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Curvature Properties of g-natural Contact Metric Structures on Unit Tangent Sphere Bundles

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Abstract. We study some curvature properties of a three-parameter family of contact metric structures on T_1M introduced in [1]. The results we obtain generalize classical theorems on the standard contact metric structure of T_1M.

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1. Introduction and main results

The study of the geometry of a Riemannian manifold (M, g) through the properties of its unit tangent sphere bundle T_1M represents a well known and interesting research field in Riemannian geometry. Traditionally, T_1M has been equipped with one of the following Riemannian metrics:

- either the Sasaki metric $\widetilde{g_S}$, induced by the Sasaki metric g_S of the tangent bundle TM (or the metric $\bar{g} = \frac{1}{4} g_S$ of the standard contact metric structure $(\eta, \bar{g})$ of T_1M), or

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- the metric $\widetilde{g_{CG}}$, induced by the Cheeger-Gromoll metric g_{CG} on TM.

Since $\bar{g}$ is homothetic to $\widetilde{g_S}$, these Riemannian metrics share essentially the same curvature properties. As concerns $(T_1M, \widetilde{g_{CG}})$, it is isometric to the tangent sphere bundle T_rM, with radius $r = \frac{1}{\sqrt{2}}$, equipped with the metric induced by the Sasaki metric of TM, the isometry being explicitly given by $\Phi : T_1M \to T_{\frac{1}{\sqrt{2}}}M$: $(x,u) \mapsto (x, u/\sqrt{2})$.

Several curvature properties on T_1M, equipped with one of the metrics above, turn out to correspond to very rigid properties for the base manifold M. We can refer to [12] for a survey on the geometry of $(T_1M, \widetilde{g_S})$. A survey on the contact metric geometry of $(T_1M, \eta, \bar{g})$ was made by the second author in [13].

In [4], the first author and M. Sarih investigated geometric properties of the tangent bundle TM, equipped with the most general "g-natural" metric. On unit tangent sphere bundles, the restrictions of g-natural metrics possess a simpler form. Precisely, it was proved in [3] that for every Riemannian metric $\tilde{G}$ on T_1M induced by a Riemannian g-natural metric G on TM, there exist four constants a, b, c and d, with

$$a > 0,\ \alpha := a(a+c) - b^2 > 0, \text{ and } \phi := a(a+c+d) - b^2 > 0, \tag{1.1}$$

such that $\tilde{G} = a.\widetilde{g^s} + b.\widetilde{g^h} + c.\widetilde{g^v} + d.\widetilde{k^v}$, where

* k is the natural F-metric on M defined by

$$k(u;X,Y) = g(u,X)g(u,Y), \quad \text{for all} \quad (u,X,Y) \in TM \oplus TM \oplus TM,$$

* $\widetilde{g^s}$, $\widetilde{g^h}$, $\widetilde{g^v}$ and $\widetilde{k^s}$ are the metrics on T_1M induced by the three lifts g^s, g^h, g^v and k^v, respectively (we refer to Section 2 for the definitions of F-metrics and their lifts).

In this paper, using curvature expressions for $(T_1M, \tilde{G})$ obtained in [2], we will study contact metric conditions, expressible in terms of the curvature tensor, of the g-natural contact metric structures $(\tilde{\eta}, \tilde{G})$ on T_1M we introduced in [1]. Throughout the paper, we shall assume that (M,g) is a Riemannian manifold of dimension ≥ 3. We report here the main results we obtained. They generalize classical theorems on the standard contact metric structure of T_1M, which may be found in Section 9.2 of [7]. Preliminary information about contact metric manifolds and g-natural contact metric structures will be given at the beginning of Section 4.

Theorem 1. *Let $\tilde{G}$ be a Riemannian g-natural metric on T_1M. $(T_1M, \tilde{\eta}, \tilde{G})$ has constant ξ-sectional curvature $\widetilde{K}$ if and only if the base manifold (M,g) has constant sectional curvature $\bar{c}$ either equal to $\frac{d}{a}$ or to $\frac{a+c}{a} > 0$.*

Theorem 2. *Let $\tilde{G}$ be a Riemannian g-natural metric on T_1M. If $(T_1M, \tilde{\eta}, \tilde{G})$ has constant φ-sectional curvature, then the base manifold (M,g) is locally isometric to a two-point homogeneous space.*

Theorem 3. *Let (M,g) be a Riemannian manifold of constant sectional curvature $\bar{c}$ and $\dim M \geq 3$, and $\tilde{G}$ a Riemannian g-natural metric on T_1M. $(T_1M, \tilde{\eta}, \tilde{G})$ has constant φ-sectional curvature $\widetilde{K}$ if and only one of the following cases occurs:*

(i) $\bar{c} = 0$, $b = \pm\sqrt{(a+c)(a-\frac{1}{8})}$ *and* $d = -\frac{a+c}{2}$. *In this case,* $\widetilde{K} = 5$.

