ARTÍCULOS

The transversality condition for infinite dimensional control systems.

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Abstract. In this paper we provide a definition of transversality for the following infinite dimensional control system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \quad t > 0 \\ x(0) = x_0 \in X; \quad u(t) \in \mathcal{U} \subset U, \quad x(t^*) \in G(t^*), \end{cases}$$

for $t^\star > 0$ minimum; where the state $x(t) \in X$, X and U are Banach spaces, A is the infinitesimal generator of a strongly continuous group $\{S(t)\}_{t\in\mathbb{R}}$ in X, $B \in L(U, X)$, the target set $G \subset X$ and the control values set $\mathcal{U}$ are convex and weakly compact. For this system we give a necessary condition for a control satisfying the transversality condition to be optimal. Finally, as an application we consider the optimal control problem governed by the wave equation

$$\begin{cases} y_{tt} - \Delta y = u(t,x), \ x \in \Omega, \, t \in \mathbb{R} \\ y = 0, \quad \text{on} \quad \mathbb{R} \times \partial\Omega, \\ y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x), \quad x \in \Omega, \\ \|u(t,\cdot)\|_{L^2} \leq 1, \quad t \in \mathbb{R}, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$, the distributed control $u \in L^2(0,t_1;L^2(\Omega))$; for this problem we compute the extremal control.

Resumen. En este trabajo se ofrece una definición de transversalidad para el siguiente sistema de control infinito dimensional

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \quad t > 0 \\ x(0) = x_0 \in X; \quad u(t) \in \mathcal{U} \subset U, \quad x(t^*) \in G(t^*), \end{cases}$$

para $t^\star > 0$ mínimo; en el estado $x(t) \in X$, X y U son espacios de Banach, A es el generador infinitesimal de un grupo fuertemente

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continuo $\{S(t)\}_{t\in\mathbb{R}}$ en X, $B \in L(U,X)$, el conjunto objetivo $G \subset X$ y el conjunto control $\mathcal{U}$ son convexos y débilmente compactos. Para este sistema damos una condición necesaria para que un control que satisfaga la condición de transversalidad sea óptimo. Por último, como aplicación se considera el problema de control óptimo gobernado por la ecuación de onda

$$\begin{cases} y_{tt} - \Delta y = u(t,x), \ x \in \Omega, \, t \in \mathbb{R} \\ y = 0, \quad \text{on} \quad \mathbb{R} \times \partial\Omega, \\ y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x), \quad x \in \Omega, \\ \|u(t,\cdot)\|_{L^2} \leq 1, \quad t \in \mathbb{R}, \end{cases}$$

donde Ω es un dominio acotado en $\mathbb{R}^n$, el control distribuido $u \in L^2(0,t_1,L^2(\omega))$; para este problema calculamos el control extremo.

1 Introduction and preliminaries

For finite dimensional linear systems a sufficient condition for the optimal control is given in [4] and [5], this condition is referred to as transversality condition and it is contained in Theorem 19 from [5], page 132. That is to say, if X is a finite dimensional Banach space, then under certain hypothesis the extremal control satisfying the transversality condition is unique optimal. Also, it is proved in Theorem 18 of [5] that, if $u^\star(t) \in \Omega$ on $0 \leq t \leq t^\star$ is a minimal time-optimal controller for the finite dimensional system

$$\begin{cases} \dot{x}(t) = A(t)x + B(t)u, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \\ x(0) = x_0 \in \mathbb{R}^n, \quad u(t) \in \Omega \subset \mathbb{R}^m, \quad x(t^*) \in G(t^*) \subset \mathbb{R}^n, \end{cases} \tag{1.1}$$

then $u^\star(t)$ is extremal; that is to say; the following maximum principle holds

$$m(t) = \max_{v\in\Omega}\Big\langle \eta(t), B(t)v \Big\rangle = \Big\langle \eta(t), B(t)u^\star(t) \Big\rangle$$

and

$$M(t) = \max_{v\in\Omega}\Big\langle \eta(t), A(t)x^\star(t) + B(t)v \Big\rangle = \Big\langle \eta(t), A(t)x^\star(t) + B(t)u^\star(t) \Big\rangle$$

almost every where $0 \leq t \leq t^\star$.

