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Minimum-Area Axially Symmetric Convex Bodies containing a Triangle and its Measure of Axial Symmetry

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Abstract. Denote by K_m the mirror image of a planar convex body K in a straight line m. It is easy to show that $K_m^* = \mathrm{conv}(K \cup K_m)$ is the smallest (by inclusion) convex body whose axis of symmetry is m and which contains K. The ratio $\mathrm{axs}(K)$ of the area of K to the minimum area of K_m^* is a measure of axial symmetry of K. A question is how to find a line m in order to guarantee that K_m^* be of the smallest possible area. A related task is to estimate $\mathrm{axs}(K)$ for the family of all convex bodies K. We give solutions for the classes of triangles, right-angled triangles and acute triangles. In particular, we prove that $\mathrm{axs}(T) > \frac{1}{2}\sqrt{2}$ for every triangle T, and that this estimate cannot be improved in general.

MSC 2000: 52A10, 52A38
Keywords: triangle, convex body, axially symmetric body, mirror image, area, measure of axial symmetry

1. Introduction

Denote by E^2 the Euclidean plane, and let $K \subset E^2$ be a *convex body*, i.e., a compact, convex set with interior points. The mirror image of K in a straight line m is denoted by K_m. We call m the *mirror line* and put $K^*_m = \mathrm{conv}(K \cup K_m)$. It is easy to show that K^*_m is the smallest (by inclusion) convex body containing K whose axis of symmetry is m. We omit proofs of the following claims.

Claim 1. *Let $K \subset E^2$ be a convex body. If the position of a straight line m changes continuously, then* $\mathrm{area}(K^*_m)$ *changes continuously.*

Claim 2. *Let $K \subset E^2$ be a convex body and let m and n be two parallel straight lines such that only m passes through K. Then* $\mathrm{area}(K^*_m) < \mathrm{area}(K^*_n)$.

By the above claims and by compactness arguments we conclude that the infimum of the area of K^*_m over all straight lines m is attained. So using the term minimum instead of infimum is correct here (the same remark concerns many other places of the paper where we consider compact families of straight lines m). If $\mathrm{area}(K^*_m)$ attains the minimum value for a line m, we call it *a best mirror line of K*. The number

$$\mathrm{axs}(K) = \frac{\mathrm{area}(K)}{\min\limits_{m} \mathrm{area}(K^*_m)}$$

is the *measure of axial symmetry of K*, mainly studied in this paper.

We conjecture that $\mathrm{axs}(K) > \frac{1}{2}\sqrt{2}$ for every convex body $K \subset E^2$ and that this value cannot be improved. In [10] it is shown that for every K we have $\mathrm{axs}(K) \geq \frac{16}{31}$. The papers [1] and [2] refer to the related question of finding an axially symmetric set of possibly small area containing K; the approach is algorithmic. Miscellaneous measures of axial symmetry are discussed in [1–7], [9] and [11]. We also refer to the well known survey article [8], concerning mostly the measures of central symmetry of convex bodies. Moreover, Part 4.2 of the survey article [9] considers measures of symmetry of convex bodies, and in particular their measures of axial symmetry.

Denote by T an arbitrary triangle. In Sections 2–6 we find the minima of the area of T^*_m when m belongs to an arbitrary pencil of parallel lines and to pencils of lines which give the angles of T. It allows us to find the best mirror line (or lines) for T, and we obtain a formula for the minimum area of T^*_m. In Section 7 we present a formula for $\mathrm{axs}(T)$. Next we prove that $\mathrm{axs}(T) > \frac{1}{2}\sqrt{2}$ for every triangle T and that $\mathrm{axs}(T) \geq \frac{1}{2}\sqrt[3]{4}$ for acute and right-angled triangles T. We show that both the estimates cannot be improved.

Let $T = abc$ be a triangle and let $|bc| \leq |ac| \leq |ab|$. We put $A = \angle bac$, $B = \angle cba$, and $C = \angle acb$. The measures of A, B, and C are denoted by α, β, and γ, respectively. For every other angle the same symbol denotes the angle and its measure, and instead of "measure of angle" we simply say "angle" as well. Clearly, $\alpha \leq \beta \leq \gamma$. In order to shorten considerations, right-angled triangles are treated as obtuse triangles.

By $\mathcal{A}$ (respectively, $\mathcal{B}$ and $\mathcal{C}$) we mean the pencil of straight lines through a (respectively, through b and through c) and through a different point of T (over all such points).

Let D be any of the angles from amongst the angles A, B, C of T. Denote by $\mathrm{bi}(D)$ the straight line containing the bisectrix of D and by $\mathrm{per}(D)$ the straight line through the vertex of the angle D perpendicular to the opposite side of T.

2. Translating the mirror line m in order to minimize area(T_m^*)

Proposition 1. *Consider a triangle T and the pencil of all straight lines m parallel to a fixed straight line. The family of the straight lines from this pencil for which* area(T_m^*) *is minimal forms a strip. A straight line from $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ is in this strip.*

Proof. Having in mind Claim 2, we consider only the lines m from our pencil which have nonempty intersection with T. It is easy to show that there is exactly one line $l \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ in the pencil. Let us assume that $l \in \mathcal{C}$. If $l \in \mathcal{A}$ or $l \in \mathcal{B}$, the considerations are similar (what is more, Case 1 is then impossible).

If p is a point, then by p_m denote the mirror image of p in the line m.

For the need of further Subcases 1.2 and 2.2 denote by a' the intersection of the straight line through a perpendicular to m with the line containing bc, and for the need of the remaining subcases, by b' the intersection of the line through b perpendicular to m with the line containing ac. Let a'' be the midpoint of aa', and b'' of bb'.

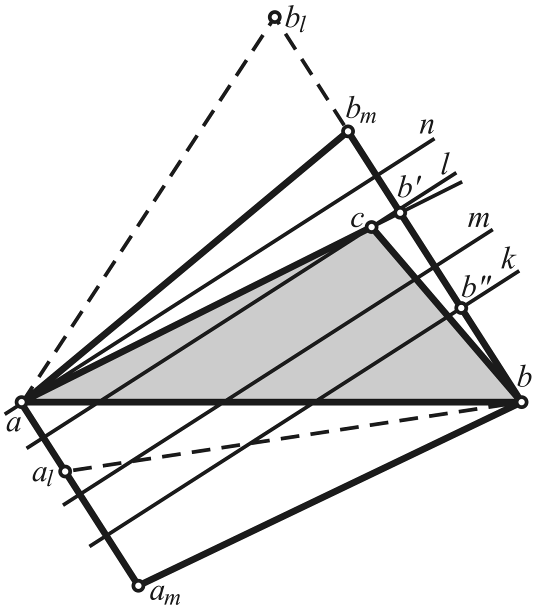

Figure 1.

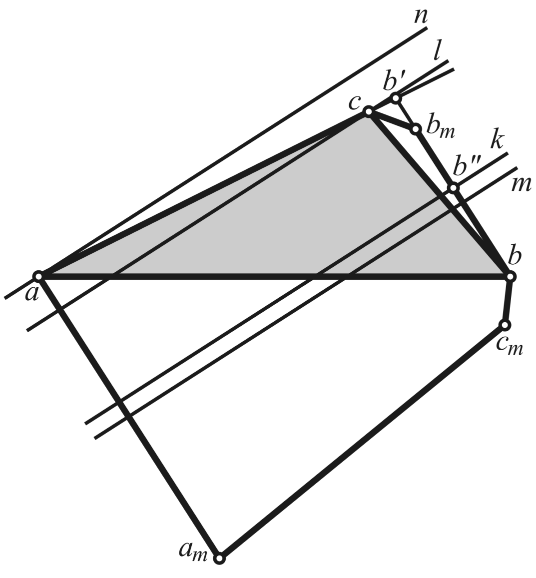

Figure 2.

