^2}\right| = 0$.

Thus $\gamma_i(x-i)e_i(x)||_2 = 1$.

Theorem 5. $||(x-n)^2e_n(x)||_2 = 1$.

Proof. $||\phi_2(x-i)^2 \cdot e_i(x)||_s =$

$$\sup_{n\in\mathbb{N}} \left| \frac{\frac{(n-i)^2e_i(n)-(n-\delta_n-i)^2e_i(n-\delta_n)}{\delta_n} + \frac{(n-i)^2e_i(n)-(n+p^k\delta_n-i)^2e_i(n+p^k\delta_n)}{p^k\delta_n}}{-(p^k+1)\delta_n} \right|$$

1) $i \lhd n-\delta_n$

1.1) $i \lhd n$ then $i \lhd n+p^k\delta_n$ and therefore

$$\left| \frac{\frac{(n-i)^2-(n-\delta_n-i)^2}{\delta_n} + \frac{(n-i)^2-(n+p^k\delta_n-i)^2}{p^k\delta_n}}{-(p^k+1)\delta_n} \right| = \left| \frac{\frac{2\delta_n(n-i)-\delta_n^2}{\delta_n} + \frac{-2p^k\delta_n(n-i)-p^{2k}\delta_n^2}{p^k\delta_n}}{-(p^k+1)\delta_n} \right|$$

$$= \left|\frac{-\delta_n - p^k\delta_n}{-(p^k+1)\delta_n}\right| = 1.$$

1.2) $i \ntriangleleft n$ then $i \ntriangleleft n + p^k\delta_n$ $(i = n - \delta_n)$ and the valuation is 0.

2) $i \ntriangleleft n - \delta_n$

2.1) $i \triangleleft n$ then $i \triangleleft n + p^k\delta_n$ for $k \neq 0$ $(i = n)$ and thus

$$\Big|\frac{p^{2k}\delta_n^2}{p^k(p^k+1)\delta_n^2}\Big| = |p^k| \leq 1.$$

For $k = 0$, $i \ntriangleleft n + \delta_n$ and the valuation is 0.

2.2) $i \ntriangleleft n$

2.2.1) $i \ntriangleleft n + p^k\delta_n$ then the valuation is 0.

2.2.2) $i \triangleleft n + p^k\delta_n$ then $i = n + p^k\delta_n$ and the valuation is 0.
Thus $||(x-i)^2 e_i(x)||_2 = 1$.

3 Characterization of $\mathcal{C}^2$-functions

Theorem 6. *Let* $f(x) = \sum_{n=0}^{\infty} a_n e_n(x) + \sum_{n=0}^{\infty} b_n(x-n)e_n(x) \in C^1(\mathbb{Z}_p \to K)$.
$f \in C^2(\mathbb{Z}_p \to K)$ *if and only if* $\lim_{n\to a}\frac{a_n}{\gamma_n^2}$ *and* $\lim_{n\to a}\frac{b_n}{\gamma_n}$ *exist for all* $a \in \mathbb{Z}_p$*, and* $\lim_{n\to a}\frac{b_n}{\gamma_n} = 2\lim_{n\to a}\frac{a_n}{\gamma_n^2}$.

Proof. Suppose f is C^2, then there exists a continuous function $R_2 : \mathbb{Z}_p \times \mathbb{Z}_p \to K$ such that $f(x) = f(y) + (x-y)f'(y) + (x-y)^2 R_2(x,y)$ for all $x, y \in \mathbb{Z}_p$.

Thus $\lim_{(x,y)\to(a,a)} \frac{f(x) - f(y) - (x-y)f'(y)}{(x-y)^2} = R_2(a,a)$ exists for all $a \in \mathbb{Z}_p$.

In particular, $\lim_{(n,n_-)\to(a,a)} \frac{f(n) - f(n_-) - (n-n_-)f'(n_-)}{(n-n_-)^2} = R_2(a,a)$ exists for all $a \in \mathbb{Z}_p$, and $\lim_{n\to a}\frac{a_n}{\gamma_n^2} = R_2(a,a)$ exists for all $a \in \mathbb{Z}_p$.