(ii) $\bar{c} \neq 0$, $a = \frac{1}{4}$, $b = d = 0$ *and* $c = -\frac{1}{4} - \frac{2\pm\sqrt{5}}{4}\bar{c}$. *In this case,* $\widetilde{K} = (2 \pm \sqrt{5})^2$.

Theorem 4. *Let $\tilde{G}$ be a Riemannian g-natural metric on T_1M. $(T_1M, \tilde{\eta}, \tilde{G})$ is a (k,μ)-space if and only if (M,g) has constant sectional curvature $\bar{c}$. In this case, if $(T_1M, \tilde{\eta}, \tilde{G})$ is not Sasakian, then*

$$k = \frac{1}{16\alpha^2}\left[-a^2\bar{c}^2 + 2(\alpha - b^2)\bar{c} + d(2(a+c)+d)\right], \qquad \mu = \frac{1}{2\alpha}(d - a\bar{c}). \quad (1.2)$$

Theorem 5. *A g-natural contact metric structure $(\tilde{\eta}, \tilde{G})$ on T_1M is locally symmetric if and only if $(\tilde{\eta}, \tilde{G}) = (\bar{\eta}, \bar{g})$ is the standard contact metric structure of T_1M and (M,g) is flat.*

The paper is organized in the following way. In Section 2 we shall recall the definition and properties of g-natural metrics on TM. Section 3 will be devoted to Riemannian g-natural metrics on T_1M and their curvature tensor. Finally, Theorems 1–5 and further curvature results will be proved in Section 4.

2. Basic formulae on tangent bundles

Let (M,g) be an n-dimensional Riemannian manifold and ∇ its Levi-Civita connection. At any point (x,u) of its *tangent bundle* TM, the tangent space of TM splits into the horizontal and vertical subspaces with respect to ∇:

$$(TM)_{(x,u)} = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}.$$

For any vector $X \in M_x$, there exists a unique vector $X^h \in \mathcal{H}_{(x,u)}$ (the *horizontal lift* of X to $(x,u) \in TM$), such that $\pi_* X^h = X$, where $\pi : TM \to M$ is the natural projection. The *vertical lift* of a vector $X \in M_x$ to $(x,u) \in TM$ is a vector $X^v \in \mathcal{V}_{(x,u)}$ such that $X^v(df) = Xf$, for all functions f on M. Here we consider 1-forms df on M as functions on TM (i.e., $(df)(x,u) = uf$). The map $X \to X^h$ is an isomorphism between the vector spaces M_x and $\mathcal{H}_{(x,u)}$. Similarly, the map $X \to X^v$ is an isomorphism between M_x and $\mathcal{V}_{(x,u)}$. Each tangent vector $\tilde{Z} \in (TM)_{(x,u)}$ can be written in the form $\tilde{Z} = X^h + Y^v$, where $X, Y \in M_x$ are uniquely determined vectors. Horizontal and vertical lifts of vector fields on M can be defined in an obvious way and are uniquely defined vector fields on TM.

The Sasaki metric g^s has been the most investigated among all possible Riemannian metrics on TM. However, in many different contexts such metric showed a very "rigid" behaviour. Moreover, g^s represents only one possible choice inside a wide family of Riemannian metrics on TM, known as *Riemannian g-natural metrics*, which depend on several independent smooth functions from

$\mathbb{R}^+$ to $\mathbb{R}$. As their name suggests, those metrics arise from a very "natural" construction starting from a Riemannian metric g over M. The introduction of g-natural metrics moves from the description of all first order natural operators $D : S^2_+T^* \rightsquigarrow (S^2T^*)T$, transforming Riemannian metrics on manifolds into metrics on their tangent bundles, where $S^2_+T^*$ and S^2T^* denote the bundle functors of all Riemannian metrics and all symmetric $(0,2)$-tensors over n-manifolds respectively. For more details about the concept of naturality and related notions, we can refer to [16].

We shall call *g-natural metric* a metric G on TM, coming from g by a first order natural operator $S^2_+T^* \rightsquigarrow (S^2T^*)T$ [4]. Given an arbitrary g-natural metric G on the tangent bundle TM of a Riemannian manifold (M,g), there are six smooth functions α_i, $\beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1,2,3$, such that for every u, X, $Y \in M_x$, we have

$$\left.\begin{array}{rcl} G_{(x,u)}(X^h,Y^h) & = & (\alpha_1+\alpha_3)(r^2)g_x(X,Y) + (\beta_1+\beta_3)(r^2)g_x(X,u)g_x(Y,u), \\ G_{(x,u)}(X^h,Y^v) & = & \alpha_2(r^2)g_x(X,Y) + \beta_2(r^2)g_x(X,u)g_x(Y,u), \\ G_{(x,u)}(X^v,Y^h) & = & \alpha_2(r^2)g_x(X,Y) + \beta_2(r^2)g_x(X,u)g_x(Y,u), \\ G_{(x,u)}(X^v,Y^v) & = & \alpha_1(r^2)g_x(X,Y) + \beta_1(r^2)g_x(X,u)g_x(Y,u), \end{array}\right\} \tag{2.1}$$

where $r^2 = g_x(u,u)$. For $n = 1$, the same holds with $\beta_i = 0$, $i = 1,2,3$. Put