Below by the angle between ca (respectively, between cb) and l we mean the angle $\angle acd$ (respectively, $\angle dcb$), where d is the intersection point of l with ab.

Case 1. The straight line through c perpendicular to m intersects the segment ab.

Subcase 1.1. The angle between ca and l is smaller than that between cb and l.

In this subcase denote by $\mathcal{S}$ the strip of all straight lines between the following two straight lines from our pencil: the line n through a and the line k through b''.

Observe that for $m \in \mathcal{S}$ we have $T_m^* = aa_mbb_m$ (see Figure 1) and that $|aa_m| + |bb_m|$ is constant here. So area(T_m^*) is constant here. In particular, it equals area(T_l^*).

If $m \notin \mathcal{S}$, then m and l are on the opposite sides of k and thus $T_m^* = aa_mc_mbb_mc$ (see Figure 2). Imagine T_m^* as the union of two trapezia: the first with bases aa_m and cc_m, and the second with bases cc_m and b_mb. Hence area$(T_m^*) = \frac{1}{2}(|aa_m| + |cc_m|)h_1 + \frac{1}{2}(|cc_m| + |b_mb|)h_2$, where h_1 and h_2 are heights of the first and the second trapezium, respectively. If the distance of m from $\mathcal{S}$ increases, $|aa_m| + |cc_m|$ increases and $|cc_m| + |b_mb|$ is constant. This and the fact that h_1 and h_2 are constant imply that area(T_m^*) increases. So area$(T_l^*) <$ area(T_m^*).

Subcase 1.2. The angle between ca and l is at least the angle between cb and l.

Clearly, the first angle must be greater than the second. We repeat the considerations of Subcase 1.1 taking this time in the part of $\mathcal{S}$ the pencil of straight lines between the following two lines parallel to l: the line through b and the line k through a''.

Case 2. The line through c perpendicular to m does not intersect ab.

Subcase 2.1. The angle between ca and l is smaller than the angle between cb and l.

By $\mathcal{S}$ we mean the pencil of lines between l and the parallel line k through b''.

First assume that m and l are weakly on one side of k (see Figure 3). Since b_m does not belong to the interior of T, we conclude that T_m^* is the hexagon $aa_mbc_mcb_m$ when $m \in \mathcal{S}$ and the hexagon $aa_mbcc_mb_m$ in the opposite case. The hexagon T_m^* is the union of two trapezia: the first with bases aa_m and bb_m, and the second with bases b_mb and c_mc.

Consequently, area$(T_m^*) = \frac{1}{2}(|aa_m| + |bb_m|)h_1 + \frac{1}{2}(|bb_m| + |cc_m|)h_2$, where h_1 is the height of the first trapezium and h_2 is the height of the second. Observe that $|bb_m| + |cc_m|$ and $|aa_m| + |bb_m|$ are constant for $m \in \mathcal{S}$. Moreover, if the distance of m from $\mathcal{S}$ increases,then $|aa_m| + |bb_m|$ does not change and $|bb_m| + |cc_m|$ increases. Thus from the fact that h_1 and h_2 are constant we obtain that area(T_m^*) is constant for $m \in \mathcal{S}$ (since $l \in \mathcal{S}$, it is equal to area(T_l^*)) and larger if $m \notin \mathcal{S}$.

Now assume that m and l are strictly on the opposite sides of k (so $m \notin \mathcal{S}$). Then $T_m^* = aa_mc_mc$ (see Figure 4). Clearly, $|aa_m|$ and $|cc_m|$ grow as the distance of m from $\mathcal{S}$ grows. Since the height of T_m^* does not change, we see that area(T_m^*) grows. This, the preceding paragraph, and $k \in \mathcal{S}$ give area$(T_l^*) =$ area$(T_k^*) <$ area(T_m^*) for every $m \notin \mathcal{S}$.

Subcase 2.2. The angle between ca and the straight line l is at least the angle between cb and l and simultaneously at most $\frac{\pi}{2} - \alpha$.

Now the strip $\mathcal{S}$ is the pencil of straight lines between l and the parallel straight line k through a''. If $k = l$ (that is, if l is the bisectrix of C), then only l belongs to

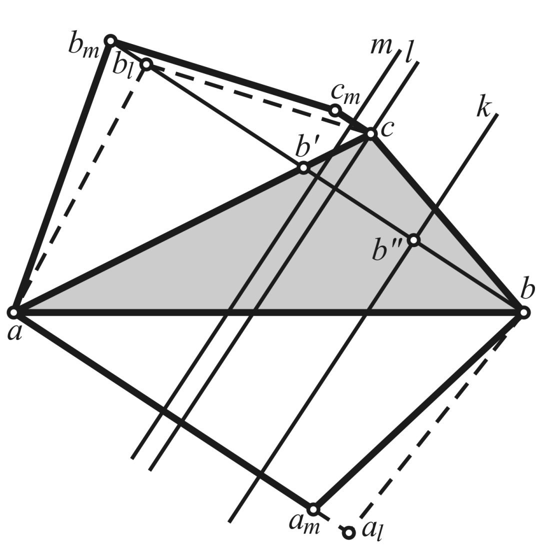

Figure 3.

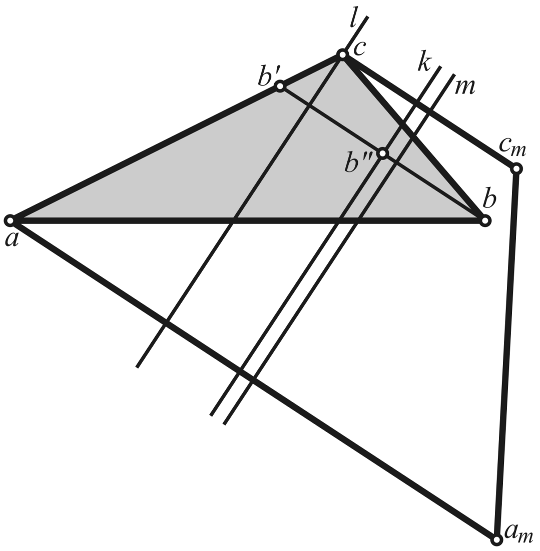

Figure 4.

$\mathcal{S}$. For this special situation the proof is left to the reader (hint: look to Figures 5 and 6 with $k = l$). Since now assume that $k \neq l$ (that is, l is not the bisectrix of C).

First assume that m and l are weakly on one side of k (observe that this side is the one which does not contain a). Hence T_m^* is the trapezium aa_mc_mc if $m \notin \mathcal{S}$ (see Figure 5), and aa_mcc_m if $m \in \mathcal{S}$. Thus $\text{area}(T_m^*) = \frac{1}{2}(|aa_m| + |cc_m|)h$, where h stands for the height of this trapezium. Observe that $|aa_m| + |cc_m|$ is constant for $m \in \mathcal{S}$ and that it grows when the distance of m from $\mathcal{S}$ grows. Moreover, h is constant. Thus $\text{area}(T_m^*)$ is constant for $m \in \mathcal{S}$, and it grows when the distance of m from $\mathcal{S}$ grows. So from $l \in \mathcal{S}$ we obtain that $\text{area}(T_l^*) \leq \text{area}(T_m^*)$ with equality if and only if $m \in \mathcal{S}$.