We also have that $f(y) = f(x) + (y-x)f'(x) + (y-x)^2 R_2(y,x)$. Hence, $f(x) + f(y) = f(y) + f(x) + (x-y)(f'(y) - f'(x)) + (x-y)^2(R_2(x,y) + R_2(y,x))$. This is equivalent to: $(x-y)\Big(f'(x) - f'(y)\Big) = (x-y)^2\Big(R_2(x,y) + R_2(y,x)\Big)$. Therefore $\lim_{(x,y)\to(a,a)} \frac{f'(x) - f'(y)}{x-y} = 2R_2(a,a)$ exists for all $a \in \mathbb{Z}_p$.

In particular, $\lim_{(n,n_-)\to(a,a)} \frac{f'(n) - f'(n_-)}{n-n_-} = 2R_2(a,a)$ exists for all $a \in \mathbb{Z}_p$ and thus $\lim_{n\to a}\frac{b_n}{\gamma_n} = 2R_2(a,a)$ exists for all $a \in \mathbb{Z}_p$.

It follows also that $\lim_{n\to a}\frac{b_n}{\gamma_n} = 2\lim_{n\to a}\frac{a_n}{\gamma_n^2}$.

Now assume this to be the case.

If $\lim_{n\to a}\frac{b_n}{\gamma_n} = \lim_{n\to a}\frac{f'(n)-f'(n_-)}{n-n_-} = 2g(a)$ exists for all $a \in \mathbb{Z}_p$, then also $\lim_{(x,y)\to(a,a)}\frac{f'(x)-f'(y)}{x-y} = 2g(a)$ exists ([3], lemma 63.1).

And if $\lim_{n\to a}\frac{a_n}{\gamma_n^2} = \lim_{n\to a}\frac{f(n)-f(n_-)-(n-n_-)f'(n_-)}{(n-n_-)^2} = g(a)$ exists for all $a \in \mathbb{Z}_p$, then also $\lim_{(x,y)\to(a,a)}\frac{f(x)-f(y)-(x-y)f'(y)}{(x-y)^2} = g(a)$ exists.

It suffices to prove this for $x, y \in \mathbb{N}$ since $\mathbb{N}$ is dense in $\mathbb{Z}_p$.

$$\frac{f(x)-f(y)-(x-y)f'(y)}{(x-y)^2} = \frac{f(x)-f(z)-(x-z)f'(z)}{(x-z)^2}\cdot\Big(\frac{x-z}{x-y}\Big)^2$$

$$-\frac{f(y)-f(z)-(y-z)f'(z)}{(y-z)^2}\cdot\Big(\frac{y-z}{x-y}\Big)^2 + \frac{f'(y)-f'(z)}{y-z}\cdot\frac{z-y}{x-y}.$$

Let z be the common initial part of x and y. Then

$$\Big|\frac{f(x)-f(y)-(x-y)f'(y)}{(x-y)^2} - g(a)\Big|$$

$$\le \max\Big(\Big|\frac{f(x)-f(z)-(x-z)f'(z)}{(x-z)^2} - g(a)\Big|\Big|\frac{x-z}{x-y}\Big|^2,$$

$$\Big|\frac{f(y)-f(z)-(y-z)f'(z)}{(y-z)^2} - g(a)\Big|\Big|\frac{y-z}{x-y}\Big|^2, \Big|\frac{f'(y)-f'(z)}{y-z} - 2\cdot g(a)\Big|\Big|\frac{z-y}{x-y}\Big|\Big).$$

So it suffices to prove that $\lim_{(x,y)\to(a,a)}\frac{f(x)-f(y)-(x-y)f'(y)}{(x-y)^2} = g(a)$ for $y \lhd x$, since $\left|\frac{y-z}{x-y}\right|$ and $\left|\frac{z-y}{x-y}\right|$ are less than or equal to 1.

