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Second Order Parallel Tensors on Generalized Sasakian Space Forms and Semi Parallel Hypersurfaces in Sasakian Space Forms

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Abstract. In this paper, we show that a second order parallel symmetric tensor in a generalized Sasakian space form is proportional to the metric tensor and we deduce that there is no semi parallel hypersurface in a Sasakian space form.

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1. Introduction

In 1926, Levy [4] has proved that a second order parallel symmetric non singular tensor in real space forms is proportional to the metric tensor. In 1989, Sharma [8] has proved that a second order parallel tensor in a Kähler space of constant holomorphic sectional curvature is a linear combination with constant coefficients of the Kählerian metric and the fundamental 2-form. Recently, Das [6] has established the same result for an α-Sasakian manifold ($\alpha \in \mathbb{R}_0$). In this paper we generalize this result to generalized Sasakian space form and we prove that there is no semi parallel hypersurface in a Sasakian space form.

2. Preliminaries

Let M denote an n-dimensional Riemannian manifold with its metric tensor g and Levi-Civita connection $\widetilde{\nabla}$. Let $\tilde{R}$ denote the Riemann curvature tensor of M. If B is a $(0,2)$ tensor which is parallel with respect to $\widetilde{\nabla}$ then we can show that

$$B\left(\tilde{R}\left(X,Y\right)Z,W\right)+B\left(Z,\tilde{R}\left(X,Y\right)W\right)=0. \qquad (1)$$

Das has proved that

Theorem 2.1. [6] *On an $\alpha-K$ contact manifold $(\alpha \in R_0)$ a second order symmetric parallel tensor is a constant multiple of the associated positive definite Riemannian metric tensor.*

The first purpose of this paper is to present a similar result for a generalized Sasakian space form. Let (M^{2n+1}, g) be a $2n+1$ dimensional differentiable manifold and let (ϕ, ξ, η) be tensor fields of type $(1,1)$, $(1,0)$ and $(0,1)$ respectively on M, such that

$$\eta\left(\xi\right)=1 \qquad \phi^2=-I+\xi\otimes\eta$$

which implies

$$\eta\circ\phi=0 \qquad \phi\left(\xi\right)=0 \qquad rank\left(\phi\right)=2n.$$

If M admits a Riemannian metric g, such that

$$\begin{aligned} g\left(\phi X,\phi Y\right) &= g\left(X,Y\right)-\eta\left(X\right)\eta\left(Y\right) \\ g\left(X,\xi\right) &= \eta(X) \end{aligned}$$

then (ϕ, ξ, η, g) is called an almost contact metric structure on M. If moreover

$$\left(\tilde{\nabla}_X\phi\right)Y=g\left(X,Y\right)\xi-\eta\left(Y\right)X$$

where $\tilde{\nabla}$ denotes the Riemannian connection of g, then (M, ϕ, ξ, η, g) is called a Sasakian manifold [10].

The sectional curvature of the plane section spanned by the unit tangent vector field X orthogonal to ξ and ϕX is called a ϕ-sectional curvature. If M has a constant ϕ-sectional curvature c, then M is called a Sasakian space form and denoted by $M^{2n+1}\left(c\right)$. The Riemannian curvature of a Sasakian space form is given by the following formula

$$\begin{aligned} R\left(X,Y\right)Z &= \frac{c+3}{4}\left[g(Y,Z)X-g(X,Z)Y\right] \\ &+\frac{c-1}{4}\left[\eta\left(X\right)\eta\left(Z\right)Y-\eta\left(Y\right)\eta\left(Z\right)X\right] \\ &+\frac{c-1}{4}[g(X,Z)\eta\left(Y\right)\xi-g(Y,Z)\eta\left(X\right)\xi+g(Z,\phi Y)\phi X \\ &-g(Z,\phi X)\phi Y+2g(X,\phi Y)\phi Z]. \end{aligned}$$