Now assume that m and l are strictly on the opposite sides of k (so $m \notin \mathcal{S}$). Clearly, $T_m^* = aa_mbcc_mb_m$ is the union of the trapezia aa_mbb_m and b_mbcc_m (see Figure 6). Thus $\text{area}(T_m^*) = \frac{1}{2}(|aa_m| + |bb_m|)h_1 + \frac{1}{2}(|bb_m| + |cc_m|)h_2$, where h_1 denotes the height of the first trapezium and h_2 denotes the height of the second. If the distance of m from $\mathcal{S}$ increases, $|aa_m|+|bb_m|$ does not change and $|bb_m|+|cc_m|$ increases. Hence the first trapezium has constant area, and the area of the second

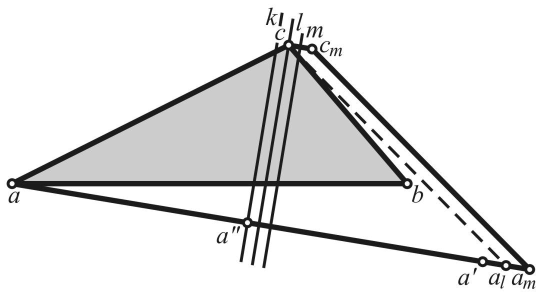

Figure 5.

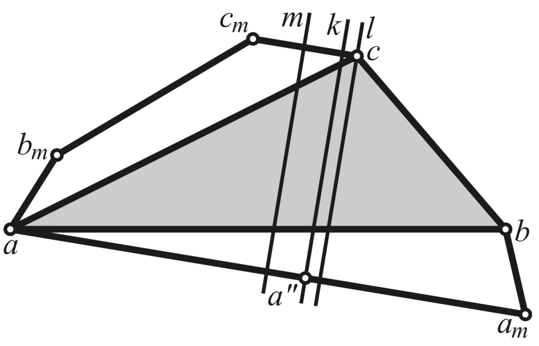

Figure 6.

one grows. Consequently, area(T^*_m) grows and thus area(T^*_l) < area(T^*_m).
Subcase 2.3. The angle between the side ca and the straight line l is over $\frac{\pi}{2} - \alpha$. We repeat the considerations of Subcase 2.1 exchanging always the letters a and b (so, for instance, exchanging also the symbols a_m and b_m). □

Corollary 1. *For every triangle T we have* $\min_m \text{area}(T^*_m) = \min_{m \in \mathcal{A}\cup\mathcal{B}\cup\mathcal{C}} \text{area}(T^*_m)$.

Remark 1. Let P be a convex n-gon. An algorithm for finding a mirror line m parallel to a given line which minimizes area(P^*_m) is presented in [2]. It asks to choose a line m from amongst a finite number of specific straight lines which minimizes area(P^*_m). Our proof of Proposition 1 shows all such best straight lines for a triangle T. What is more, without evaluating area(T^*_m). But as far as we see, our approach works only for $n = 3$.

3. Minimizing area(T^*_m) for m from the pencil $\mathcal{A}$

Proposition 2. *For every triangle T we have* $\min_{m \in \mathcal{A}} \text{area}(T^*_m) = \frac{\sin(\alpha+\beta)}{\sin\beta} \cdot \text{area}(T)$. *This minimum is attained only for $m = \text{bi}(A)$.*

Proof. In the proof, besides the last paragraph, for simplicity we assume that $|ab| = 1$. Let $m \in \mathcal{A}$. The angle between ab and m (smaller than $\frac{\pi}{2}$) is denoted by ψ.
Case 1. (for all T): $\psi \in [0, \frac{\alpha}{2}]$.
Of course, T^*_m is the pentagon ac_mbb_mc (see Figure 7). Clearly, $T^*_m = ac_mb \cup abb_m \cup ab_mc$. Since b_m is symmetric to b with respect to m, we have $\angle bab_m = 2\psi$. This and $|ab| = |ab_m| = 1$ imply that area(abb_m) $= \frac{1}{2}\sin 2\psi$. Obviously, $\angle b_mac = \angle bac - \angle bab_m$. Hence $\angle b_mac = \alpha - 2\psi$. From this, from $|ab_m| = 1$ and from $|ac| = \frac{\sin\beta}{\sin(\alpha+\beta)}$ it follows that area(ab_mc) $= \frac{1}{2}\frac{\sin\beta}{\sin(\alpha+\beta)}\sin(\alpha - 2\psi)$. By axial symmetry, area(ac_mb) has the same value. These facts lead imply that area(T^*_m) equals $f_1(\psi) = \frac{1}{2}\sin 2\psi + \frac{\sin\beta}{\sin(\alpha+\beta)}\sin(\alpha - 2\psi)$. The second derivative of this function is $f_1''(\psi) = -2\sin 2\psi - 4\frac{\sin\beta}{\sin(\alpha+\beta)}\sin(\alpha - 2\psi)$. Since 2ψ and $\alpha - 2\psi$ belong to $[0, \pi]$, we get $f_1''(\psi) < 0$. So $f_1(\psi)$ is concave in the interval $[0, \frac{\alpha}{2}]$.
Case 2. (for all T): $\psi \in [\frac{\alpha}{2}, \frac{\pi}{2} - \beta]$ for acute T, and $\psi \in [\frac{\alpha}{2}, \alpha]$ for obtuse T.
We easily conclude that $T^*_m = abb_m$ (see Figure 8). Hence $\angle bab_m = 2\psi$. So area(T^*_m) $= \frac{1}{2}\sin 2\psi$. The function $f_2(\psi) = \frac{1}{2}\sin 2\psi$ is concave in each of the intervals $[\frac{\alpha}{2}, \frac{\pi}{2} - \beta]$ and $[\frac{\alpha}{2}, \alpha]$ because its second derivative $f_2''(\psi) = -2\sin 2\psi$ is negative in both.
Case 3. (only for acute T): $\psi \in [\frac{\pi}{2} - \beta, \alpha]$.
In this case T^*_m is the pentagon abc_mcb_m (see Figure 9). Observe that $T^*_m = abc_m \cup ac_mc \cup acb_m$. Of course, $\angle bab_m = 2\psi$. This and $\angle cab_m = \angle bab_m - \angle bac$ imply that $\angle cab_m = 2\psi - \alpha$. Since the triangles acb_m and abc_m are symmetric with respect to m, we see that $\angle cab_m = \angle bac_m = 2\psi - \alpha$. This, $\angle c_mac =$

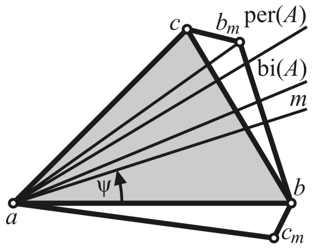

Figure 7.

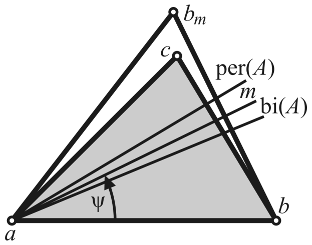

Figure 8.

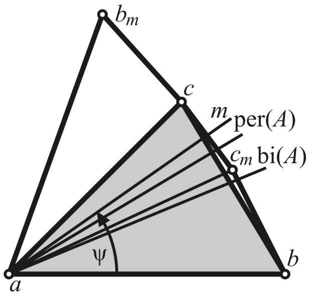

Figure 9.