There exist $t_1 \lhd t_2 \lhd \ldots \lhd t_n$ so that $y = t_1$, $x = t_n$ and $(t_j)_- = t_{j-1}$.

$$\frac{f(x)-f(y)-(x-y)f'(y)}{(x-y)^2} = \sum_{j=2}^{n}\lambda_j\frac{f(t_j)-f(t_{j-1})-(t_j-t_{j-1})f'(t_{j-1})}{(t_j-t_{j-1})^2}$$

$$+\sum_{j=2}^{n-1}\mu_j\frac{f'(t_1)-f'(t_j)}{t_1-t_j}$$

with $\lambda_j = \left(\frac{t_j-t_{j-1}}{t_n-t_1}\right)^2$ and $\mu_j = \frac{(t_j-t_{j+1})(t_1-t_j)}{(t_n-t_1)^2}$ and $\sum_{j=2}^{n}\lambda_j + 2\sum_{j=2}^{n-1}\mu_j = 1$.

(This may be shown using induction on n). Now

$$\Big|\frac{f(x)-f(y)-(x-y)f'(y)}{(x-y)^2} - g(a)\Big| \le$$

$$\max\Big(\max_{j=2,\ldots,n}\Big|\frac{f(t_j)-f(t_{j-1})-(t_j-t_{j-1})f'(t_{j-1})}{(t_j-t_{j-1})^2}-g(a)\Big||\lambda_j|,$$

$$\max_{j=2,\ldots,n-1}\Big|\frac{f'(t_1)-f'(t_j)}{t_1-t_j}-2\cdot g(a)\Big||\mu_j|\Big)$$

which tends to zero by assumption.
Furthermore,

$$\phi_2 f(x,y,z)=\frac{f(x)-f(y)-(x-y)f'(y)}{(x-y)^2}\frac{x-y}{y-z}-$$

$$-\frac{f(x)-f(z)-(x-z)f'(z)}{(x-z)^2}\frac{x-z}{y-z}+\frac{f'(y)-f'(z)}{y-z}.$$

Because of the symmetry in the variables of $\phi_2 f$ we may assume that $|y-a|\geq |x-a|\geq |z-a|$.
Therefore $\lim_{(x,y,z)\to(a,a,a)}\phi_2 f(x,y,z)$ exists for all $a\in\mathbb{Z}_p$ and f is a C^2-function since

$$|\phi_2 f(x,y,z)-g(a)|\leq\max\Big(\Big|\frac{f(x)-f(y)-(x-y)f'(y)}{(x-y)^2}-g(a)\Big|\Big|\frac{x-y}{y-z}\Big|,$$

$$\Big|\frac{f(x)-f(z)-(x-z)f'(z)}{(x-z)^2}-g(a)\Big|\Big|\frac{x-z}{y-z}\Big|,\Big|\frac{f'(y)-f'(z)}{y-z}-2\cdot g(a)\Big|\Big).$$

Generalization. *Let* $f(x)=\sum_{i=0}^{\infty}a_{i,0}e_i(x)+\sum_{i=0}^{\infty}a_{i,1}(x-i)e_i(x)+\ldots+\sum_{i=0}^{\infty}a_{i,n}\frac{(x-i)^n}{n!}e_i(x)\in C^n(\mathbb{Z}_p\to K)$, *then* $f\in C^{n+1}(\mathbb{Z}_p\to K)$ *if and only if* $\lim_{i\to a}\frac{a_{i,n}}{\gamma_i}=2!\lim_{i\to a}\frac{a_{i,n-1}}{\gamma_i^2}=3!\lim_{i\to a}\frac{a_{i,n-2}}{\gamma_i^3}=\ldots=(n+1)!\lim_{i\to a}\frac{a_{i,0}}{\gamma_i^{n+1}}$ *exist for all* $a\in\mathbb{Z}_p$.

4 An orthonormal base for $\mathcal{C}^2$-functions

Before we prove our main theorem, we prove one more proposition that we will use in the proof of our final theorem.