Here $A(t)$ and $B(t)$ are matrices of order $n \times n$ and $n \times m$ respectively, Ω is compact, the target set $G(t)$ is also compact and varies continuously on $[0,\infty]$; and $\eta(t)$ is a nontrivial solution of the adjoint system

$$\dot{\eta} = -\eta A(t),$$

and $\eta(t^\star)$ is an outwards unit normal vector to a supporting hyperplane to the set of attainability $K(t^\star)$ at $x^\star(t^\star)$ in $\partial K(t^\star)$. Furthermore, if $G(t) = G$ is constat, then $x^\star(t^\star)$ lies in the new frontier of $K(t^\star)$. In this case, provided $A(t)$ and $B(t)$ are continuous, the normal $\eta(t^\star)$ can be selected so that

$$M(t^\star) \geq 0. \tag{1.2}$$

If in addition G is convex, then $\eta(t^\star)$ can be selected satisfying the **transversality condition**; namely, $\eta(t^\star)$ is normal to a common supporting hyperplane separating $K(t^\star)$ and G.

In this paper we generalize these results to the following infinite dimensional optimal control system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \quad t > 0 \\ x(0) = x_0 \in X; \quad u(t) \in \mathcal{U} \subset U, \quad x(t^*) \in G, \end{cases} \tag{1.3}$$

where the state $x(t) \in X$; X and U are separable Banach spaces with X been reflexive, $B \in L(U, X)$, the controls $u \in L^1_{loc}(\mathbb{R}_+, U)$, the target set $G \subset X$ and the control values set $\mathcal{U}$ are convex and weakly compact, and A is the infinitesimal generator of a strongly continuous group $\{S(t)\}_{t\in\mathbb{R}}$ of bounded linear operators in X. A mild solution of (1.3) is a function $x_u(\cdot) : [0, \infty) \longrightarrow X$ defined by

$$x_u(t) = S(t)x_0 + \int_0^t S(t-\alpha)Bu(\alpha)d\alpha, \quad t \geq 0, \tag{1.4}$$

where $u \in L^1_{loc}(\mathbb{R}_+, U)$.

Definition 1.1. For $t_1 > 0$ the set of **admissible controls** on $[0, t_1]$ is defined by

$$C(t_1) = \{u \in L^1(0, t_1; U) : u(t) \in \mathcal{U} \quad \text{a.e in} \quad [0, t_1]\}$$

and the corresponding set of **attainable points** by

$$K(t_1) = \{x_u(t_1) : x_u(\cdot) \quad \text{is mild solution of (1.3)}, u \in C(t_1)\}.$$

Since $\mathcal{U}$ is convex and weakly compact and U is separable, $BC(t_1)$ can be considered as a mensurable multifunction taking weakly compact values and so $K(t_1)$ is weakly compact in X ([1], [7]).

Through this work we suppose that $\text{int}K(t_1) \neq \emptyset$, where $\text{int}K(t_1)$ and $\partial K(t_1)$ denote the interior and the boundary of $K(t_1)$ respectively.

The following definition is a generalization of a similar one given in [5] page 73.

Definition 1.2. A control $u \in C(t_1)$ is called an **extremal control** if the corresponding solution x_u of (1.3) satisfies $x_u(t_1) \in \partial K(t_1)$.

Definition 1.3. If $t^\star \geq 0$ and $u^\star \in C(t^\star)$ with corresponding solution $x^\star(\cdot)$ of (1.4) satisfying $x^\star(t^\star) \in G$, then $u^\star$ is called an **optimal control** if

$$t^\star = inf\{t \in [0,\infty) : K(t) \cap G \neq \emptyset\}.$$

As an application of our result we shall consider the following optimal control problem governed by the wave equation

$$\begin{cases} y_{tt} - \Delta y = u(t,x), \; x \in \Omega, \, t \in \mathbb{R} \\ y = 0, \quad \text{on} \quad \mathbb{R} \times \partial\Omega, \\ y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x), \quad x \in \Omega, \\ \|u(t,\cdot)\|_{L^2} \leq 1, \quad t \in \mathbb{R}, \end{cases} \tag{1.5}$$

and prove that the optimal control is given by:

$$u(t,x) = \begin{cases} \frac{\eta_2(t,x)}{\|\eta_2(t,\cdot)\|_{L^2}} & \text{if} \quad \|\eta_2(t,\cdot)\|_{L^2} \neq 0 \\ 1 & \text{if} \quad \|\eta_2(t,\cdot)\|_{L^2} = 0 \end{cases}$$

where

$$\eta_2(s,x) = \sum_{i=1}^{\infty}(-\lambda_j^{\frac{1}{2}} \sin(\sqrt{\lambda_j} s) < \phi_j, x_1^\star > \phi_j(x) + \cos(\sqrt{\lambda_j} s) < \phi_j, x_2^\star > \phi_j(x)),$$

and $x_1^\star \in H_0^1(\Omega)$, $x_2^\star \in L^2(\Omega)$, where λ_j and ϕ_j are the eigenvalues and the eigenfunctions of $-\Delta$ respectively and $\langle \phi_j, x_i^\star \rangle = \int_\Omega \phi_j(x) x_i^\star(x) dx$.

2 Main results

Following the lead of Lee-Markus [5] we state the forthcoming results.

Theorem 2.1. *A control $u \in C(t_1)$ is extremal if, and only if, there is $x^\star \in X^\star \backslash \{0\}$ such that*

$$\max_{v \in \mathcal{U}} \big\langle \eta(s), Bv \big\rangle = \big\langle \eta(s), Bu(s) \big\rangle \quad a.e \quad on \quad [0, t_1] \tag{2.1}$$

where

$$\eta(s) = S^\star(-s)x^\star, \quad 0 \leq s \leq t_1.$$

Proof. By Theorem 4.3 of [1] a control $u \in C(t_1)$ is extremal if and only if, there exists $y^\star \in X^\star \backslash \{0\}$ such that for almost every $s \in [0, t_1]$

$$\begin{aligned} \max_{v \in \mathcal{U}} \big\langle y^\star, S(t_1 - s)Bv \big\rangle &= \big\langle y^\star, S(t_1 - s)Bu(s) \big\rangle = \big\langle S^\star(t_1 - s)y^\star, Bu(s) \big\rangle \\ &= \big\langle S^\star(-s)S^\star(t_1)y^\star, Bu(s) \big\rangle \quad \text{a.e.} \end{aligned}$$

Since $S^\star$ is a group, $S^\star(t_1)$ is invertible; then $x^\star = S^\star(t_1)y^\star \neq 0$, and by putting $\eta(s) = S^\star(-s)x^\star$ we get the result. $\square$

Definition 2.2. An extremal control $u \in C(t_1)$ satisfies the **transversality condition** if the hyperplane

$$\Pi(t_1) = \Big\{x \in X : \big\langle \eta(t_1), x - x_u(t_1)\big\rangle = 0\Big\}$$

separates $K(t_1)$ and G at the point $x_u(t_1)$.

Theorem 2.3. *Suppose $x_0 \in D(A)$ and $u^\star \in C(t^\star)$ is an* **optimal control** *which satisfies*

$$\begin{aligned} &S(-s)Bu^\star \in D(A) \quad a.e \quad on \quad [0,t^\star], \quad t^\star > 0 \\ &AS(t-\cdot)Bu^*(\cdot) \in L^1(0,t;X), \quad \forall t \in [0,t^\star). \end{aligned} \tag{2.2}$$

Then

$$M(t) = \max_{v\in\Omega}\Big\langle \eta(t), A(t)x^\star(t) + B(t)v\Big\rangle = \Big\langle \eta(t), A(t)x^\star(t) + B(t)u^\star(t)\Big\rangle$$

is defined a.e on $[0,t^\star]$, where $x^\star(\cdot) = x_{u^\star}(\cdot)$ and $\eta(t) = S(-t)x^\star$ with $x^\star \neq 0$ according to Theorem 2.1. Moreover, we can choose $\eta(t^\star)$ such that

$$M(t^\star) \geq 0$$

and satisfying the ***transversality condition****.*

Proof. If hypothesis (2.2) is satisfied, then Lemma 2.22 of [2] implies that $x^\star$ is differentiable almost every where on $[0,t^\star]$ because X is reflexive and

$$\dot{x}^\star(t) = Ax^\star(t) + Bu^\star(t) \quad \text{a.e} \quad \text{on} \quad [0,t^\star],$$

which implies $M(t)$ is well defined. Since $x^\star(\cdot)$ may not be differentiable in every point of $[0,t^\star]$, we will use a limit process to prove $M(t^\star) \geq 0$.