$\angle bac - \angle bac_m$ and $\angle bac = \alpha$ imply that $\angle c_m ac = 2\alpha - 2\psi$. Clearly, $|ab| = 1$ and $|ac| = \frac{\sin\beta}{\sin(\alpha+\beta)}$. The above considerations show that $\mathrm{area}(T^*_m)$ is given by the function $f_3(\psi) = \frac{1}{2}(\frac{\sin\beta}{\sin(\alpha+\beta)})^2 \sin(2\alpha - 2\psi) + \frac{\sin\beta}{\sin(\alpha+\beta)} \sin(2\psi - \alpha)$. We have $f''_3(\psi) = -2(\frac{\sin\beta}{\sin(\alpha+\beta)})^2 \sin(2\alpha - 2\psi) - 4\frac{\sin\beta}{\sin(\alpha+\beta)} \sin(2\psi - \alpha)$. By the assumption of Case 3 we get $0 \le 2\alpha - 2\psi \le \pi$ and $0 \le 2\psi - \alpha \le \pi$. By both inequalities, and also since $2\alpha - 2\psi$ and $2\psi - \alpha$ cannot be simultaneously 0 or π, we have $f''_3(\psi) < 0$. So the function $f_3(\psi)$ is concave in the interval $[\frac{\pi}{2} - \beta, \alpha]$. This finishes the considerations of Case 3.

The functions $f_1(\psi)$, $f_2(\psi)$ (and $f_3(\psi)$ for acute T) are concave in the considered intervals. So the smallest value of each of them is attained at an end-point, or at both, of the corresponding interval. Since the three (for acute T) and two (for obtuse T) intervals are neighboring, $\min\limits_{m \in \mathcal{A}} \mathrm{area}(T^*_m)$ is attained at at least one end-point of these intervals.

From Cases 1–3 and from the above explanation we see that in order to find the smallest value of $\mathrm{area}(T^*_m)$ for acute T and $m \in \mathcal{A}$ we choose the smallest from the numbers $f_1(0) = \frac{\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$, $f_1(\frac{\alpha}{2}) = f_2(\frac{\alpha}{2}) = \frac{1}{2}\sin\alpha$, $f_2(\frac{\pi}{2} - \beta) = f_3(\frac{\pi}{2} - \beta) = \frac{1}{2}\sin(\pi - 2\beta)$ and $f_3(\alpha) = \frac{\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$. Clearly, $\frac{\sin\alpha\sin\beta}{\sin(\alpha+\beta)} = 2 \cdot \mathrm{area}(T)$. The remaining two values $\frac{1}{2}\sin\alpha$ and $\frac{1}{2}\sin(\pi - 2\beta)$ are smaller. Which of them is smaller? From $\alpha + \beta + \gamma = \pi$ and $\beta \le \gamma$ we get $\alpha + 2\beta \le \pi$. So $\alpha \le \pi - 2\beta$. Moreover, since T is acute, $\alpha + \beta > \frac{\pi}{2}$, and thus $\beta > \frac{\pi}{4}$, which implies $\pi - 2\beta < \frac{\pi}{2}$. Hence $\frac{1}{2}\sin\alpha \le \frac{1}{2}\sin(\pi - 2\beta)$. We see that $\frac{1}{2}\sin\alpha$ is the smallest possible area of T^*_m. It is attained for $\psi = \frac{\alpha}{2}$, i.e. for $m = \mathrm{bi}(A)$ (if $\frac{1}{2}\sin\alpha = \frac{1}{2}\sin(\pi - 2\beta)$, then $\beta = \gamma$, and thus $\mathrm{bi}(A)$ and $\mathrm{per}(A)$ are the best mirror lines which coincide).

In order to find the smallest value of $\mathrm{area}(T^*_m)$ for obtuse T over all $m \in \mathcal{A}$ we choose the smallest from the numbers $f_1(0) = \frac{\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$, $f_1(\frac{\alpha}{2}) = f_2(\frac{\alpha}{2}) = \frac{1}{2}\sin\alpha$ and $f_2(\alpha) = \frac{1}{2}\sin 2\alpha$. It is $\frac{1}{2}\sin\alpha$. Here is why. The inequality $\frac{1}{2}\sin\alpha < \frac{1}{2}\sin 2\alpha$ follows from $0 < \alpha \le \frac{\pi}{4}$. Since $\frac{\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$ is the double area of T, the value $\frac{1}{2}\sin\alpha$ is smaller.

We see that for both types of triangles, namely acute and obtuse ones, with $|ab| = 1$ the line $\mathrm{bi}(A)$ in the part of m minimizes $\mathrm{area}(T^*_m)$, which is $\mathrm{area}(T^*_{\mathrm{bi}(A)}) = \frac{1}{2}\sin\alpha$.

In the general situation when $|ab|$ is arbitrary, the area of T^*_m is $|ab|^2$ times larger than for a homothetic image with $|ab| = 1$. Since $\mathrm{area}(T) = \frac{1}{2}\frac{\sin\alpha\sin\beta}{\sin(\alpha+\beta)}|ab|^2$, we obtain $|ab|^2 = 2\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta} \cdot \mathrm{area}(T)$. So for arbitrary $|ab|$ we have $\mathrm{area}(T^*_{\mathrm{bi}(A)}) = \frac{1}{2}\sin\alpha \cdot 2\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta} \cdot \mathrm{area}(T) = \frac{\sin(\alpha+\beta)}{\sin\beta} \cdot \mathrm{area}(T)$. Clearly, it minimizes $\mathrm{area}(T^*_m)$ in the general situation, again with $m = \mathrm{bi}(A)$ as the only best mirror line in $\mathcal{A}$. □

4. The minimum of area(T^*_m) for $m \in \mathcal{B}$ is at least the minimum for $m \in \mathcal{A}$

Proposition 3. *For every triangle T we have* $\min\limits_{m\in\mathcal{A}} \mathrm{area}(T^*_m) \leq \min\limits_{m\in\mathcal{B}} \mathrm{area}(T^*_m)$.

Proof. Having in mind Proposition 2, it is sufficient to prove that $\mathrm{area}(T^*_{\mathrm{bi}(A)}) \leq \mathrm{area}(T^*_m)$ for every $m \in \mathcal{B}$. The angle between ab and m (smaller than $\frac{\pi}{2}$) is denoted by ψ. Without loss of generality we may assume that $|ab| = 1$.

Case 1. When $\psi \in [0, \frac{\alpha}{2}]$.