- $\phi_i(t) = \alpha_i(t) + t\beta_i(t)$,
- $\alpha(t) = \alpha_1(t)(\alpha_1+\alpha_3)(t) - \alpha_2^2(\mathrm{t})$,
- $\phi(t) = \phi_1(t)(\phi_1+\phi_3)(t) - \phi_2^2(t)$,

for all $t \in \mathbb{R}^+$. Then, a g-natural metric G on TM is Riemannian if and only if the following inequalities hold:

$$\alpha_1(t) > 0, \quad \phi_1(t) > 0, \quad \alpha(t) > 0, \quad \phi(t) > 0, \tag{2.2}$$

for all $t \in \mathbb{R}^+$. (For $n = 1$, system (2.2) reduces to $\alpha_1(t) > 0$ and $\alpha(t) > 0$, for all $t \in \mathbb{R}^+$.)

Convention 1. a) In the sequel, when we consider an arbitrary Riemannian g-natural metric G on TM, we implicitly suppose that it is defined by the functions α_i, $\beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1,2,3$, satisfying (2.1)–(2.2).

b) Unless otherwise stated, all real functions α_i, β_i, ϕ_i, α and ϕ and their derivatives are evaluated at $r^2 := g_x(u,u)$.

c) We shall denote respectively by R and Q the curvature tensor and the Ricci operator of a Riemannian manifold (M,g). The tensor R is taken with the sign convention

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

for all vector fields X, Y, Z on M.

Next, as it is well known, the *tangent sphere bundle of radius $\rho > 0$* over a Riemannian manifold (M,g) is the hypersurface $T_\rho M = \{(x,u) \in TM | g_x(u,u) = \rho^2\}$. The tangent space of $T_\rho M$, at a point $(x,u) \in T_\rho M$, is given by

$$(T_\rho M)_{(x,u)} = \{X^h + Y^v / X \in M_x, Y \in \{u\}^\perp \subset M_x\}. \tag{2.3}$$

When $\rho = 1$, T_1M is called *the unit tangent (sphere) bundle.*

Let $G = a.g^s + b.g^h + c.g^v + \beta.k^v$ be a Riemannian g-natural metric on TM and $\tilde{G}$ the metric on T_1M induced by G. Then, $\tilde{G}$ only depends on a, b, c and $d := \beta(1)$, and these coefficients satisfy (1.1) (see also [3]).

Using the Schmidt's orthonormalization process, a simple calculation shows that the vector field on TM defined by

$$N^G_{(x,u)} = \frac{1}{\sqrt{(a+c+d)\phi}}\,[-b.u^h + (a+c+d).u^v], \tag{2.4}$$

for all $(x, u) \in TM$, is normal to T_1M and unitary at any point of T_1M. Here ϕ is, by definition, the quantity $\phi(1) = a(a + c + d) - b^2$.

Now, we define the "tangential lift" X^{t_G} – with respect to G – of a vector $X \in M_x$ to $(x, u) \in T_1M$ as the tangential projection of the vertical lift of X to (x, u) – with respect to N^G –, that is,

$$X^{t_G} = X^v - G_{(x,u)}(X^v, N^G_{(x,u)})\; N^G_{(x,u)} = X^v - \sqrt{\tfrac{\phi}{a+c+d}}\,g_x(X, u)\; N^G_{(x,u)}. \tag{2.5}$$

If $X \in M_x$ is orthogonal to u, then $X^{t_G} = X^v$.

The tangent space $(T_1M)_{(x,u)}$ of T_1M at (x, u) is spanned by vectors of the form X^h and Y^{t_G}, where $X, Y \in M_x$. Hence, the Riemannian metric $\tilde{G}$ on T_1M, induced from G, is completely determined by the identities

$$\left.\begin{array}{lcl}
\tilde{G}_{(x,u)}(X^h, Y^h) & = & (a+c)g_x(X, Y) + dg_x(X, u)g_x(Y, u),\\
\tilde{G}_{(x,u)}(X^h, Y^{t_G}) & = & bg_x(X, Y),\\
\tilde{G}_{(x,u)}(X^{t_G}, Y^{t_G}) & = & ag_x(X, Y) - \frac{\phi}{a+c+d}g_x(X, u)g_x(Y, u),
\end{array}\right\} \tag{2.6}$$

for all $(x, u) \in T_1M$ and $X, Y \in M_x$. It should be noted that, by (3.4), horizontal and vertical lifts are orthogonal with respect to $\tilde{G}$ if and only if $b = 0$.

Convention 2. For any $(x, u) \in T_1M$, the tangential lift to (x, u) of the vector u is given by $u^{t_G} = \frac{b}{a+c+d}\,u^h$, that is, it is a horizontal vector. Hence, the tangent space $(T_1M)_{(x,u)}$ coincides with the set

$$\{X^h + Y^{t_G} / X \in M_x, Y \in \{u\}^\perp \subset M_x\}. \tag{2.7}$$

Then, the operation of tangential lift from M_x to a point $(x, u) \in T_1M$ will be always applied only to vectors of M_x which are orthogonal to u.