It is clear that $x^\star(t^\star) \notin K(t_1)$ if $0 \leq t_1 < t^\star$. Thus, by Theorem 6.3 of [4] there is $\eta(t_1) \in X^\star$, with $\|\eta(t_1)\| = 1$ and

$$0 < \inf_{x\in K(t_1)} \|x^\star(t^\star) - x\| = \inf_{x\in K(t_1)} \big\langle \eta(t_1), x^\star(t^\star) - x\big\rangle. \tag{2.3}$$

Since $K(t_1)$ is weakly compact, there is $x(t_1) \in K(t_1)$ such that

$$0 < \inf_{x\in K(t_1)} \|x^\star(t^\star) - x\| = \big\langle \eta(t_1), x^\star(t^\star) - x(t_1)\big\rangle. \tag{2.4}$$

From (2.3) and (2.4) we get

$$\big\langle \eta(t_1), x - x^\star(t_1)\big\rangle \leq 0 \quad \text{for every} \quad x \in K(t_1).$$

Thus $\eta(t_1)$ separates $x^\star(t^\star)$ from $K(t_1)$ at $x(t_1) \in K(t_1)$.

This implies that

$$\big\langle \eta(t_1), x^\star(t^\star) - x(t_1) \big\rangle > 0 \quad \textit{and} \quad \big\langle \eta(t_1), x - x(t_1) \big\rangle \leq 0, \quad x \in K(t_1) \qquad (2.5)$$

Now, we will prove that there is $\hat{t}_1 \in (t, t^\star)$ such that $\big\langle \eta(t_1), \dot{x}(\hat{t}_1) \big\rangle > 0$. Otherwise,

$$\big\langle \eta(t_1), \dot{x}^\star(t) \big\rangle \leq 0 \quad t \in [t_1, t^\star],$$

whenever $\dot{x}^\star(t)$ exist on $[0, t^\star]$.

Since $x^\star(\cdot)$ is absolutely continuous,

$$\int_{t_1}^{t^\star} \big\langle \eta(t_1), \dot{x}^\star(t) \big\rangle dt \leq 0 \Longleftrightarrow \big\langle \eta(t_1), x^\star(t^\star) - x^\star(t_1) \big\rangle \leq 0$$

which implies

$$\begin{aligned} 0 < \big\langle \eta(t_1), x^\star(t^\star) - x(t_1) \big\rangle &= \big\langle \eta(t_1), x^\star(t^\star) - x^\star(t_1) \big\rangle \\ &+ \big\langle \eta(t_1), x^\star(t_1) - x(t_1) \big\rangle \leq 0 \end{aligned}$$

which contradicts (2.5).

In this way we can choose a sequence

$$0 < t_1 < \hat{t}_1 < t_2 < \hat{t}_2 < \cdots < t_n < \hat{t}_n < \cdots < t^\star$$

with $\|\eta(t_n)\| = 1$, for all n, and satisfying

$$\big\langle \eta(t_n), \dot{x}^\star(\hat{t}_n) \big\rangle > 0 \Longleftrightarrow \big\langle \eta(t_n), Ax^\star(t_n) + Bu^\star(\hat{t}_n) > 0. \qquad (2.6)$$

Since $\mathcal{U}$ weakly compact, and $K(t)$ is weakly compact and uniformly bounded for $0 < t \leq t^\star$, we can suppose:

$$w - \lim_{n\to\infty} u^\star(\hat{t}_n) = u \in \mathcal{U}, \qquad w - \lim_{n\to\infty} \eta(t_n) = \eta(t^\star) \qquad (2.7)$$

and

$$\lim_{n)\to\infty} \big\langle (t_n), x(t_n) \big\rangle = \alpha \in \mathbb{R}. \qquad (2.8)$$

Consider the hyperplane $\Pi(t^\star)$ given by

$$\Pi(t^\star) = \{x \in X : \big\langle \eta(t^\star), x - x^\star(t^\star) \big\rangle = 0\}.$$