Clearly, T^*_m is the pentagon ac_mbca_m, and thus $T^*_m = ac_mb \cup aba_m \cup a_mbc$. Observe that $\angle a_mba = 2\psi$. The last fact and $|ba| = |ba_m| = 1$ imply that $\mathrm{area}(aba_m) = \frac{1}{2}\sin 2\psi$. By $|ba_m| = 1$ and $|bc| = \frac{\sin\alpha}{\sin(\alpha+\beta)}$, from $\angle cba_m = \angle cba - \angle a_mba = \beta - 2\psi$ we get $\mathrm{area}(a_mbc) = \frac{1}{2}\frac{\sin\alpha}{\sin(\alpha+\beta)}\sin(\beta-2\psi)$. Since triangles a_mbc and ac_mb are symmetric with respect to m, $\mathrm{area}(a_mbc) = \mathrm{area}(ac_mb)$. The above facts show that $\mathrm{area}(T^*_m)$ is given by the function $f_1(\psi) = \frac{1}{2}\sin 2\psi + \frac{\sin\alpha}{\sin(\alpha+\beta)}\sin(\beta-2\psi)$. From $0 \leq 2\psi \leq \pi$ and $0 \leq \beta - 2\psi \leq \pi$ we get that $f_1''(\psi) = -2\sin 2\psi - 4\frac{\sin\alpha}{\sin(\alpha+\beta)}\sin(\beta-2\psi)$ is negative in the interval $[0, \frac{\alpha}{2}]$. So $f_1(\psi)$ is concave here and the smallest value of $f_1(\psi)$ is attained at least at one end-point of the interval $[0, \frac{\alpha}{2}]$. We have $f_1(0) = \frac{\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$, $f_1(\frac{\alpha}{2}) = \frac{1}{2}\sin\alpha + \frac{\sin\alpha}{\sin(\alpha+\beta)}\sin(\beta-\alpha)$. Clearly $\frac{\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$ is the double area of T and $\mathrm{area}(T^*_{\mathrm{bi}(A)}) = \frac{1}{2}\sin\alpha$ is smaller. Moreover, from $0 \leq \frac{\sin\alpha}{\sin(\alpha+\beta)}\sin(\beta-\alpha)$ we get $\frac{1}{2}\sin\alpha \leq \frac{1}{2}\sin\alpha + \frac{\sin\alpha}{\sin(\alpha+\beta)}\sin(\beta-\alpha)$. Consequently, $\mathrm{area}(T^*_{\mathrm{bi}(A)}) \leq \mathrm{area}(T^*_m)$ for every $m \in \mathcal{B}$ and every $\psi \in [0, \frac{\alpha}{2}]$.

Case 2. When $\psi \in [\frac{\alpha}{2}, \beta]$.

In this case T^*_m contains the triangle aba_m. From Case 1 we know that $\mathrm{area}(aba_m) = \frac{1}{2}\sin 2\psi$. Since $\psi \in [\frac{\alpha}{2}, \beta]$, we obtain that $\alpha \leq 2\psi \leq 2\beta$. From $\alpha + \beta + \gamma = \pi$ and $\beta \leq \gamma$ we get $2\beta \leq \pi - \alpha$. The last fact implies that $\alpha \leq 2\psi \leq \pi - \alpha$. Hence $\frac{1}{2}\sin\alpha \leq \frac{1}{2}\sin 2\psi \leq \frac{1}{2}\sin(\pi-\alpha)$. Thus $\mathrm{area}(T^*_{\mathrm{bi}(A)}) \leq \mathrm{area}(aba_m)$. Since $aba_m \subset T^*_m$, we obtain the inequality $\mathrm{area}(T^*_{\mathrm{bi}(A)}) \leq \mathrm{area}(T^*_m)$ for every $m \in \mathcal{B}$ and every $\psi \in [\frac{\alpha}{2}, \beta]$.

5. Minimizing area(T^*_m) for m from the pencil $\mathcal{C}$

Proposition 4. *Let T be a triangle. We have* $\min\limits_{m\in\mathcal{C}} \mathrm{area}(T^*_m) = \frac{2\cos\alpha\sin\beta}{\sin(\alpha+\beta)} \cdot \mathrm{area}(T)$ *for $3\alpha + \beta < \pi$ and* $\min\limits_{m\in\mathcal{C}} \mathrm{area}(T^*_m) = \frac{\sin\beta}{\sin\alpha} \cdot \mathrm{area}(T)$ *for $3\alpha + \beta > \pi$. The first minimum is attained only for $m = \mathrm{per}(C)$, and the second only for $m = \mathrm{bi}(C)$.*

If $3\alpha + \beta = \pi$, *then* $\min_{m \in \mathcal{C}} \operatorname{area}(T_m^*)$ *is given by both formulas and holds only for* $m = \operatorname{per}(C)$ *and* $m = \operatorname{bi}(C)$.

Proof. Besides the last paragraph, for simplicity assume that $|ab| = 1$. Let $m \in \mathcal{C}$. By ψ denote the angle $\angle acd$, where d is the intersection point of m with ab. Clearly, $\gamma = \pi - \alpha - \beta$.

Case 1. (only for obtuse T): $\psi \in [0, \gamma - \frac{\pi}{2}]$.

In that case $T_m^* = aa_mbb_m$. Of course, the angle between ab and a_mb_m (that is, $\angle bdb_m$) is twice the angle $\angle bdc$. From $\angle bdc + \angle cbd + \angle dcb = \pi$ and from $\angle cbd = \beta$ and $\angle dcb = \gamma - \psi$ we obtain that $\angle bdc = \alpha + \psi$. Consequently, $\angle bdb_m = 2(\alpha + \psi)$. This and $|ab| = |a_mb_m| = 1$ imply that $\operatorname{area}(T_m^*) = \frac{1}{2}\sin(2\alpha + 2\psi)$. The function $f_1(\psi) = \frac{1}{2}\sin(2\alpha + 2\psi)$ is concave in the interval $[0, \gamma - \frac{\pi}{2}]$ because $f_1''(\psi) = -2\sin(2\alpha + 2\psi)$ is negative here.

Case 2. (for all T): $\psi \in [0, \frac{\gamma}{2}]$ for acute T, and $\psi \in [\gamma - \frac{\pi}{2}, \frac{\gamma}{2}]$ for obtuse T.

In this case T_m^* is the pentagon cb_maa_mb. We have $T_m^* = ca_mb \cup cb_ma \cup caa_m$. From $\angle aca_m = 2\psi$ and $|ca| = |ca_m| = \frac{\sin\beta}{\sin\gamma}$ we get that $\operatorname{area}(caa_m) = \frac{1}{2}(\frac{\sin\beta}{\sin\gamma})^2 \sin 2\psi$. Clearly, $\angle a_mcb = \angle acb - \angle aca_m = \gamma - 2\psi$. This, $|ca_m| = \frac{\sin\beta}{\sin\gamma}$ and $|bc| = \frac{\sin\alpha}{\sin\gamma}$ imply that $\operatorname{area}(ca_mb) = \frac{1}{2}\frac{\sin\alpha}{\sin\gamma}\frac{\sin\beta}{\sin\gamma}\sin(\gamma - 2\psi)$. Of course, triangles ca_mb and cb_ma are symmetric with respect to m. Hence $\operatorname{area}(ca_mb) = \operatorname{area}(cb_ma)$. All this leads to the conclusion that $\operatorname{area}(T_m^*)$ is given by the function $f_2(\psi) = \frac{1}{2}(\frac{\sin\beta}{\sin\gamma})^2 \sin 2\psi + \frac{\sin\alpha}{\sin\gamma}\frac{\sin\beta}{\sin\gamma}\sin(\gamma - 2\psi)$. Here is the second derivative: $f_2''(\psi) = -2(\frac{\sin\beta}{\sin\gamma})^2 \sin 2\psi - 4\frac{\sin\alpha}{\sin\gamma}\frac{\sin\beta}{\sin\gamma}\sin(\gamma - 2\psi)$. From the assumptions of Case 2 we obtain that $0 \le 2\psi \le \pi$ and $0 \le \gamma - 2\psi \le \pi$. Thus for every $\psi \in [0, \frac{\gamma}{2}]$ we have $f_2''(\psi) < 0$. Consequently, the function $f_2(\psi)$ is concave in the interval $[0, \frac{\gamma}{2}]$ (and, in particular, in the interval $[\gamma - \frac{\pi}{2}, \frac{\gamma}{2}]$ for obtuse triangles T).

Case 3. (for all T): $\psi \in [\frac{\gamma}{2}, \frac{\pi}{2} - \alpha]$.