The Levi-Civita connection and the curvature tensor of $(T_1M, \tilde{G})$ were respectively calculated by the authors in [1] and [2]. In particular, we have the following

Proposition 1. [2] *Let (M, g) be a Riemannian manifold and let $G = a.g^s + b.g^h + c.g^v + \beta.k^v$, where a, b and c are constants and $\beta : [0, \infty) \to \mathbb{R}$ is a function satisfying* (1.1). *Denote by ∇ and R the Levi-Civita connection and the Riemannian curvature tensor of (M, g), respectively. If we denote by $\tilde{R}$ the Riemannian curvature tensor of $(T_1M, \tilde{G})$, then:*

(i)

$$\begin{aligned}
&\tilde{R}(X^h,Y^h)Z^h \\
&= \Big\{R(X,Y)Z + \tfrac{ab}{2\alpha}[2(\nabla_u R)(X,Y)Z - (\nabla_Z R)(X,Y)u] + \tfrac{a^2}{4\alpha}[R(R(Y,Z)u,u)X \\
&\quad -R(R(X,Z)u,u)Y - 2R(R(X,Y)u,u)Z] + \tfrac{a^2b^2}{4\alpha^2}\,[R(X,u)R(Y,u)Z \\
&\quad -R(Y,u)R(X,u)Z + R(X,u)R(Z,u)Y - R(Y,u)R(Z,u)X] \\
&\quad +\tfrac{ad(\alpha-b^2)}{4\alpha^2}\,[g(Z,u)R(X,Y)u + g(Y,u)R(X,u)Z - g(X,u)R(Y,u)Z] \\
&\quad +\tfrac{ab^2}{2\alpha^2}\left[-\tfrac{ad+b^2}{a+c+d}\,g(R(Y,u)Z,u)\; + d\,g(Y,u)g(Z,u)\right] R_u X \\
&\quad -\tfrac{ab^2}{2\alpha^2}\left[-\tfrac{ad+b^2}{a+c+d}\,g(R(X,u)Z,u) + d\,g(X,u)g(Z,u)\right] R_u Y \\
&\quad +\tfrac{d}{4\alpha}\left[-\tfrac{2b^2}{a+c+d}\,g(R(Y,u)Z,u) + d\,g(Y,u)g(Z,u)\right] X \\
&\quad -\tfrac{d}{4\alpha}\left[-\tfrac{2b^2}{a+c+d}\,g(R(X,u)Z,u) + d\,g(X,u)g(Z,u)\right] Y \\
&\quad +\tfrac{d}{4\alpha(a+c+d)}\{-4abg((\nabla_u R)(X,Y)Z,u) + a^2\,[g(R(Y,Z)u,R(X,u)u) \\
&\quad -g(R(X,Z)u,R(Y,u)u) - 2g(R(X,Y)u,R(Z,u)u)] + \tfrac{a^2b^2}{\alpha}\,[g(R(Y,u)Z \\
&\quad +R(Z,u)Y,R(X,u)u) - g(R(X,u)Z + R(Z,u)X,R(Y,u)u)] \\
&\quad -\left[\tfrac{ad(b^2-\alpha)}{\alpha} + \tfrac{2b^2d(\phi+2b^2)}{\phi(a+c+d)} + \tfrac{4b^2\alpha}{\phi}\right]\,[g(X,u)g(R(Y,u)Z,u) \\
&\quad -g(Y,u)g(R(X,u)Z,u)] - 3a(a+c)\,g(R(X,Y)Z,u) \\
&\quad +(a+c)d\,[g(X,u)g(Y,Z) - g(Y,u)g(X,Z)]\}u\Big\}^h \\
&\quad +\Big\{-\tfrac{b^2}{\alpha}\,(\nabla_u R)(X,Y)Z + \tfrac{a(a+c)}{2\alpha}(\nabla_Z R)(X,Y)u - \tfrac{ab}{4\alpha}\,[R(R(Y,Z)u,u)X \\
&\quad -R(R(X,Z)u,u)Y - 2R(R(X,Y)u,u)Z - R(X,R(Y,u)Z)u \\
&\quad -R(X,R(Z,u)Y)u + R(Y,R(X,u)Z)u + R(Y,R(Z,u)X)u] \\
&\quad -\tfrac{ab^3}{4\alpha^2}\,[R(X,u)R(Y,u)Z - R(Y,u)R(X,u)Z + R(X,u)R(Z,u)Y \\
&\quad -R(Y,u)R(Z,u)X] - \tfrac{bd(3\alpha-b^2)}{4\alpha^2}\,[g(Z,u)R(X,Y)u + g(Y,u)R(X,u)Z \\
&\quad -g(X,u)R(Y,u)Z] + \tfrac{b(b^2-\alpha)}{2\alpha^2}\,\left[\tfrac{ad+b^2}{a+c+d}\,g(R(Y,u)Z,u)\right. \\
&\quad \left.-d\,g(Y,u)g(Z,u)\right] R_u X - \tfrac{b(b^2-\alpha)}{2\alpha^2}\,\left[\tfrac{ad+b^2}{a+c+d}\,g(R(X,u)Z,u) - d\,g(X,u)g(Z,u)\right] \\
&\quad R_u Y + \tfrac{(a+c)bd}{2\alpha(a+c+d)}\,[g(R(Y,u)Z,u)X - g(R(X,u)Z,u)Y]\Big\}^{t_G},
\end{aligned}$$