Claim. $\Pi(t^\star)$ separates $x^\star(t^\star)$ and $K(t^\star)$. In fact,

$$\big\langle \eta(t^\star), x - x^\star(t^\star) \big\rangle \leq 0, \quad \forall x \in K(t).$$

Otherwise, there is $x_0 \in K(t^\star)$ such that

$$\big\langle \eta(t^\star), x_0 - x^\star(t^\star) \big\rangle > 0. \qquad (2.9)$$

Since by (2.5)

$$\big\langle \eta(t_n), x^\star(t^\star) - x(t_n)\big\rangle \geq 0 \quad n = 1, 2, \cdots ;$$

then, by inequality

$$\big\langle \eta(t_n), x^\star(t^\star) - x(t_n)\big\rangle \leq \|x^\star(t^\star) - x^\star(t_n)\|,$$

we get

$$\lim_{n\to\infty} \big\langle \eta(t_n), x(t_n)\big\rangle = \big\langle \eta(t^\star), x^\star(t^\star)\big\rangle.$$

Since $\lim_{n\to\infty} h(K(t_n), K(t^\star)) = 0$, where h denotes the Hausdorff metric (see [1]), there is a sequence $\{\overline{x}_n\} \subset K(t_n)$, $n = 1, 2, \cdots$ with $\lim_{n\to\infty} \|\overline{x}_n - x_0\| = 0$ and by (2.9) we get

$$\big\langle \eta(t_n), \overline{x}_n - x(t_n)\big\rangle \leq 0 \quad n = 1, 2, \cdots . \tag{2.10}$$

But, however

$$\begin{aligned} \lim_{n\to\infty} \big\langle \eta(t_n), \overline{x}_n - x(t_n)\big\rangle &= \lim_{n\to\infty} \big(\big\langle \eta(t_1), \overline{x}_n - x_0\big\rangle \\ &+ \big\langle \eta(t_n), x_0\big\rangle - \big\langle \eta(t_n), x(t_n)\big\rangle\big) \\ &= \big\langle \eta(t^\star), x_0 - x^\star(t^\star)\big\rangle > 0, \end{aligned}$$

which is a contradiction with (2.10). So $\Pi(t^\star)$ separates $x^\star(t^\star)$ and $K(t^\star)$.

Since A is closed, our hypothesis together with Theorem II.2.6 of [3] imply

$$\begin{aligned} Ax^\star(\hat{t}_n) &= AS(\hat{t}_n)x_0 + AS(\hat{t}_n)\int_0^{\hat{t}_n} S(-s)Bu^\star(s)ds \\ &= S(\hat{t}_n)Ax_0 + S(\hat{t}_n)\int_0^{\hat{t}_n} S(-s)ABu^\star(s)ds \end{aligned}$$

which implies that

$$\begin{aligned} \lim_{n\to\infty} Ax^\star(\hat{t}_n) &= S(t^\star)Ax_0 + S(t^\star)\int_0^{t^\star} S(-s)ABu^\star(s)ds \\ &= Ax^\star(t^\star). \end{aligned}$$

Hence, by taking limit in (2.6) we obtain $M(t^\star) \geq 0$. □

The proof of Corollary 2.4 and Theorem 2.5 are similar to proof of Corollary and Theorem 18 from [5], pages 131 and 132, respectively.

Corollary 2.4. *If A is a bounded operator and $u \in C(t^\star)$ is an extremal control, then*

$$\begin{aligned} M(t) &= \max_{v\in\mathcal{U}} \big\langle \eta(t), Ax(t) + Bv\big\rangle \\ &= \big\langle \eta(t), Ax(t) + Bu(t)\big\rangle \quad a.e \ on \quad [0, t^\star]. \end{aligned} \tag{2.11}$$

is well defined and constant.