Proposition 7. *If* $f(x)=\sum_{n=0}^{\infty}a_n e_n(x)$ *then* $P_2P_1f(x)=\sum_{n=0}^{\infty}a_n\frac{(x-n)^2}{2}e_n(x)$.

Proof. $P_1e_n(x)=\sum_{m=s}^{\infty}e_n(x_m)\cdot(x_{m+1}-x_m)=e_n(x)\sum_{m=s}^{\infty}(x_{m+1}-x_m)=(x-n)\cdot e_n(x)$.
Indeed, $e_n(x)=1$ if and only if there exist an $s\in\mathbb{N}$ such that $n=x_s$, or equivalently, if there exist an $s\in\mathbb{N}$ such that $e_n(x_m)=0$ for all $m<s$ and $e_n(x_m)=1$ for all

$m \geq s$.
In a similar way, we have for P_2:

$$P_2 e_n(x) = (x-n) \cdot e_n(x)$$

and
$P_2\Big((x-n)\cdot e_n(x)\Big)$

$$\begin{aligned} &= e_n(x)\sum_{m=s}^{\infty}(x_m - x_s)\cdot(x_{m+1}-x_m) + \frac{x_{m+1}^2}{2} + \frac{x_m^2}{2} - x_m x_{m+1} \\ &= e_n(x)\Big(\frac{x^2}{2} - \frac{x_s^2}{2} - x_s(x-x_s)\Big) \\ &= e_n(x)\frac{(x-n)^2}{2}. \end{aligned}$$

Finally,

$$P_2P_1 f(x) = P_2\Big(\sum_{n=0}^{\infty} a_n(x-n)e_n(x)\Big) = \sum_{n=0}^{\infty} a_n \frac{(x-n)^2}{2} e_n(x).$$

Theorem 8. $\{\gamma_n^2 e_n(x), \gamma_n(x-n)e_n(x), (x-n)^2 e_n(x) \mid n \in \mathbb{N}\}$ *is an orthonormal base for* $C^2(\mathbb{Z}_p \to K)$.

Proof. We know that $\{e_n(x) \mid n \in \mathbb{N}\}$ is an orthonormal base for $C(\mathbb{Z}_p \to K)$. Let $T = 2P_2P_1 : C(\mathbb{Z}_p \to K) \to C^2(\mathbb{Z}_p \to K) : \sum a_n e_n(x) \to \sum a_n (x-n)^2 e_n(x)$. For all elements $f(x) = \sum a_n e_n(x)$ of $C(\mathbb{Z}_p \to K)$ we now have $||Tf||_2 \leq \max|a_n| = ||f||_s = \left|\left|\frac{(Tf)''}{2}\right|\right|_s \leq ||Tf||_2$ (see [2] p. 89). So T is an isometry and thus $(x-n)^2 \cdot e_n(x)$ $(n \in \mathbb{N})$ is orthonormal in $C^2(\mathbb{Z}_p \to K)$.
W. Schikhof ([2]) proved that for every $f \in C^1(\mathbb{Z}_p \to K)$, $\phi_2 f$ can be extended to a continuous function $\tilde{\phi}_2 f$ on $\mathbb{Z}_p^{n+1}\backslash\{(x,\ldots,x) \in \mathbb{Z}_p^{n+1} \mid x \in \mathbb{Z}_p\}$ with

$$\tilde{\phi}_2 f(x,y,z) = \frac{\overline{\phi_1 f}(x,z) - \overline{\phi_1 f}(y,z)}{x-y}$$

for $x \neq y$. Let us take $z = y$, then

$$\begin{aligned} \tilde{\phi}_2 f(x,y,y) &= \frac{\overline{\phi_1 f}(x,y) - \overline{\phi_1 f}(y,y)}{x-y} \\ &= \frac{\phi_1 f(x,y) - f'(y)}{x-y} \\ &= \frac{f(x) - f(y) - (x-y)f'(y)}{(x-y)^2}. \end{aligned}$$