Example 2.2. [10] We consider $\mathbb{R}^{2n+1}$ with the coordinates (x^i, y^i, z), $i=1,\dots,n$ and its usual contact form $\eta = \frac{1}{2}(dz - \sum_{i=1}^{n} y^i dx^i)$. The characteristic field ξ is given by $\xi = 2\frac{\partial}{\partial z}$, the tensor field ϕ is given by the matrix $\begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y^j & 0 \end{pmatrix}$ and the Riemannian metric $g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{n} (dx^i)^2 + (dy^i)^2$ is an associated metric for η. In this case $\mathbb{R}^{2n+1}$ is a Sasakian space form with ϕ-sectional curvature $c = -3$ denoted by $\mathbb{R}^{2n+1}(-3)$.

Given an almost contact metric (M, ϕ, ξ, η, g), M is called generalized Sasakian space form if there exist three functions f_1, f_2 and f_3 such that the Riemannian curvature tensor is given by the following formula

$$\begin{aligned} R(X,Y)Z \;=\; & f_1[g(Y,Z)X - g(X,Z)Y] + f_2[g(X,\phi Z)\phi Y \\ & -g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z] + f_3[\eta(X)\eta(Z)Y \\ & -\eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi]. \end{aligned} \tag{2}$$

In such a case, we will write $M(f_1, f_2, f_3)$. This kind of manifold appears as natural generalization of the Sasakian space form by taking:

$$f_1 = \frac{c+3}{4} \text{ and } f_2 = f_3 = \frac{c-1}{4}.$$

The ϕ-sectional curvature of a generalized Sasakian space form $M(f_1, f_2, f_3)$ is $f_1 + 3f_2$ [9].

Let N^{2n} be an immersed hypersurface of $M^{2n+1}(f_1, f_2, f_3)$. We denote the Levi-Civita connection of M by $\widetilde{\nabla}$ and the Levi-Civita connection of N by ∇. Then we have the formulas of Gauss and Weingarten

$$\begin{aligned} \widetilde{\nabla}_X Y &= \nabla_X Y + h(X,Y)r \\ \widetilde{\nabla}_X r &= -SX. \end{aligned}$$

X and Y are tangent vector fields, r a unit normal vector normal to N and h the second fundamental form of N and S the shape operator of N. Note that h and S are related by $h(X,Y) = g(SX,Y)$. In a hypersurface the $(0,4)$ tensor field $\tilde{R}.h$ is defined by

$$\tilde{R}.h(X,Y,Z,W) = -h\left(\tilde{R}(X,Y)Z, W\right) - h\left(Z, \tilde{R}(X,Y)W\right).$$

In [2] J. Deprez has defined semi parallel immersions which satisfy the condition $\tilde{R}.h = 0$. The authors F. Dillen, J. Fastenakels, S. Haesen, G. Van Der Veken and L. Verstraelen gave a geometrical interpretation of semi parallel submanifolds.

Proposition 2.3. [16] *A submanifold N of M is semi parallel if, $\forall p \in M$, the normal vectors $h(u,v)^{*\perp}$ and $h(u^*, v^*)$ coincide for all $u, v \in T_P M$ and for every coordinate parallelogram in M, up to second order. Where u^* and v^* are the parallel transport of u and v with respect to ∇ and $h(u,v)^{*\perp}$ is the parallel transport of $h(u,v)$ with respect to the normal connection $\nabla^{\perp}$.*

We have proved in [3] that

Theorem 2.4. *There is not a parallel connected hypersurface in a Sasakian space form* $M^{2n+1}(c)$ *with* $n \geq 2$ *and* $c \neq 1$.

The Ricci tensor is given by Kim [13]

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)\, g(X,Y) - (3f_2 + (2n-1) f_3)\, \eta(X)\, \eta(Y).$$

3. Main results

Theorem 3.1. *In a generalized Sasakian space form* $M(f_1, f_2, f_3)$ *with* $f_1 \neq f_3$, *a second order parallel symmetric tensor* B *is a constant multiple of the associated positive definite metric tensor.*

Proof. If B is parallel $\left(\widetilde{\nabla} B = 0\right)$, it follows that

$$B\left(\tilde{R}(X,Y)Z, W\right) + B\left(Z, \tilde{R}(X,Y)W\right) = 0 \tag{3}$$

for X, Y, Z and W vector fields on M.

