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Remarks on Numerically Positive Line Bundles on Normal Surfaces

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Abstract. Let L be a numerically positive Cartier divisor on a normal complete algebraic surface X. We prove that L is ample if $g(L) \leq 1$.

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1. Introduction

Let k be an algebraically closed field of $\mathrm{char}(k) \geq 0$, which we fix as the ground field throughout the present article. Let X be a normal complete algebraic surface and L a Cartier divisor on X. Then the sectional genus $g(L)$ of L defined by $g(L) = 1 + L(K_X + L)/2$, where K_X is the canonical divisor of X. Since L is Cartier, $g(L)$ is an integer.

A Cartier divisor L on a normal complete algebraic surface X is said to be *numerically positive* or *nup* for shortness if $LC > 0$ for any irreducible curve C on X. It is clear that an ample Cartier divisor is nup. Examples of nup non-ample divisors on smooth projective surfaces were constructed by Mumford (see [3, p. 56]) and by Lanteri-Rondena (see [7, §3]). Nevertheless, it seems that nup non-ample Cartier divisors are very rare. In fact, Lanteri-Rondena [7] proved that a nup divisor L on a smooth complex projective surface with $g(L) \leq 1$ is ample and

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gave some necessarly conditions that surfaces containing nup non-ample Cartier divisors.

In the present article, we study nup Cartier divisors on normal complete algebraic surfaces and attempt to generalize some results of Lanteri-Rondena [7]. In Section 3, we prove the following result which is the main result of the present article.

Theorem 1.1. *Let L be a nup Cartier divisor on a normal complete algebraic surface. Then we have:*

(1) $g(L) \geq 0$.
(2) *If $g(L) \leq 1$, then L is ample.*

As easy consequences of Theorem 1.1, we obtain the following corollaries. Corollary 1.2 is a special case of [5, Theorem] (see also [8]).

Corollary 1.2. *Let X be a normal complete $\mathbb{Q}$-Gorenstein algebraic surface. Then $-K_X$ is ample if and only if it is nup.*

Corollary 1.3. *Let L be a nef Cartier divisor on a normal complete $\mathbb{Q}$-Gorenstein algebraic surface X. Then $K_X + L$ is ample if and only if it is nup.*

Throughout the present article, we employ the following notations:

$\kappa(X)$: The Kodaira dimension of a smooth projective variety X,
$\sim$: the linear equivalence of Cartier divisors,
$\equiv$: the numerically equivalence of $\mathbb{Q}$-divisors.

2. Preliminaries

A semipolarized normal surface is, by definition, a pair (X, L) of a normal complete algebraic surface X and a nef Cartier divisor L on X. A semipolarized normal surface (X, L) is said to be a *scroll* over a smooth curve B if X is a $\mathbb{P}^1$-bundle over B and $L\ell = 1$ for a fiber ℓ of the ruling $p : X \to B$.

Lemma 2.1. *Let (X, L) be a scroll over a smooth curve B of genus g. Then $g(L) = g$.*

Proof. See [9, Lemma 3.2]. □

Now, let X be a normal complete algebraic surface and $\pi : Y \to X$ the minimal resolution of X. Let L be a nup Cartier divisor on X and set $M := \pi^*L$. Then M is nef.

Lemma 2.2. *Let C be an irreducible curve on Y. Then $MC = 0$ if and only if C is π-exceptional.*

Proof. Since L is nup, we have $MC = L\pi_*C > 0$ provided C is not π-exceptional. Hence the assertion follows. □

The following lemma is a special case of [9, Theorem 1].

Lemma 2.3. *With the same notation as above, the pair (Y, M) satisfies one of the following:*

(1) $K_Y + M$ *is nef.*

(2) $(Y, M) \cong (X, L) \cong (\mathbb{P}^2, \mathcal{O}(r))$, $r = 1$ *or* 2.

(3) (Y, M) *is a scroll over a smooth curve.*

Proof. By Lemma 2.2, $(K_Y + M)\ell \geq 0$ for any (-1)-curve ℓ on Y. Hence, by using the same argument as in [9, §2], we know that one of the assertions (2) and (3) holds true if $K_Y + M$ is not nef. □

3. Proofs

In this section, we prove the results stated in the introduction.

Let L be a nup Cartier divisor on a normal complete algebraic surface X. Let $\pi : Y \to X$ be the minimal resolution of X and set $M := \pi^* L$. Then $g(L) = g(M)$. In Lemmas 3.1–3.3 below, we retain this situation.

Lemma 3.1. *If $g(L) \leq 0$, then $g(L) = 0$ and L is ample.*

Proof. Assume that $g(L)(= g(M)) \leq 0$. Then $M(K_Y + M) = 2g(M) - 2 \leq -2$, so that $K_Y + M$ is not nef because M is nef. Hence one of the cases (2) and (3) in Lemma 1.4 takes place. In the case (2), we can easily see that $L = M$ is ample and $g(L) = 0$.