Let $f(x) = \sum a_n \gamma_n^2 e_n(x) + \sum b_n \gamma_n (x-n) e_n(x) \in C^2(\mathbb{Z}_p \to K) \subset C^1(\mathbb{Z}_p \to K)$. We know ([2], p. 89) that $||f||_2 \geq ||\tilde{\phi}_2 f||_s \geq |\tilde{\phi}_2 f(n, n_-, n_-)| = |a_n|$ and also that $||f||_2 \geq ||\phi_1 f'||_s \geq |\phi_1 f'(n,n_-)| = |b_n|$ for all $n \neq 0$. Furthermore, we have that $||f||_2 \geq ||f||_s \geq |f(0)| = |a_0|$ and $||f||_2 \geq ||f'||_s \geq |f'(0)| = |b_0|$. Therefore $||f||_2 \geq \max(|a_n|,|b_n|)$ and it follows that $\gamma_n^2 e_n(x)$, $\gamma_n(x-n)\cdot e_n(x)$ $(n \in \mathbb{N})$ are orthonormal in $C^2(\mathbb{Z}_p \to K)$. Let

$$\begin{aligned} N^2(\mathbb{Z}_p \to K) &= \{f \in C^2(\mathbb{Z}_p \to K) | f' = 0\} \qquad \text{and} \\ N_2^2(\mathbb{Z}_p \to K) &= \{f \in C^2(\mathbb{Z}_p \to K) | f'' = 0\}. \end{aligned}$$

For $f \in N_2^2(\mathbb{Z}_p \to K)$ and $g \in C(\mathbb{Z}_p \to K)$ we have:

$$||f+Tg||_2 \geq \Big|\Big|\frac{(f+Tg)''}{2}\Big|\Big|_s = \Big|\Big|\frac{(Tg)''}{2}\Big|\Big|_s = ||g||_s = ||Tg||_2.$$

Also $||f+Tg||_2 \geq ||f||_2$, since

$$||f||_2 = ||f+Tg-Tg||_2 \leq \max(||f+Tg||_2, ||Tg||_2) = ||f+Tg||_2.$$

It follows that $N_2^2(\mathbb{Z}_p \to K) \perp \operatorname{Im} T$ (the image of T). In particular

$$[[e_0(x), e_1(x), \ldots, xe_0(x), (x-1)e_1(x), \ldots]] \perp [[x^2 e_0(x), (x-1)^2 e_1(x), \ldots]].$$

So, the set $\{\gamma_n^2 e_n(x), \gamma_n(x-n)e_n(x), (x-n)^2 e_n(x) \mid n \in \mathbb{N}\}$ is orthonormal. It is also a base for $C^2(\mathbb{Z}_p \to K)$. Indeed, for every $f = f - P_2 f' + P_2 f' \in C^2(\mathbb{Z}_p \to K)$: $f'(x) = f'(0)\cdot e_0(x) + \sum_{n=1}^{\infty}(f'(n) - f'(n_-) - (n-n_-)f''(n_-))\cdot e_n(x) + f''(0)\cdot x \cdot e_0(x) + \sum_{n=1}^{\infty}(f''(n) - f''(n_-))\cdot(x-n)\cdot e_n(x)$ and $P_2 f'(x) = f'(0)\cdot x \cdot e_0(x) + \sum_{n=1}^{\infty}(f'(n) - f'(n_-) - (n-n_-)f''(n_-))\cdot(x-n)\cdot e_n(x) + f''(0)\cdot\frac{x^2}{2}\cdot e_0(x) + \sum_{n=1}^{\infty}(f''(n) - f''(n_-))\cdot\frac{(x-n)^2}{2}\cdot e_n(x)$. Consider $g = f - P_2 f' \in N^2(\mathbb{Z}_p \to K)$ and let $g(x) = g(0)\cdot e_0(x) + \sum_{n=1}^{\infty}(g(n) - g(n_-))\cdot e_n(x)$ be its representation in $C^1(\mathbb{Z}_p \to K) \subset C(\mathbb{Z}_p \to K)$. This is also an identity in $C^2(\mathbb{Z}_p \to K)$, since $\sum_{n=0}^{\infty} a_n e_n(x) \in C(\mathbb{Z}_p \to K)$ belongs to $N^2(\mathbb{Z}_p \to K)$ if and only if $\lim_{n\to\infty} |a_n|\cdot n^2 = 0$ ([3], p. 195) or equivalently $\lim_{n\to\infty}\frac{a_n}{\gamma_n^2} = 0$. We have therefore shown that f can be written as a convergent linear combination of $e_0(x), e_1(x), \ldots, xe_0(x), (x-1)e_1(x), \ldots, \frac{x^2}{2}e_0(x), \frac{(x-1)^2}{2}e_1(x), \ldots$
This finishes the proof of theorem 8.