By taking $Y = \xi$ and $Z = W$ and using equation (2), we have

$$(f_1 - f_3)(\eta(Z) B(X,Z) - g(X,Z) B(\xi,Z) + \eta(Z) B(Z,X) - g(X,Z) B(Z,\xi)) = 0$$

since $f_1 \neq f_3$ and B is symmetric we have

$$\eta(Z) B(X,Z) = g(X,Z) B(Z,\xi)$$

so

$$B(Z,\xi) = \eta(Z) B(\xi,\xi)$$

which implies that

$$\eta(Z)(B(X,Z) - g(X,Z) B(\xi,\xi)) = 0.$$

If $\eta(Z) \neq 0$, we have

$$B(X,Z) = g(X,Z) B(\xi,\xi). \tag{4}$$

If $\eta(Z) = 0$, so $B(\xi, Z) = 0$ and by substituting $Y = W = \xi$ in equation (4) we get the above equation. □

Corollary 3.2. *If the Ricci tensor of a generalized Sasakian space form* $M(f_1, f_2, f_3)$ *with* $f_1 \neq f_3$ *is parallel, then* M *is Einstein.*

We also have

Theorem 3.3. *There are no semi parallel hypersurfaces in a Sasakian space form* $M^{2n+1}(c)$ *with* $c \neq 1$ *and* $n \geq 2$.

Proof. If N is a semi parallel hypersurface and h the second fundamental form of N, we have:

$$-h\left(\tilde{R}(X,Y)Z,W\right)-h\left(Z,\tilde{R}\left(X,Y\right)W\right)=0$$

by using the same argument as in Theorem 3.1 we deduce that

$$h=\lambda g$$

where λ is constant. Consequently

$$\tilde{\nabla}h=0$$

which contradicts Theorem 2.3. □

Corollary 3.4. *There are no semi parallel hypersurfaces in* $\mathbb{R}^{2n+1}(-3)$ *with* $n\geq 2$.

Remark 3.5. Let us consider the $(2n+1)$-dimensional unit sphere, i.e., $S^{2n+1}=\{z\in\mathbb{C}^{n+1}:|z|=1\}$. Any point z of S^{2n+1} can be identified to $(x^1,\ldots,x^n,y^1,\ldots,y^n)\in\mathbb{R}^{2n+2}$. We put $Jz=(-y^1,\ldots,-y^n,x^1,\ldots,x^n)$, where J is the usual complex structure on $\mathbb{C}^{n+1}$. We define the characteristic vector field ξ, the 1-form η and the $(1,1)$ tensor ϕ by

$$\xi=-Jz,\ \ \eta(X)=g(X,\xi)\ \ \text{and}\ \phi=s\circ J$$

where g is the induced metric of $\mathbb{C}^{n+1}$ on S^{2n+1} and s is the orthogonal projection of $T_x\mathbb{C}^{n+1}$ on T_xS^{2n+1}. So, S^{2n+1} is a Sasakian space form with ϕ-sectional curvature equal to 1.

Now we consider the Clifford hypersurface $M_{p,q}$ defined by

$$M_{p,q}=S^{2p+1}\left(\sqrt{\frac{p}{2n}}\right)\times S^{2q+1}\left(\sqrt{\frac{q}{2n}}\right),\quad p+q=n-1.$$

Then, $M_{p,q}$ is a minimal hypersurface of S^{2n+1} tangent to the structure field ξ of S^{2n+1} and $M_{p,q}$ has a parallel second fundamental form. Therefore the assumption in Theorem 2.4 and Theorem 3.3 on the ϕ-sectional curvature $c\neq 1$ of the ambient space is essential.

Remark 3.6. The condition $n\geq 2$ in Theorem 2.4 and Theorem 3.3 is also essential, there exist parallel surfaces for $n=1$ [14] and [15].

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