(ii)

$$\begin{aligned}
&\tilde{R}(X^h,Y^{t_G})Z^h \\
&= \Big\{-\tfrac{a^2}{2\alpha}\,(\nabla_X R)(Y,u)Z + \tfrac{ab}{2\alpha}[R(X,Y)Z + R(Z,Y)X] \\
&\quad +\tfrac{a^3b}{4\alpha^2}\,[R(X,u)R(Y,u)Z - R(Y,u)R(X,u)Z - R(Y,u)R(Z,u)X] \\
&\quad +\tfrac{a^2bd}{4\alpha^2}\,[g(X,u)R(Y,u)Z - g(Z,u)R(X,Y)u] \\
&\quad -\tfrac{ab}{4\alpha^2(a+c+d)}\,[a(ad+b^2)\,g(R(Y,u)Z,u) + \alpha d\,g(Y,Z)]R_u X \\
&\quad +\tfrac{a^2b}{2\alpha^2}\,\left[\tfrac{ad+b^2}{a+c+d}\,g(R(X,u)Z,u) - d\,g(X,u)g(Z,u)\right] R_u Y \\
&\quad -\tfrac{bd}{4\alpha(a+c+d)}\,[a\,g(R(Y,u)Z,u) + (2(a+c)+d)\,g(Y,Z)]X \\
&\quad +\tfrac{b}{\alpha}\,\left[-\tfrac{ad+b^2}{2(a+c+d)}\,g(R(X,u)Z,u) + d\,g(X,u)g(Z,u)\right] Y
\end{aligned}$$

$$
\begin{aligned}
&-\tfrac{bd}{2\alpha}\, g(X,Y)Z + \tfrac{d}{4\alpha(a+c+d)}\Big\{ 2a^2\, g((\nabla_X R)(Y,u)Z,u) \\
&+\tfrac{a^3 b}{\alpha}\,[g(R(Y,u)Z,R(X,u)u) - g(R(X,u)Z + R(Z,u)X, R(Y,u)u)] \\
&+ab\,\left[-\tfrac{\alpha+\phi}{\alpha} + \tfrac{d}{a+c+d}\right]\, g(X,u)g(R(Y,u)Z,u) \\
&-2ab\,[2g(R(X,Y)Z,u) + g(R(Z,Y)X,u)] \\
&+bd\,\left[\left(3 - \tfrac{d}{a+c+d}\right)\, g(X,u)g(Y,Z) + 2\, g(Z,u)g(X,Y)\right]\Big\}\, u\Big\}^h \\
&+\Big\{ \tfrac{ab}{2\alpha}\,(\nabla_X R)(Y,u)Z + \tfrac{a^2}{4\alpha}\, R(X,R(Y,u)Z)u - \tfrac{a^2b^2}{4\alpha^2}\,[R(X,u)R(Y,u)Z \\
&-R(Y,u)R(X,u)Z - R(Y,u)R(Z,u)X] - \tfrac{b^2}{\alpha}\, R(X,Y)Z + \tfrac{a(a+c)}{2\alpha} R(X,Z)Y \\
&+\tfrac{ad(\alpha-b^2)}{4\alpha^2}\,[g(X,u)R(Y,u)Z - g(Z,u)R(X,Y)u] \\
&-\tfrac{\alpha-b^2}{4\alpha^2(a+c+d)}\,[a(ad+b^2)\, g(R(Y,u)Z,u) + \alpha d\, g(Y,Z)]R_u X \\
&+\tfrac{ab^2}{2\alpha^2}\,\left[-\tfrac{ad+b^2}{a+c+d}\, g(R(X,u)Z,u) + d\, g(X,u)g(Z,u)\right] R_u Y \\
&+\tfrac{(a+c)d}{4\alpha(a+c+d)}\,[a\, g(R(Y,u)Z,u) + (2(a+c)+d)\, g(Y,Z)]X \\
&+\tfrac{1}{4\alpha}\,\left[2b^2\left(2 - \tfrac{d}{a+c+d}\right)\, g(R(X,u)Z,u) - d(4(a+c)+d)\, g(X,u)g(Z,u)\,\right]Y \\
&+\tfrac{(a+c)d}{2\alpha}\, g(X,Y)Z\Big\}^{t_G},
\end{aligned}
$$