We consider the case (3). If (Y, M) is a scroll over a smooth curve B of genus g, then $0 \leq g = g(M)$ by Lemma 2.1. So $B \cong \mathbb{P}^1$ and $g(M) = 0$. In particular, Y is the Hirzebruch surface $\mathbb{F}_n$ of degree $n(\geq 0)$. Let ℓ be a fiber of the fixed ruling on $\mathbb{F}_n$ and M_n a minimal section of $\mathbb{F}_n$. Then $M \sim M_n + b\ell$ for some integer b. If X is smooth, then $0 < LM_n = b - n$. So $b > n$ and hence L is ample by [4, Proposition V.2.20]. We assume that X is not smooth. Then X is the rational cone obtained from $\mathbb{F}_n$ ($n \geq 2$) by contracting the minimal section M_n. Since $MM_n = 0$, $b = n$ and so $L^2 = M^2 = (M_n + n\ell)^2 = n > 0$. Hence L is ample by the Nakai-Moishezon criterion. □

Lemma 3.2. *If $g(L) = 1$ and $K_Y + M$ is not nef, then L is ample.*

Proof. By Lemmas 2.1 and 2.3, (Y, M) is a scroll over a smooth elliptic curve B. Then $Y \cong \mathbb{P}_B(\mathcal{E})$, where $\mathcal{E}$ is a normalized rank two vector bundle on B. Set $e := -\deg(\det \mathcal{E})$. Then $e \geq -1$ by [4, p. 384]. Let ℓ be a fiber of the ruling $p : Y \to B$ and C_0 a minimal section. Then $M \equiv C_0 + b\ell$ for some integer b. Since M is nef, it follows from [9, Lemmas 1.4 and 1.5] that $b \geq e$ (resp. $b \geq 0$) if $e \geq 0$ (resp. $e = -1$). So, if $e = -1$ then $M \equiv C_0 + b\ell$ is ample by [4, Proposition V.2.21].

Assume that $e \geq 0$. If $b > e$, then $M \equiv C_0 + b\ell$ is ample by [4, Proposition V.2.20]. So we may assume that $b = e$. If $e = 0$ then $MC_0 = (C_0)^2 = 0$, which contradicts Lemma 2.2. So, $e > 0$. Then $L^2 = M^2 = (C_0 + e\ell)^2 = e > 0$ and hence L is ample by the Nakai-Moishezon criterion. □

Lemma 3.3. *If $g(L) = 1$ and $K_Y + M$ is nef, then L is ample.*

Proof. By the Nakai-Moishezon criterion, it suffices to show that $L^2 > 0$. Suppose that $L^2 (= M^2) = 0$. Then $K_Y M = 0$ because $g(L) = 1$. Since $K_Y + M$ is nef, we have $0 \leq (K_Y + M)^2 = (K_Y)^2$. If $(K_Y)^2 > 0$, then $M \equiv 0$ by the Hodge index theorem. This contradicts the assumption that L is nup. Hence $(K_Y)^2 = 0$.

We consider the following two cases separately.

Case 1: K_Y is not nef. Since $(K_Y)^2 = 0$, one of the following holds by [11, Theorem 2.1].

(i) Y is a $\mathbb{P}^1$-bundle over a smooth elliptic curve B.

(ii) Y contains (-1)-curves.

We shall consider the above two subcases separately.

Subcase 1-(i). There exists a normalized rank two vector bundle $\mathcal{E}$ on B such that $Y \cong \mathbb{P}_B(\mathcal{E})$. Set $e := -\deg(\det \mathcal{E})$. Let C_0 be a minimal section and ℓ a fiber of the ruling $p : Y \to B$. Then $K_Y \equiv -2C_0 - e\ell$. Now we write $M \equiv aC_0 + b\ell$. Then $K_Y + M \equiv (a-2)C_0 + (b-e)\ell$. Since $K_Y + M$ is nef, we have $a \geq 2$ by [9, Lemmas 1.4 and 1.5]. Moreover, $0 = M^2 = a(2b - ae)$, and so $2b = ae$. If $e \geq 0$, then $b \geq ae \geq 0$ because M is nef. Hence $b = e = 0$. However, this contradicts Lemma 2.2 because $MC_0 = (C_0)^2 = 0$. Assume that $e = -1$. By [10, Theorem 4], there exists an elliptic fibration $f : Y \to \mathbb{P}^1$ onto $\mathbb{P}^1$. By [1, Theorem 2], we have $K_Y \equiv \alpha F$, where $\alpha < 0$ and F is a fiber of f. This also contradicts Lemma 2.2 because $MF = (1/\alpha)MK_Y = 0$ and $F^2 = 0$. Thus we know that Subcase 1-(i) does not take place.

Subcase 1-(ii). We prove the following claim.

