A remark on the ϕ-norms

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Abstract. We introduce the concept of ϕ-norm on (complex or real) linear spaces for a class of functions ϕ which are J-convex, nondecreasing and having a fixed point on the set of nonnegative real numbers. The q-norms introduced by H. Belbachir, M. Mirzavaziri and M. S. Moslehian, (see A.J.M.A.A., Vol. 3, No. 1, Art. 2, (2006)) are particular cases of ϕ-norms. We establish that every ϕ-norm is a norm in the usual sense, and that the converse is true as well.

Resumen. Introducimos el concepto de ϕ-norma sobre espacios lineales (complejos o reales) para una clase de funciones ϕ que es J-convexa, no decreciente y con un punto fijo en el conjunto de los números reales no negativos. Las q-normas introducidas por H. Belbachir, Mirzavaziri M. y Moslehian MS, (ver AJMAA, vol. 3, No. 1, del art. 2 (2006)) son casos particulares de ϕ-normas. Probamos que cada ϕ-norma es una norma en el sentido usual, y que el converso también es cierto.

1 Introduction

In [3], S. Saitoh noticed that in any arbitrary linear space E, the so-called parallelogram inequality $\|x+y\|^2 \le 2(\|x\|^2 + \|y\|^2)$, for vectors x, y in E, may be more suitable than the usual triangle inequality. He considered this inequality in the setting of a natural sum Hilbert space for two arbitrary Hilbert spaces.

It is easy to see that any arbitrary norm $\|.\|$ on E satisfies the parallelogram inequality.

The reader is referred to [2] for undefined terms and notations.

In [1], the authors have introduced an extension of the triangle inequality by using the concept of a q-norm.

Definition 1.1. *Let $\mathcal{X}$ be a real or complex linear space and $q \in [1, \infty)$. A mapping $\|\cdot\| : \mathcal{X} \to [0, \infty)$ is called a q-norm on $\mathcal{X}$ if it satisfies the following conditions:*

1. $\|x\| = 0 \Leftrightarrow x = 0,$

2. $\|\lambda x\| = |\lambda| \, \|x\|$ *for all* $x \in \mathcal{X}$ *and all scalar* λ,

3. $\|x+y\|^q \le 2^{q-1} \left(\|x\|^q + \|y\|^q\right)$ *for all* $x, y \in \mathcal{X}$.

In [1], the following result was proved.

Theorem 1.1. *Every q-norm is a norm in the usual sense.*

The purpose of this paper is to extend the result above by using the notion of a ϕ-norm which is more general than the notion of a q-norm.

2 Definitions and preliminaries

Let Φ be the set of functions $\phi : [0, +\infty) \to [0, +\infty)$ which are not identically zero and satisfying the following properties:
(ϕ_1) ϕ is non-decreasing on $[0, +\infty)$.
(ϕ_2) $\phi\left(\frac{s+t}{2}\right) \le \frac{\phi(s)+\phi(t)}{2}$, for all $s, t \in [0, +\infty)$.
(ϕ_3) There exists a positive number $r > 0$ such that $\phi(r) = r$.

Example 1. For each $q > 1$, the function $\phi_q(t) = t^q$ satisfies the properties (ϕ_1), (ϕ_2) and (ϕ_3) with $r = 1$.

Example 2. Consider $\phi(t) = \exp(t-1)$. Then ϕ satisfies the properties (ϕ_1), (ϕ_2) and (ϕ_3) with $r = 1$.

Definition 2.1. *Let* $\mathcal{X}$ *be a real or complex linear space and* $\phi \in \Phi$*. A mapping* $\|\cdot\| : \mathcal{X} \to [0, \infty)$ *is called a* ϕ*-norm on* $\mathcal{X}$ *if it satisfies the following conditions:*

1. $\|x\| = 0 \Leftrightarrow x = 0$,

2. $\|\lambda x\| = |\lambda| \, \|x\|$ *for all* $x \in \mathcal{X}$ *and all scalar* λ,

3. $\phi\left(\frac{\|x+y\|}{2}\right) \le \frac{\phi(\|x\|)+\phi(\|y\|)}{2}$, *for all* $x, y \in \mathcal{X}$.

When ϕ is continuous, We observe that the property (3) above is equivalent to say that the function $x \longrightarrow \phi(\|x\|)$ is convex on $\mathcal{X}$.

Remark 1. For evey number $q > 1$, a mapping $\|\cdot\| : \mathcal{X} \to [0, \infty)$ is a q-norm on $\mathcal{X}$ if and only if $\|\cdot\|$ is a ϕ_q-norm, where $\phi_q(t) = t^q$.

We have the following result.

Proposition 2.1. *Every norm in the usual sense is a* ϕ*-norm for every* $\phi \in \Phi$.

Proof. Let $x, y \in \mathcal{X}$. Since $\|.\|$ is a norm, then we have

$$\|x+y\| \le \|x\| + \|y\|.$$

Since ϕ is non-decreasing and convex on $[0,\infty)$, it follows that

$$\phi\left(\frac{\|x+y\|}{2}\right) \leq \phi\left(\frac{\|x\|+\|y\|}{2}\right) \leq \frac{\phi(\|x\|)+\phi(\|y\|)}{2}.$$

So, $\|.\|$ is a ϕ-norm.