Proof. Since A is a bounded operator, $D(A) = X$ and $S(t) = \exp(tA)$, then the hypothesis of Theorem 5.3 of [1] are satisfied. Therefore $M(\cdot)$ is well defined and it is absolutely continuous, and so M is differentiable almost every where on $[0, t^\star]$. We will estimate the derivative of $M(t)$ at $t = \tau_1$, if it exists. Suppose $\tau_2 > \tau_1$. Then

$$\begin{aligned} \frac{M(\tau_2) - M(\tau_1)}{\tau_2 - \tau_1} &\geq \frac{\langle \eta(\tau_2), Ax(\tau_2) + Bu(\tau_1) \rangle - \langle \eta(\tau_1), A(\tau_1) + Bu(\tau_1) \rangle}{\tau_2 - \tau_1} \\ &= \langle \eta(\tau_2), A\frac{x(\tau_2) - x(\tau_1)}{\tau_2 - \tau_1} \rangle + \langle \frac{\eta(\tau_2) - \eta(\tau_1)}{\tau_2 - \tau_1}, Ax(\tau_1) \rangle \\ &+ \langle \frac{\eta(\tau_2) - \eta(\tau_1)}{\tau_2 - \tau_1}, Bu(\tau_1) \rangle. \end{aligned}$$

Without loss generality we can suppose that $\dot{x}(\tau_1)$ exists. If $x^\star \in X^\star \backslash \{0\}$ satisfies the equation $\eta(t) = \exp(-A^\star t)x^\star$, then

$$\dot{\eta}(t) = -A^\star \exp(-A^\star t)x^\star = -A^\star \eta(t).$$

Thus, by taking limit as $\tau_2 \longrightarrow \tau_1$, we have

$$\begin{aligned} \frac{dM}{dt}(\tau_1) &\geq \langle \eta(\tau_1), A\dot{x}(\tau_1) \rangle + \langle \dot{\eta}(\tau_1), Ax(\tau_1) \rangle + \langle \dot{\eta}(\tau_1), Bu(\tau_1) \rangle \\ &= \langle \eta(\tau_1), A(Ax(\tau_1) + Bu(\tau_1)) \rangle - \langle A^\star \eta(\tau_1), Ax(\tau_1) \rangle \\ &- \langle A^\star \eta(\tau_1), Bu(\tau_1) \rangle = 0. \end{aligned}$$

Similar computation shows that $\frac{dM}{dt}(\tau_1) \leq 0$. Consequently M is constant on $[0, t^\star]$. □

The following theorem prove that, under normality conditions, the Maximum Principle is sufficient for optimality, provided that the optimal control exists and is the unique extremal control which satisfies the transversality condition.

Theorem 2.5. *Let A be a bounded linear operator such that the following conditions are satisfied:*
a) The system is normal for $t > 0$.
b) G is a convex and weakly compact.
c) If $\overline{t} > 0$, $u \in C(\overline{t})$ and $x_u(\overline{t}) \in G$, then there exists a control $\overline{u}$ such that $x_{\overline{u}}(t) \in G$ with $t \geq \overline{t}$ and $\overline{u}$ is not extremal for any $t > \overline{t}$.

If $u_1 \in C(t_1), u_2 \in C(t_2)$ satisfies the transversality conditions, then $t_1 = t_2 = t^\star$ and $u_2(t) = u_1(t)$ a.e on $[0, t^\star]$. In particular $u_1 = u^\star$ in the unique extremal control.

Proof. Consider first the case $t_1 = t_2$. Since the system is normal, $K(t_1)$ is strictly convex (see [1]). Then, by the transversality condition, there is a hyperplane Π which separates $K(t_1)$ and G.

So, both $x_1(t_1)$ and $x_2(t_2)$ belong to ∂G. Hence $x_1(t_1), x_2(t_2) \in \partial K(t_1) \cap \Pi$. Since $K(t_1)$ is strictly convex, $x_1(t_1) = x_2(t_2)$. Applying again the normality condition, we conclude that

$$u_1(t) = u_2(t) \quad \text{a.e on} \quad [0, t_1].$$

Suppose now $t_1 < t_2$, by the transversality condition there is a hyperplane separating $K(t_2)$ and G, and by hypothesis (c) $K(t_2) \cap G \neq \emptyset$ which is a contradiction. □

3 Applications

In this section we shall consider the optimal control problem governed by the wave equation

$$\begin{cases} y_{tt} - \Delta y = u(t,x), \ x \in \Omega, \, t \in \mathbb{R} \\ y = 0, \quad \text{on} \quad \mathbb{R} \times \partial\Omega, \\ y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x), \quad x \in \Omega, \\ \|u(t,\cdot)\|_{L^2} \leq 1, \quad t \in \mathbb{R}, \end{cases} \tag{3.1}$$

where the distributed control $u \in L^2(0, t_1; L^2(\Omega))$. We wish to steer the initial state (y_0, y_1) to the origin in minimal time. The system (3.1) can be written as an abstract second order equation in the Hilbert space $X = L^2(\Omega)$. But before that, we shall consider the following properties of the operator $-\Delta$.