Observe that $T_m^* = caa_m$ and that $\angle aca_m = 2\psi$. So $\operatorname{area}(T_m^*)$ equals $f_3(\psi) = \frac{1}{2}(\frac{\sin\beta}{\sin\gamma})^2 \sin 2\psi$. By $0 \le 2\psi \le \pi$ we see that $f_3''(\psi) = -2(\frac{\sin\beta}{\sin\gamma})^2 \sin 2\psi$ is negative in our interval. Thus the function $f_3(\psi)$ is concave in the interval $[\frac{\gamma}{2}, \frac{\pi}{2} - \alpha]$.

Case 4. (for all T): $\psi \in [\frac{\pi}{2} - \alpha, \gamma]$ for acute T, and $\psi \in [\frac{\pi}{2} - \alpha, \frac{\pi}{2}]$ for obtuse T.

Since T_m^* is the pentagon cab_mba_m (see Figure 10), we have $T_m^* = cba_m \cup cab_m \cup cb_mb$.

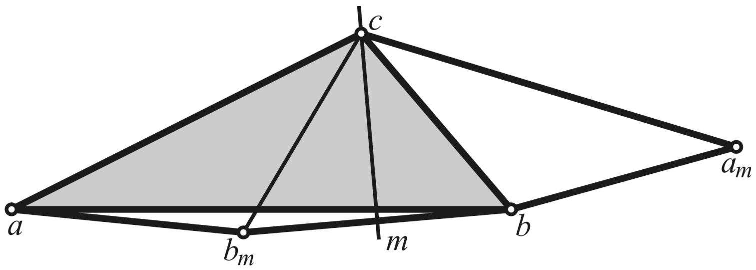

Figure 10.

From $\angle aca_m = 2\psi$ and $\angle bca_m = \angle aca_m - \angle acb$ we obtain $\angle bca_m = 2\psi - \gamma$. The symmetry of triangles cba_m and cab_m with respect to m implies that $\angle bca_m =$

$\angle acb_m = 2\psi - \gamma$. This, $\angle b_m cb = \angle acb - \angle acb_m$ and $\angle acb = \gamma$ give $\angle b_m cb = 2\gamma - 2\psi$. From these facts, from $|cb| = \frac{\sin\alpha}{\sin\gamma}$ and $|ca| = \frac{\sin\beta}{\sin\gamma}$ we obtain that area(T_m^*) equals $f_4(\psi) = \frac{1}{2}(\frac{\sin\alpha}{\sin\gamma})^2 \sin(2\gamma - 2\psi) + \frac{\sin\alpha}{\sin\gamma}\frac{\sin\beta}{\sin\gamma}\sin(2\psi - \gamma)$. Since $2\gamma - 2\psi$ and $2\psi - \gamma$ are between 0 and π, we see that the second derivative $f_4''(\psi) = -2(\frac{\sin\alpha}{\sin\gamma})^2 \sin(2\gamma - 2\psi) - 4\frac{\sin\alpha}{\sin\gamma}\frac{\sin\beta}{\sin\gamma}\sin(2\psi - \gamma)$ is negative for acute and obtuse T. Hence $f_4(\psi)$ is concave in the intervals $[\frac{\pi}{2} - \alpha, \gamma]$ and $[\frac{\pi}{2} - \alpha, \frac{\pi}{2}]$.

Case 5. (only for obtuse T): $\psi \in [\frac{\pi}{2}, \gamma]$.

We easily conclude that T_m^* is the quadrangle $ab_m ba_m$. Similarly like in Case 1 we show that the angle between ab and $a_m b_m$ is $2(\gamma - \psi + \beta)$. The last fact and $|ab| = |a_m b_m| = 1$ imply that area(T_m^*) is given by the function $f_5(\psi) = \frac{1}{2}\sin(2\gamma - 2\psi + 2\beta)$. Of course, $f_5''(\psi) = -2\sin(2\gamma - 2\psi + 2\beta)$. Since $0 \le 2\gamma - 2\psi + 2\beta \le \pi$, we have $f_5''(\psi) < 0$. Hence $f_5(\psi)$ is concave in the interval $[\frac{\pi}{2}, \gamma]$. This finishes our considerations of Case 5.

The functions $f_i(\psi)$ for $i = 2, 3, 4$ (for acute T) and for $i = 1, \dots, 5$ (for obtuse T) are concave in the intervals considered in corresponding cases. So the smallest value of each of them is attained at an end-point (or at both) of the corresponding interval. Since the three (for acute T) and five (for obtuse T) intervals are neighboring, the smallest value of area(T_m^*) is attained at least at one end-point of at least one of the intervals.

Our proposition is true if $\beta = \alpha$ (see Remark 2 below). So assume that $\beta \ne \alpha$.

In order to find the smallest value of area(T_m^*) for acute T when $m \in \mathcal{C}$, we look at Cases 2–4. We choose the smallest from the numbers $f_2(0) = \frac{\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$, $f_2(\frac{\gamma}{2}) = f_3(\frac{\gamma}{2}) = \frac{1}{2}\frac{\sin^2\beta}{\sin(\alpha+\beta)}$, $f_3(\frac{\pi}{2} - \alpha) = f_4(\frac{\pi}{2} - \alpha) = \frac{1}{2}\frac{\sin^2\beta\sin 2\alpha}{\sin^2(\alpha+\beta)}$ and $f_4(\gamma) = \frac{\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$. Clearly, $\frac{\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$ is the double area of T. Hence $\frac{1}{2}\frac{\sin^2\beta\sin 2\alpha}{\sin^2(\alpha+\beta)}$ is smaller. It remains to compare $\frac{1}{2}\frac{\sin^2\beta}{\sin(\alpha+\beta)}$ and $\frac{1}{2}\frac{\sin^2\beta\sin 2\alpha}{\sin^2(\alpha+\beta)}$. If $\frac{1}{2}\frac{\sin^2\beta}{\sin(\alpha+\beta)} = \frac{1}{2}\frac{\sin^2\beta\sin 2\alpha}{\sin^2(\alpha+\beta)}$, then $3\alpha + \beta = \pi$. We have $\frac{1}{2}\frac{\sin^2\beta\sin 2\alpha}{\sin^2(\alpha+\beta)} < \frac{1}{2}\frac{\sin^2\beta}{\sin(\alpha+\beta)}$ if and only if $3\alpha + \beta < \pi$. So if $3\alpha + \beta < \pi$, the area of T_m^* is minimum for $\psi = \frac{\pi}{2} - \alpha$, i.e., for $m = \mathrm{per}(C)$, and it is $\frac{1}{2}\frac{\sin^2\beta\sin 2\alpha}{\sin^2(\alpha+\beta)}$. We also conclude that if $3\alpha + \beta > \pi$, the area of T_m^* is minimum for $\psi = \frac{\gamma}{2}$, this is for $m = \mathrm{bi}(C)$. It is $\frac{1}{2}\frac{\sin^2\beta}{\sin(\alpha+\beta)}$. If $3\alpha + \beta = \pi$, then we have two minima: for $\psi = \frac{\pi}{2} - \alpha$ (i.e., for $m = \mathrm{per}(C)$) and for $\psi = \frac{\gamma}{2}$ (i.e., for $m = \mathrm{bi}(C)$).