The following lemma will be used in the proof of the main result of this note.

Lemma 2.1. *Let $\mathcal{X}$ be a real or complex linear space. Let $\|\cdot\| : \mathcal{X} \to [0,\infty)$ be a mapping satisfying (1) and (2) in the definition of a ϕ-norm. Then the following assertions are equivalent:*
(i) $\|\cdot\|$ is a norm.
(ii) The set $B_r = \{x \in \mathcal{X} : \|x\| \leq r\}$ is convex, for any arbitrary number $r > 0$.
(iii) The set $B_1 = \{x \in \mathcal{X} : \|x\| \leq 1\}$ is convex.
(iv) There exists a positive number $r > 0$ such that the set $B_r = \{x \in \mathcal{X} : \|x\| \leq r\}$ is convex.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious. It remains to show the implication (iv) $\Rightarrow$ (i). Let $r > 0$ be such that the set B_r is convex. Let $x, y \in \mathcal{X}$. We can suppose that $x \neq 0$ and $y \neq 0$. We put $x' = r\frac{x}{\|x\|}$ and $y' = r\frac{y}{\|y\|}$. We have $x', y' \in B_r$. By assumption, we know that $\lambda x' + (1-\lambda)\, y' \in B_r$ for all $0 \leq \lambda \leq 1$. In particular, for $\lambda = \frac{\|x\|}{\|x\|+\|y\|}$ we obtain

$$\left\| \frac{rx}{\|x\|+\|y\|} + \frac{ry}{\|x\|+\|y\|} \right\| = \|\lambda x' + (1-\lambda)\, y'\| \leq r.$$

So that $r\|x+y\| \leq r[\|x\|+\|y\|]$, which implies that $\|x+y\| \leq \|x\|+\|y\|$. So $\|.\|$ is a norm on $\mathcal{X}$. This ends the proof.

3 The result

Now we are ready to state and prove the main result of this note.

Theorem 3.1. *Let $\mathcal{X}$ be a real or complex linear space. Let $\|\cdot\| : \mathcal{X} \to [0,\infty)$ be a mapping satisfying the following conditions:*

1. $\|x\| = 0 \Leftrightarrow x = 0$,
2. $\|\lambda x\| = |\lambda|\, \|x\|$ *for all $x \in \mathcal{X}$ and all scalar λ.*

Then the following assertions are equivalent:
(i) $\|\cdot\|$ is a norm (in the usual sense on $\mathcal{X}$).
(ii) $\|\cdot\|$ is a ϕ-norm on $\mathcal{X}$ for every $\phi \in \Phi$.
(iii) There exists a $\phi \in \Phi$ such that $\|\cdot\|$ is a ϕ-norm on $\mathcal{X}$.

Proof. The implication (i) $\Rightarrow$ (ii) is Proposition 2.1. The implication (ii) $\Rightarrow$ (iii) is evident. We have to prove the implication (iii) $\Rightarrow$ (i). Suppose that $\|\cdot\|$ is a ϕ-norm on $\mathcal{X}$ for some $\phi \in \Phi$. By Property (ϕ_3), there exists a positive number $r > 0$ such that $\phi(r) = r$. By Lemma 2.1, it is sufficient to prove that the set $B_r = \{x \in \mathcal{X} : \|x\| \leq r\}$ is convex.

Let $x, y \in B_r$. Then, by the properties (ϕ_1), (ϕ_2) and (ϕ_3) of ϕ, we have

$$\phi\left(\frac{\|x+y\|}{2}\right) \leq \frac{\phi(\|x\|)+\phi(\|y\|)}{2} \leq \frac{\phi(r)+\phi(r)}{2} = \phi(r) = r.$$

So $\frac{1}{2}x + \left(1-\frac{1}{2}\right)y \in B_r$. Thus if $D = \left\{\frac{k}{2^n} \mid n = 1,2,\ldots; k = 0,1,\ldots,2^n\right\}$, then for each $\lambda \in D$ we have $\lambda x + (1-\lambda)\,y \in B_r$.

Let $0 \leq \lambda \leq 1$ and $z = \lambda x + (1-\lambda)\,y$. We can suppose that $0 < \lambda < 1$. Since D is dense in $[0,1]$, there exists a sequence $\{\rho_n\}$ of points of D satisfying $\rho_n \geq \lambda$, for every nonnegative integer n, such that $\lim_n \rho_n = \lambda$. We put $\beta_n = \frac{1-\rho_n}{1-\lambda}$. Obviously, we have $0 \leq \beta_n \leq 1$ and $\lim_n \beta_n = 1$. For every nonnegative integer n, we observe that

$$0 \leq \frac{\lambda}{\rho_n}\beta_n \leq \beta_n \leq 1.$$

Therefore $\frac{\lambda}{\rho_n}\beta_n x \in B_r$. Since $\rho_n \in D$ we conclude that

$$\beta_n z = \lambda\beta_n\, x + (1-\lambda)\,\beta_n y = \rho_n\,\frac{\lambda}{\rho_n}\beta_n\, x + (1-\rho_n)\,y \in B_r.$$

Thus $\beta_n\|z\| = \|\beta_n z\| \leq r$ for all n. By letting n tend to infinity, we get $\|z\| \leq r$, i.e. $z \in B_r$.

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References

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