Let $X = L^2(\Omega)$ and consider the linear unbounded operator $A : D(A) \subset X \to X$ defined by $A\phi = -\Delta\phi$, where

$$D(A) = H_0^1(\Omega) \cap H^2(\Omega). \tag{3.2}$$

The operator A has the following very well known properties, the spectrum of A consists of only eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j \to \infty,$$

each one with multiplicity γ_j equal to the dimension of the corresponding eigenspace.

a) There exists a complete orthonormal set $\{\phi_j\}$ of eigenvectors of A.

b) For all $x \in D(A)$ we have

$$Ax = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < \xi, \phi_{j,k} > \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j \xi, \tag{3.3}$$

where $< \cdot, \cdot >$ is the inner product in X and

$$E_j x = \sum_{k=1}^{\gamma_j} < \xi, \phi_{j,k} > \phi_{j,k}. \tag{3.4}$$

So, $\{E_j\}$ is a family of complete orthogonal projections in X and $x = \sum_{j=1}^{\infty} E_j x, \ x \in X$.

c) $-A$ generates an analytic semigroup $\{e^{-At}\}$ given by

$$e^{-At}x = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j x. \tag{3.5}$$

d) The fractional powered spaces X^r are given by:

$$X^r = D(A^r) = \{x \in X : \sum_{j=1}^{\infty} (\lambda_j)^{2r} \|E_j x\|^2 < \infty\}, \ r \geq 0,$$

with the norm

$$\|x\|_r = \|A^r x\| = \left\{ \sum_{j=1}^{\infty} \lambda_j^{2r} \|E_j x\|^2 \right\}^{1/2}, \quad x \in X^r,$$

and

$$A^r x = \sum_{j=1}^{\infty} \lambda_j^r E_j x. \tag{3.6}$$

Also, for $r \geq 0$ we define $Z_r = X^r \times X$, which is a Hilbert Space with norm given by

$$\left\| \begin{bmatrix} w \\ v \end{bmatrix} \right\|_{Z_r} = \left(\|w\|_r^2 + \|v\|^2 \right)^{\frac{1}{2}}.$$

$$\begin{cases} y'' = -Ay + u(t) \\ y(0) = y_0, \quad y'(0) = y_1 \\ |u(t)| \leq 1, \end{cases} \tag{3.7}$$

where the operator $-A$ is the Laplacian operator defined above.
Using the change of variables $y' = v$, the second order equation (3.7) can be written as a first order system of ordinary differential equations in the Hilbert space $Z = Z_{1/2} = X^{1/2} \times X$ as

$$\begin{cases} z' = \mathcal{A}z + Bu(t), z \in Z \\ z(0) = z_0, \\ |u(t)| \leq 1, \end{cases} \tag{3.8}$$

where

$$z_0 = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}, \quad z = \begin{bmatrix} y \\ v \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I_X \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} 0 & I_X \\ -A & 0 \end{bmatrix}, \tag{3.9}$$

$\mathcal{A}$ is an unbounded linear operator with domain $D(\mathcal{A}) = D(A) \times X$ and $u \in L^2(0, \tau, U)$ with $U = X$.

The proof of the following theorem follows from Theorem 3.1 from [6] by putting $c = 0$ and $d = 1$.