In order to find the smallest value of area(T_m^*) for obtuse T when $m \in \mathcal{C}$, we look at all Cases 1–5, and thus we choose the smallest from the numbers $f_1(0) = \frac{1}{2}\sin 2\alpha$, $f_1(\gamma - \frac{\pi}{2}) = f_2(\gamma - \frac{\pi}{2}) = \frac{1}{2}\sin 2\beta$, $f_2(\frac{\gamma}{2}) = f_3(\frac{\gamma}{2}) = \frac{1}{2}\frac{\sin^2\beta}{\sin(\alpha+\beta)}$, $f_3(\frac{\pi}{2} - \alpha) = f_4(\frac{\pi}{2} - \alpha) = \frac{1}{2}\frac{\sin^2\beta\sin 2\alpha}{\sin^2(\alpha+\beta)}$, $f_4(\frac{\pi}{2}) = f_5(\frac{\pi}{2}) = \frac{1}{2}\sin 2\alpha$ and $f_5(\gamma) = \frac{1}{2}\sin 2\beta$. Since $\frac{1}{2}\sin 2\alpha$ and $\frac{1}{2}\sin 2\beta$ are at least the double area of T, we obtain that $\frac{1}{2}\frac{\sin^2\beta\sin 2\alpha}{\sin^2(\alpha+\beta)}$ is smaller. From $\alpha + \beta \le \frac{\pi}{2}$ and from $\alpha < \beta$ we get $2\alpha < \alpha + \beta \le \frac{\pi}{2}$ and thus $\frac{1}{2}\frac{\sin^2\beta\sin 2\alpha}{\sin^2(\alpha+\beta)} < \frac{1}{2}\frac{\sin^2\beta}{\sin(\alpha+\beta)}$. From the above considerations we conclude that the smallest from the four considered numbers is $\frac{1}{2}\frac{\sin^2\beta\sin 2\alpha}{\sin^2(\alpha+\beta)}$. This means that the minimum area of T_m^* is attained for $m = \mathrm{per}(C)$.

In the general situation when $|ab|$ is arbitrary, we proceed as in the last paragraph of the proof of Proposition 2. So now $\min_{m\in\mathcal{C}} \operatorname{area}(T^*_m) = \operatorname{area}(T^*_{\mathrm{bi}(C)}) = \frac{1}{2}\frac{\sin^2\beta}{\sin(\alpha+\beta)} \cdot 2\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta} \cdot \operatorname{area}(T) = \frac{\sin\beta}{\sin\alpha} \cdot \operatorname{area}(T)$ provided $3\alpha+\beta > \pi$, and $\min_{m\in\mathcal{C}} \operatorname{area}(T^*_m) = \operatorname{area}(T^*_{\mathrm{per}(C)}) = \frac{1}{2}\frac{\sin^2\beta\sin 2\alpha}{\sin^2(\alpha+\beta)} \cdot 2\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta} \cdot \operatorname{area}(T) = \frac{2\cos\alpha\sin\beta}{\sin(\alpha+\beta)} \cdot \operatorname{area}(T)$ provided $3\alpha+\beta < \pi$. The last part of Proposition 4 results from the preceding paragraph. □

Remark 2. If $\beta = \alpha$, then $\mathrm{per}(C)$ and $\mathrm{bi}(C)$ coincide and thus both formulas for $\min_{m\in\mathcal{C}} \operatorname{area}(T^*_m)$ from Proposition 4 give the same greatest value.

6. A formula for the minimum of area(T^*_m) over all mirror lines m

Corollary 1 and Propositions 2–4 imply the following proposition.

Proposition 5. *We have* $\min_m \operatorname{area}(T^*_m) = \min\{\frac{\sin(\alpha+\beta)}{\sin\beta}, \frac{2\cos\alpha\sin\beta}{\sin(\alpha+\beta)}, \frac{\sin\beta}{\sin\alpha}\} \cdot \operatorname{area}(T)$ *for any triangle T such that $\alpha \le \beta \le \gamma$. This minimum equals* $\frac{\sin(\alpha+\beta)}{\sin\beta} \cdot \operatorname{area}(T)$ *if and only if* $\mathrm{bi}(A)$ *is a best mirror line, equals* $\frac{2\cos\alpha\sin\beta}{\sin(\alpha+\beta)} \cdot \operatorname{area}(T)$ *if and only if* $\mathrm{per}(C)$ *is a best mirror line, and equals* $\frac{\sin\beta}{\sin\alpha} \cdot \operatorname{area}(T)$ *if and only if* $\mathrm{bi}(C)$ *is a best mirror line.*

Propositions 2–5 and the proof of Proposition 1 imply the following remark.

Remark 3. If $\mathrm{per}(C)$ is a best mirror line, all parallel mirror lines between $\mathrm{per}(C)$ and the parallel line passing through the midpoint of ab are also best mirror lines. Besides this exceptional case, we never have parallel best mirror lines.

7. Estimates of the measure of axial symmetry of triangles

From the definition of $\operatorname{axs}(K)$ and from Proposition 5 we immediately get Theorem 1.

Theorem 1. *We have* $\operatorname{axs}(T) = \max\{\frac{\sin\beta}{\sin(\alpha+\beta)}, \frac{\sin(\alpha+\beta)}{2\cos\alpha\sin\beta}, \frac{\sin\alpha}{\sin\beta}\}$ *for any triangle T with measures $\alpha \le \beta \le \gamma$ of angles. Here* $\operatorname{axs}(T) = \frac{\sin\beta}{\sin(\alpha+\beta)}$ *if and only if* $\mathrm{bi}(A)$ *is a best mirror line,* $\operatorname{axs}(T) = \frac{\sin(\alpha+\beta)}{2\cos\alpha\sin\beta}$ *if and only if* $\mathrm{per}(C)$ *is a best mirror line, and* $\operatorname{axs}(T) = \frac{\sin\alpha}{\sin\beta}$ *if and only if* $\mathrm{bi}(C)$ *is a best mirror line.*

How to recognize the triangles for which each of the three formulas $f = \frac{\sin\beta}{\sin(\alpha+\beta)}$, $g = \frac{\sin(\alpha+\beta)}{2\cos\alpha\sin\beta}$ and $h = \frac{\sin\alpha}{\sin\beta}$ from Theorem 1 gives the maximum? We omit an elementary calculation which shows that $f = g$ if and only if $\beta = \arctan\frac{\sin\alpha}{\sqrt{2\cos\alpha}-\cos\alpha}$, that $f = h$ if and only if $\alpha = \frac{1}{2}\arccos(\cos\beta - 2\sin^2\beta) - \frac{1}{2}\beta$, and that $g = h$ if and only if $\beta = \pi - 3\alpha$ or $\beta = \alpha$. These equivalences permit to draw Figure 11 showing the domain of all pairs (β, α) (every pair represents all triangles with angles $\alpha \le \beta \le \gamma$). This domain is divided into three regions $R_{\mathrm{bi}(A)}$,

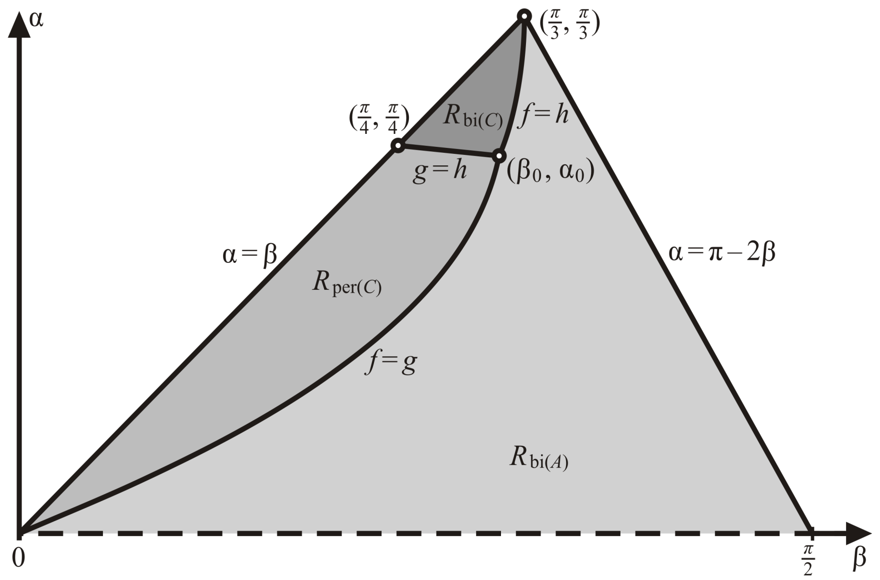

Figure 11.