(iii)

$$
\begin{aligned}
\tilde{R}(X^{t_G},Y^{t_G})Z^{t_G} &= \tfrac{1}{2\alpha(a+c+d)}\,\Big\{\big\{a^2 b\, [g(Y,Z)R_u X - g(X,Z)R_u Y] \\
&\quad - b(\alpha+\phi)[g(Y,Z)X - g(X,Z)Y]\big\}^h + \big\{-ab^2\, [g(Y,Z)R_u X - g(X,Z)R_u Y] \\
&\quad +[(a+c)(\alpha+\phi) + \alpha d]\, [g(Y,Z)X - g(X,Z)Y]\,\big\}^{t_G}\,\Big\},
\end{aligned}
$$

for all $x \in M$, $(x,u) \in T_1M$ and all arbitrary vectors $X, Y, Z \in M_x$ satisfying Convention 2*, where $R_u X = R(X,u)u$ denotes the* Jacobi operator *associated to u.*

3. Curvature of g-natural contact metric structures

We briefly recall that a *contact manifold* is a $(2n-1)$-dimensional manifold $\bar{M}$ admitting a global 1-form η (a *contact form*) such that $\eta\wedge(d\eta)^{n-1} \neq 0$ everywhere on $\bar{M}$. Given η, there exists a unique vector field ξ, called the *characteristic vector field*, such that $\eta(\xi) = 1$ and $d\eta(\xi,\cdot) = 0$. Furthermore, a Riemannian metric g is said to be an *associated metric* if there exists a tensor φ, of type (1,1), such that

$$
\eta = g(\xi,\cdot)\,, \quad d\eta = g(\cdot,\varphi\cdot)\,, \quad \varphi^2 = -I + \eta\otimes\xi\,. \tag{3.1}
$$

(η, g, ξ, φ), or (η, g), is called a *contact metric structure* and $(\bar{M}, \eta, g)$ a *contact metric manifold*.

Sasakian contact metric structures are characterized by the property that the covariant derivative of its tensor φ satisfies

$$
(\nabla_Z\varphi)W = \bar{g}(Z,W)\xi - \eta(W)Z, \tag{3.2}
$$

for all Z, W vector fields on $\bar{M}$. A *K-contact manifold* is a contact metric manifold $(\bar{M}, \eta, \bar{g})$ whose characteristic vector field ξ is a Killing vector field with respect to $\bar{g}$. Any Sasakian manifold is K-contact and the converse also holds for three-dimensional spaces. The tensor $h = \frac{1}{2}\mathcal{L}_\xi \varphi$, where $\mathcal{L}$ denotes the Lie derivative, plays a very important role in describing the geometry of a contact metric manifold $(\bar{M}, \eta, g)$. K-contact spaces are characterized by equation $h = 0$. h is symmetric and satisfies

$$\nabla\xi = -\varphi - \varphi h, \quad h\varphi = -\varphi h, \quad h\xi = 0. \tag{3.3}$$

In [1], we investigated under which conditions a Riemannian g-natural metric on T_1M may be seen as a Riemannian metric associated to a very "natural" contact form. Given an arbitrary Riemannian g-natural metric $\tilde{G} = a.\widetilde{g^s} + b.\widetilde{g^h} + c.\widetilde{g^v} + d.\widetilde{k^v}$ over T_1M, the unit vector field $N^G_{(x,u)} = \frac{1}{\sqrt{(a+c+d)\phi}}[-b.u^h + (a+c+d).u^v]$, for all $(x,u) \in TM$, is normal to T_1M at any point (cf. Section 2). The tangent space to T_1M at (x,u) splits as

$$(T_1M)_{(x,u)} = \mathrm{Span}(\tilde{\xi}) \oplus \{X^h | X \perp u\} \oplus \{X^{t_G} | X \perp u\},$$

where we put

$$\tilde{\xi}_{(x,u)} = ru^h, \tag{3.4}$$

r being a positive constant. It should be noted the special role played by u^h in the decomposition of $(T_1M)_{(x,u)}$, and its geometrical meaning: for any vector $u = \sum_i u^i(\partial/\partial x^i)_x \in M_x$, we have $u^h_{(x,u)} = \sum_i u^i(\partial/\partial x^i)^h_{(x,u)}$, that is, u^h is the *geodesic flow* on TM. Henceforth, it is a "natural" choice to assume a vector parallel to u^h, as the characteristic vector field of a suitable contact metric structure. We consider the triple $(\tilde{\eta}, \tilde{\varphi}, \tilde{\xi})$, where $\tilde{\xi}$ is defined as in (3.4), $\tilde{\eta}$ is the 1-form dual to $\tilde{\xi}$ through $\tilde{G}$, and $\tilde{\varphi}$ is completely determined by $\tilde{G}(Z, \tilde{\varphi}W) = (d\tilde{\eta})(Z, W)$, for all Z, W vector fields on T_1M. Then, simply calculations show that

$$\left.\begin{array}{l} \tilde{\eta}(X^h) = \frac{1}{r}g(X,u), \\ \tilde{\eta}(X^{t_G}) = brg(X,u) \end{array}\right\} \tag{3.5}$$

and

$$\left.\begin{array}{l} \tilde{\varphi}(X^h) = \frac{1}{2r\alpha}\left[-bX^h + (a+c)X^{t_G} + \frac{bd}{a+c+d}g(X,u)u^h\right], \\ \tilde{\varphi}(X^{t_G}) = \frac{1}{2r\alpha}\left[-aX^h + bX^{t_G} + \frac{\phi}{a+c+d}g(X,u)u^h\right], \end{array}\right\} \tag{3.6}$$

for all $X \in M_x$. If (and only if)

$$\frac{1}{r^2} = 4\alpha = a + c + d \tag{3.7}$$

holds, then $\tilde{\eta}$ is well-defined and it is a contact form on T_1M, homothetic – with homothety factor r – to the classical contact form on T_1M (see, for example, [7] for a definition).