Corollary 9. *For every $f \in C^2(\mathbb{Z}_p \to K)$ we have*

$$f(x) = \sum_{n=0}^{\infty} a_n e_n(x) + \sum_{n=0}^{\infty} b_n(x-n)e_n(x) + \sum_{n=0}^{\infty} c_n \frac{(x-n)^2}{2} e_n(x)$$

with

$$\begin{array}{ll}
a_0 = f(0) & \\
a_n = f(n) - f(n_-) - (n-n_-)\cdot f'(n_-) - \frac{(n-n_-)^2}{2} f''(n_-) & \text{for } n \neq 0 \\
b_0 = f'(0) & \\
b_n = f'(n) - f'(n_-) - (n-n_-)\cdot f''(n_-) & \text{for } n \neq 0 \\
c_0 = f''(0) & \\
c_n = f''(n) - f''(n_-) & \text{for } n \neq 0.
\end{array}$$

Proof. Let $f = g + P_2 f'$ with $g = f - P_2 f'$, then

$$f(x) = g(0)\cdot e_0(x) + \sum_{n=1}^{\infty}(g(n) - g(n_-))\cdot e_n(x) + f'(0)\cdot x\cdot e_0(x) + \sum_{n=1}^{\infty}(f'(n) -$$

$$f'(n_-))\cdot(x-n)\cdot e_n(x) + f''(0)\cdot\frac{x^2}{2}\cdot e_0(x) + \sum_{n=1}^{\infty}(f''(n) - f''(n_-))\cdot\frac{(x-n)^2}{2}\cdot e_n(x).$$

$g(0) = f(0) - P_2 f'(0) = f(0)$
$g(n) - g(n_-) = f(n) - f(n_-) - P_2 f'(n) + P_2 f'(n_-)$.
For $n = n_0 + n_1 p + \ldots + n_s p^s$, we have

$$P_2 f'(n) = f'(0) n_0 + f'(n_0) n_1 p + \ldots + f'(n_0 + n_1 p + \ldots + n_{s-1} p^{s-1}) n_s p^s + f''(0)\frac{n_0^2}{2} +$$

$$f''(n_0)\frac{n_1^2 p^2}{2} + \ldots + f''(n_0 + n_1 p + \ldots + n_{s-1} p^{s-1})\frac{n_s^2 p^{2s}}{2}$$

and

$$P_2 f'(n_-) = f'(0) n_0 + f'(n_0) n_1 p + \ldots + f'(n_0 + n_1 p + \ldots + n_{s-2} p^{s-2}) n_{s-1} p^{s-1} +$$

$$f''(0)\frac{n_0^2}{2} + f''(n_0)\frac{n_1^2 p^2}{2} + \ldots + f''(n_0 + n_1 p + \ldots + n_{s-2} p^{s-2})\frac{n_{s-1}^2 p^{2s-2}}{2}.$$

So

$$\begin{aligned}
P_2 f'(n) - P_2 f'(n_-) &= f'(n_-) n_s p^s + f''(n_-)\frac{n_s^2 p^{2s}}{2} \\
&= (n-n_-)\cdot f'(n_-) + \frac{(n-n_-)^2}{2}\cdot f''(n_-).
\end{aligned}$$

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