$R_{\text{per}(C)}$ and $R_{\text{bi}(C)}$ whose points represent triangles for which the mirror lines bi(A), per(C) and bi(C), respectively, are the best, and for which the three successive formulas from Theorem 1 give the maximum value. Moreover, by Remark 2, if $\beta = \alpha$ then per(C) and bi(C) coincide, and thus the last two formulas give the same greatest value 1. In Figure 11 we see only the pieces of the curves $f = g$, $f = h$ and $g = h$ which separate our regions. It is easy to check that the three curves meet at the point (β_0, α_0) representing each triangle with angles $\alpha_0 = \arcsin x \approx 0.73009$, where $x \approx 0.66694$ is a root of the equation $4 - 4x^2 = (3 - 4x^2)^4$ and $\beta_0 = \pi - 3\alpha_0 \approx 0.95131$.

The segment $\alpha = \frac{\pi}{2} - \beta$ (not marked in Figure 11) connecting the points $(\frac{\pi}{4}, \frac{\pi}{4})$ and $(\frac{\pi}{2}, 0)$ consists of points representing right-angled triangles and divides the "regions" of acute and obtuse triangles. The region $R_{\text{bi}(C)}$ in which $\text{axs}(T) = \frac{\sin\alpha}{\sin\beta}$ is a subset of the "region" of acute triangles. This and Theorem 1 give the following corollary.

Corollary 2. *If T is an arbitrary obtuse (in particular, right-angled) triangle, then* $\text{axs}(T) = \max\{\frac{\sin\beta}{\sin(\alpha+\beta)}, \frac{\sin(\alpha+\beta)}{2\cos\alpha\sin\beta}\}$.

Theorem 2. *For every triangle T we have* $\text{axs}(T) > \frac{1}{2}\sqrt{2}$. *This estimate cannot be improved: the infimum of* $\text{axs}(T)$ *over all triangles T equals* $\frac{1}{2}\sqrt{2}$.

Proof. The inequality $\frac{\sin(\alpha+\beta)}{2\cos\alpha\sin\beta} \le \frac{\sin\beta}{\sin(\alpha+\beta)}$ is equivalent to $\frac{\sin^2\beta}{\sin^2(\alpha+\beta)} \ge \frac{1}{2\cos\alpha}$ and thus to $\frac{\sin\beta}{\sin(\alpha+\beta)} \ge \frac{1}{\sqrt{2\cos\alpha}}$. On the other hand, $\frac{\sin(\alpha+\beta)}{2\cos\alpha\sin\beta} \ge \frac{\sin\beta}{\sin(\alpha+\beta)}$ is equivalent to $\frac{\sin^2(\alpha+\beta)}{4\cos^2\alpha\sin^2\beta} \ge \frac{1}{2\cos\alpha}$, and consequently it is equivalent to $\frac{\sin(\alpha+\beta)}{2\cos\alpha\sin\beta} \ge \frac{1}{\sqrt{2\cos\alpha}}$. From Theorem 1 and from the above equivalences we obtain $\text{axs}(T) = \max\{\frac{\sin\beta}{\sin(\alpha+\beta)}, \frac{\sin(\alpha+\beta)}{2\cos\alpha\sin\beta}, \frac{\sin\alpha}{\sin\beta}\} \ge \max\{\frac{\sin\beta}{\sin(\alpha+\beta)}, \frac{\sin(\alpha+\beta)}{2\cos\alpha\sin\beta}\} \ge \frac{1}{\sqrt{2\cos\alpha}} > \frac{1}{2}\sqrt{2}$.

Consider the family of triangles for which $\frac{\sin\beta}{\sin(\alpha+\beta)} = \frac{\sin(\alpha+\beta)}{2\cos\alpha\sin\beta}$ (i.e., $f = g$; a piece of this curve is shown in Figure 11). Their common value is $\frac{1}{\sqrt{2\cos\alpha}}$, which follows from the first two sentences of this proof. It tends to $\frac{1}{2}\sqrt{2}$ when α tends to 0. So by Corollary 2 the estimate $\mathrm{axs}(T) > \frac{1}{2}\sqrt{2}$ cannot be improved for our family, and hence in general. $\square$

Remark 4. In order to show the second part of Theorem 2 we can also use other families of triangles with α close to 0, for instance, the family with $\frac{\beta}{\alpha} = \sqrt{2}+1$, or with $\frac{\sin\beta}{\sin\alpha} = \sqrt{2}+1$, or with $\frac{\tan\beta}{\tan\alpha} = \sqrt{2}+1$ (this ratio equals $|ad|/|db|$, where d is the projection of c on ab).

Theorem 3. *For every acute triangle T we have $\mathrm{axs}(T) > \frac{1}{2}\sqrt[3]{4}$, and this estimate cannot be improved. For every right-angled triangle T we have $\mathrm{axs}(T) \geq \frac{1}{2}\sqrt[3]{4}$ with equality only for right-angled triangles with $\beta = \arcsin\sqrt[3]{1/2}$ (≈ 0.91687).*

Proof. Apply Theorem 1. Since $0 < \alpha \leq \frac{\pi}{3}$ and $\frac{\pi}{2} < \alpha+\beta < \pi$ for acute T, the derivatives of $\frac{\sin\beta}{\sin(\alpha+\beta)}$, $\frac{\sin(\alpha+\beta)}{2\cos\alpha\sin\beta}$ and $\frac{\sin\alpha}{\sin\beta}$ with respect to α are positive. So these three functions are increasing with respect to α. Thus, if β is constant and we decrease α to $\frac{\pi}{2}-\beta$ (which gives a right-angled triangle), we get $\frac{\sin\beta}{\sin(\alpha+\beta)} > \sin\beta$, $\frac{\sin(\alpha+\beta)}{2\cos\alpha\sin\beta} > \frac{1}{2\sin^2\beta}$ and $\frac{\sin\alpha}{\sin\beta} > \cot\beta$. It is easy to see that the minimum of $\max\{\sin\beta, \frac{1}{2\sin^2\beta}, \cot\beta\}$ is $\frac{1}{2}\sqrt[3]{4}$; it is attained for the root $\beta = \arcsin\sqrt[3]{1/2}$ of the equation $\sin\beta = \frac{1}{2\sin^2\beta}$. So $\mathrm{axs}(T) > \frac{1}{2}\sqrt[3]{4}$ for acute triangles and $\mathrm{axs}(T) \geq \frac{1}{2}\sqrt[3]{4}$ for right-angled triangles. The last estimate cannot be improved for right-angled triangles with $\beta = \arcsin\sqrt[3]{1/2}$. Acute triangles arbitrarily close to them show that also the preceding estimate cannot be improved. $\square$

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