From (1.1) and (3.7) it follows $d = (a+c)(4a-1) - 4b^2$. So, among Riemannian g-natural metrics on T_1M, the ones satisfying (3.7) are contact metrics associated to the contact structures described by (3.4)–(3.6). In this way, we have proved the following:

Theorem 6. [1] *The set* $(\tilde{G}, \tilde{\eta}, \tilde{\varphi}, \tilde{\xi})$*, described by* (3.4) – (3.7)*, is a family of contact metric structures over* T_1M*, depending on three real parameters* a*,* b *and* c*.*

More details can be found in [1], where we also proved that the class of g-natural contact metric structures on T_1M is invariant under D-homothetic deformations.

At any point (x,u) of the contact metric manifold $(T_1M, \tilde{\eta}, \tilde{G})$, the tensor $\tilde{h} = \frac{1}{2}\mathcal{L}_{\tilde{\xi}}\tilde{\varphi}$ is described as follows:

$$\left.\begin{aligned} \tilde{h}(X^h) &= \tfrac{1}{4\alpha}\left[-(a+c)(X-g(X,u)u)^h + a(R_uX)^h - 2b(R_uX)^{t_G}\right], \\ \tilde{h}(X^{t_G}) &= \tfrac{1}{4\alpha}\left[-2bX^h + b\left(1+\tfrac{d}{a+c+d}\right)g(X,u)u^h + (a+c)X^{t_G} - a(R_uX)^{t_G}\right], \end{aligned}\right\} \tag{3.8}$$

for all $X \in M_x$, where $R_uX = R(X,u)u$ denotes the *Jacobi operator* associated to u.

Remark 1. Some contact metric properties of $(T_1M, \tilde{\eta}, \tilde{G})$ turn out to be related to the base manifold being an Osserman space. We briefly recall here that a Riemannian manifold (M,g) is called *globally Osserman* if the eigenvalues of the Jacobi operator R_u are independent of both the unit tangent vector $u \in M_x$ and the point $x \in M$. The well-known *Osserman conjecture* states that any globally Osserman manifold is locally isometric to a *two-point homogeneous space*, that is, either a flat space or a rank-one symmetric space. The complete list of rank-one symmetric spaces is formed by $\mathbb{RP}^n$, S^n, $C\mathbb{P}^n$, $H\mathbb{P}^n$, $Cay\mathbb{P}^2$ and their non-compact duals. Actually, the Osserman conjecture has been proved to be true for all manifolds of dimension $n \neq 16$ ([15], [20], [21]). Moreover, also in dimension 16, if (M,g) is a Riemannian manifold such that R_u admits at most two distinct eigenvalues (besides 0), then it is locally isometric to a two-point homogeneous space [22].

3.1. g-natural contact structures of constant ξ-sectional curvature

Let $(\bar{M}, \eta, \bar{g})$ be a contact metric manifold. The sectional curvature of plane sections containing the characteristic vector field ξ, is called *ξ-sectional curvature* (see Section 11.1 of [7]). Clearly, if π is a plane section containing ξ, we can determine the sectional curvature of π at a point $x \in \bar{M}$ as $K(Z, \xi_x)$, where Z is a vector of π_x, orthogonal to ξ_x. As it was proved in [19] (see also Theorem 7.2 of [7]), a contact metric manifold is K-contact if and only if it has constant ξ-sectional curvature equal to 1.

Proof of Theorem 1. We first suppose that $(T_1M, \tilde{\eta}, \tilde{G})$ has constant ξ-sectional curvature $\widetilde{K}$. Let (x,u) be a point of T_1M and Y a unit vector orthogonal to u. From (i) of Proposition 1, we get

$$\tilde{R}(\tilde{\xi}_{(x,u)}, Y^h)\tilde{\xi}_{(x,u)} = r^2\left\{-\tfrac{ab}{2\alpha}(\nabla_uR)(Y,u)u + \tfrac{3a^2}{4\alpha}R_u^2Y - (1+\tfrac{ad}{2\alpha})R_uY - \tfrac{d^2}{4\alpha}Y\right\}^h$$
$$+r^2\left\{\tfrac{b^2-\alpha}{2\alpha}(\nabla_uR)(Y,u)u - \tfrac{ab}{R}{}^2_{u}Y + \tfrac{bd}{\alpha}R_uY\right\}^{t_G}.$$