Theorem 3.1. *The operator $\mathcal{A}$ given by (3.9), is the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t\in\mathbb{R}}$ given by*

$$S(t)z = \sum_{j=1}^{\infty} e^{A_j t} P_j z, \quad z \in Z, \quad t \geq 0 \tag{3.10}$$

where $\{P_j\}_{j\geq 1}$ is a complete family of orthogonal projections in the Hilbert space Z given by

$$P_j = diag[E_j, E_j], \quad j \geq 1 \tag{3.11}$$

and

$$R_j = \begin{bmatrix} 0 & 1 \\ -\lambda_j & 0 \end{bmatrix}, \qquad A_j = R_j P_j \quad t \geq 1. \tag{3.12}$$

Note that

$$R_j^* = \begin{bmatrix} 0 & -1 \\ \lambda_j & 0 \end{bmatrix}, \qquad A_j^* = R_j^* P_j, \quad j \geq 1.$$

Moreover $e^{A_j s} = e^{R_j s} P_j$ and the eigenvalues of R_j are $\sqrt{\lambda_j} i$ and $-\sqrt{\lambda_j} i$. Now

$$\begin{aligned} e^{R_j t} &= \{\cos(\sqrt{\lambda_j} t) I + \frac{1}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j} t) R_j\} \\ &= \begin{bmatrix} \cos(\sqrt{\lambda_j} t) & \frac{\sin(\sqrt{\lambda_j} t)}{\sqrt{\lambda_j}} \\ -\lambda_j^{\frac{1}{2}} \sin(\sqrt{\lambda_j} t) & \cos(\sqrt{\lambda_j} t) \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} e^{R_j^* t} &= \{\cos(\sqrt{\lambda_j} t) I + \frac{1}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j} t) R_j^*\} \\ &= \begin{bmatrix} \cos(\sqrt{\lambda_j} t) & -\frac{\sin(\sqrt{\lambda_j} t)}{\sqrt{\lambda_j}} \\ \lambda_j^{\frac{1}{2}} \sin(\sqrt{\lambda_j} t) & \cos(\sqrt{\lambda_j} t) \end{bmatrix}. \end{aligned}$$

Hence, the adjoint equation is

$$\dot{\eta} = -\mathcal{A}^{\star}\eta,$$

where

$$\mathcal{A}^{\star} = \begin{bmatrix} 0 & -I_X \\ A & 0 \end{bmatrix}$$

is infinitesimal generator of strongly continuous group $\{S^{\star}(t)\}_{t\in\mathbb{R}}$ give by

$$S^{\star}(-s)x^{\star} = \sum_{j=1}^{\infty} e^{-R_j^{\star}s} P_j\, x^{\star}, \qquad x^{\star} = \begin{bmatrix} x_1^{\star} \\ x_2^{\star} \end{bmatrix} \in X_{\frac{1}{2}} + X.$$

Therefore, a solution of the adjoint equation such that $\eta(0) = x^{\star}$ is given by

$$\eta(s) = T^{\star}(-s)x^{\star}.$$

Now, we shall apply Theorem (2.1) to find the optimal optimal control:

$$\Big\langle \eta(s), Bu \Big\rangle_{X^{\frac{1}{2}}\times X} = \Big\langle \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}, \begin{bmatrix} 0 \\ u \end{bmatrix} \Big\rangle_{X^{\frac{1}{2}}\times X} = \Big\langle \eta_2(s), u \Big\rangle_X \leq \|\eta_2(s)\|_X.$$

Hence, if we put

$$u(t,\xi) = \begin{cases} \frac{\eta_2(t,\xi)}{\|\eta_2(t,\cdot)\|_X} & \text{if} \quad \|\eta_2(t,\cdot)\|_X \neq 0 \\ 1 & \text{if} \quad \|\eta_2(t,\cdot)\|_X = 0 \end{cases},$$

then

$$\underset{v\in\mathcal{U}}{max}\big\langle \eta(s), Bv \big\rangle \leq \|\eta_2(s)\|_X = \big\langle \eta(s), B\hat{u}(s) \big\rangle, \quad \text{a.e} \quad \text{on} \quad [0, t_1].$$

On the other hand, we known that

$$\eta(s) = T^{\star}(-s)x^{\star} = \sum_{j=1}^{\infty} e^{-R_j^{\star}s} P_j\, x^{\star},$$

therefore

$$\eta_2(s,x) = \sum_{i=1}^{\infty}(-\lambda_j^{\frac{1}{2}} \sin(\sqrt{\lambda_j}s) < \phi_j, x_1^{\star} > \phi_j(x) + \cos(\sqrt{\lambda_j}s) < \phi_j, x_2^{\star} > \phi_j(x)).$$

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