Claim. *Y is a rational surface.*

Proof. Since $(K_Y)^2 = 0$ and Y is not relatively minimal, Y is either a rational surface or a surface of general type. If Y is of general type, then K_Y is numerically equivalent to an effective $\mathbb{Q}$-divisor H. Since $HM = K_Y M = 0$, $(K_Y)^2 = H^2 < 0$ by Lemma 2.2. This is a contradiction. □

Since $MK_Y = M^2 = (K_Y)^2 = 0$ and K_Y is not nef, it follows from Lemma 2.2 that $h^0(Y, mM) = h^2(Y, mM) = 0$ for any integer $m > 0$. So $\chi(mM) = -h^1(Y, mM) \leq 0$. On the other hand, by the Riemann-Roch theorem and the claim as above, we have

$$\chi(mM) = \frac{1}{2} mM(mM - K_Y) + \chi(\mathcal{O}_Y) = \chi(\mathcal{O}_Y) = 1.$$

This is a contradiction. Thus we know that Subcase 1-(ii) does not take place.

Case 2: K_Y is nef. Since $(K_Y)^2 = 0$, Y is a minimal surface of $\kappa(Y) = 0$ or 1.

Subcase 2-(i): $\kappa(Y) = 1$ (cf. Subcase 1-(i)). By the classification theory of smooth projective surfaces in any characteristic (cf. [12]), there exists an elliptic or quasi-elliptic fibration $f : Y \to B$ onto a smooth projective curve B. By [1, Theorem 2], we have $K_Y \equiv \alpha F$, where $\alpha > 0$ and F is a general fiber of f. Then $MF =$

$(1/\alpha)MK_Y = 0$ and $F^2 = 0$. This contradicts Lemma 2.2. Thus we know that Subcase 2-(i) does not take place.

Subcase 2-(ii): $\kappa(Y) = 0$. By using the same argument as in Subcase 1-(ii) and by [1, p. 25], we may assume that $\chi(\mathcal{O}_Y) = 0$. Namely, Y is an abelian, hyperelliptic, or quasi-hyperelliptic surface (cf. [12], [1] and [2]). Then Y contains no irreducible curves with negative self-intersection number. So $X \cong Y$. However, this contradicts Lemma 3.4 below. Thus we know that Subcase 2-(ii) does not take place. $\square$

The following lemma is proved in [7, Lemma (2.2)] in the case $\mathrm{char}(k) = 0$. Almost all the part of the proof of [7, Lemma (2.2)] works in arbitrary characteristic.

Lemma 3.4. *Let S be a minimal smooth projective surface of $\kappa(S) = 0$. Then every nup divisor on S is ample.*

Proof. If $\chi(\mathcal{O}_S) > 0$, then the assertion can be verified by using the same argument as Subcase 1-(ii) in the proof of Lemma 3.3. So we may assume further that $\chi(\mathcal{O}_S) = 0$, i.e., S is an abelian, hyperelliptic, or quasi-hyperelliptic surface.

If S is an abelian surface, then the assertion can be verified by using the same argument as in the proof of [14, Proposition 1.4]. For the reader's convenience, we reproduce the proof. Let L be a nup divisor on S. Given a point $x \in S$, $T_x : S \to S$ denotes translation by x according to the group law. Define

$$\phi_L : S \to \mathrm{Pic}(S)$$

as $\phi_L(x) := T_x^*(L) \otimes L^{-1}$, and write $K(L) = \mathrm{Ker}\phi_L$. The connected component Z of $K(L)$ passing through the origin is a subgroup scheme. Since L is nup, we know that $\dim Z = 0$, $K(L)$ is finite, and $\deg \phi_L > 0$. Since $\chi(L) = (1/2)L^2$ by the Riemann-Roch theorem, and also $\chi(L)^2 = \deg \phi_L$ (cf. [13, p. 150]), it follows that $L^2 \neq 0$. Hence L is ample.

We treat the case where S is a hyperelliptic or quasi-hyperelliptic surface. Then the Albanese variety $\mathrm{Alb}(S)$ of S is an elliptic curve and the Albanese map

$$f : S \to \mathrm{Alb}(S)$$

is an elliptic or quasi-elliptic fibration (see [1, Proposition]). Moreover, there exists a second structure $g : S \to \mathbb{P}^1$ of S as an elliptic surface over $\mathbb{P}^1$ by [1, Theorem 3]. Let F (resp. G) be a fiber of f (resp. g). Then $FG > 0$. Since $\mathrm{b}_2(S) = 2$, $\{F, G\}$ is a basis of $\mathrm{NS}(S) \otimes \mathbb{Q}$. Let L be a nup divisor on S. Then $L \equiv aF + bG$ for some $a, b \in \mathbb{Q}$. Since L is nup, we have $a = LG/FG > 0$ and $b = LF/FG > 0$. So, $L^2 = 2abFG > 0$ and hence L is ample. $\square$

Theorem 1.1 is thus verified.

Proof of Corollary 1.2. Let n be a positive integer such that nK_X is Cartier. Assume that $-K_X$ is nup non-ample. Then $g(-nK_X) = 1$ since $(-K_X)^2 = 0$. This contradicts Theorem 1.1. $\square$

Proof of Corollary 1.3. Assume that K_X+L is nup non-ample. Then $(K_X+L)^2 = 0$. Since $(K_X+L)L \geq 0$, we have $(K_X+L)K_X \leq 0$. Then,

$$g(K_X+L) = \frac{1}{2}(K_X+L)K_X + 1 \leq 1.$$

This contradicts Theorem 1.1. □

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