Therefore, taking into account (2.6), the sectional curvature of the plane spanned by $\tilde{\xi}_{(x,u)}$ and Y^h is given by

$$\begin{aligned}\tilde{K}(\tilde{\xi}_{(x,u)}, Y^h) = \tfrac{r^2}{a+c}\Big\{ & bg((\nabla_u R)(Y,u)u,Y) + \tfrac{a(b^2-3\alpha)}{4\alpha} g(R_u^2 Y, Y) \\ & + \Big[\big(1+\tfrac{ad}{2\alpha}\big)(a+c) - \tfrac{b^2 d}{\alpha}\Big] g(R_u Y, Y) + \tfrac{(a+c)d^2)}{4\alpha}\Big\}.\end{aligned} \tag{3.9}$$

In the same way, from (ii) of Proposition 1 and taking into account (2.6), we find that the sectional curvature of the plane spanned by $\tilde{\xi}_{(x,u)}$ and Y^{t_G} is given by

$$\tilde{K}(\tilde{\xi}_{(x,u)}, Y^{t_G}) = \tfrac{r^2}{a}\Big\{\tfrac{a^3}{4\alpha} g(R_u^2 Y, Y) - \tfrac{a^2 d}{2\alpha} g(R_u Y, Y) + \Big(d + \tfrac{ad^2}{4\alpha}\Big)\Big\}. \tag{3.10}$$

Since $(T_1 M, \tilde{\eta}, \tilde{G})$ has constant ξ-sectional curvature $\widetilde{K}$, from (3.9) and (3.10) we then have

$$\left.\begin{aligned} & bg((\nabla_u R)(Y,u)u,Y) + \tfrac{a(b^2-3\alpha)}{4\alpha} g(R_u^2 Y, Y) \\ & + \Big[\big(1+\tfrac{ad}{2\alpha}\big)(a+c) - \tfrac{b^2 d}{\alpha}\Big] ag(R_u Y, Y) + \tfrac{(a+c)d^2}{4\alpha} = 4\alpha(a+c)\widetilde{K}, \\ & \tfrac{a^3}{4\alpha} g(R_u^2 Y, Y) - \tfrac{a^2 d}{2\alpha} g(R_u Y, Y) + \Big(d + \tfrac{ad^2}{4\alpha}\Big) = 4\alpha a\widetilde{K}, \end{aligned}\right\} \tag{3.11}$$

for all orthogonal unit vectors u and Y. In particular, if Y is an eigenvector for the Jacobi operator R_u, then $R_u Y = \lambda Y$ and the second equation of (3.11) gives

$$a^3\lambda^2 - 2a^2 d\lambda + d(4\alpha + ad) = 16\alpha^2 a\widetilde{K}, \tag{3.12}$$

from which it follows that R_u has constant eigenvalues, both independent of u and the point x at M. Therefore, (M,g) is a globally Osserman space. Moreover, (3.12) also implies that R_u has at most two distinct eigenvalues and so, (M,g) is locally isometric to a two-point homogeneous space [22]. In particular, (M,g) is locally symmetric. So, from (3.11) we get that the eigenvalues λ of R_u must satisfy

$$\left.\begin{aligned} & a(b^2-3\alpha)\lambda^2 + 2\left[(2\alpha + ad)(a+c) - 4b^2\right]\lambda + (a+c)d^2 = 16\alpha^2(a+c)\widetilde{K}, \\ & a^3\lambda - 2a^2 d\lambda + d(4\alpha + ad) = 16\alpha^2 a\widetilde{K}. \end{aligned}\right\} \tag{3.13}$$

Taking into account (1.1), we can calculate $\widetilde{K}$ from both equations of (3.13) and compare these two expressions. In this way, we easily find

$$a^2\lambda^2 - a(a+c+d)\lambda + d(a+c) = 0,$$

which implies that the only possible values for λ are $\lambda_1 = \frac{a+c}{a}$ and $\lambda_2 = \frac{d}{a}$.

If λ_1 (respectively, λ_2) is the only nontrivial eigenvalue of the Jacobi operator R_u, then (M,g) has constant sectional curvature equal to λ_1 (respectively, λ_2). On the other hand, when R_u admits both eigenvalues λ_1 and λ_2, then for $\lambda = \frac{a+c}{a}$, (3.12) gives $4\alpha a\widetilde{K} = d + \frac{a(a+c-d)}{4\alpha}$, while for $\lambda = \frac{d}{a}$, (3.12) implies $4\alpha a\widetilde{K} = d$. Since the value of $\widetilde{K}$ is uniquely determined, we necessarily have $a+c-d=0$, that is, $d = a+c$ and so, $\lambda_1 = \lambda_2$ and (M,g) has again constant sectional curvature.

Conversely, assume that (M, g) has constant sectional curvature equal to $\frac{d}{a}$ (respectively, $\frac{a+c}{a}$). Then, (3.9) and (3.10) imply at once that $(T_1M, \tilde{\eta}, \tilde{G})$ has constant ξ-sectional curvature $\widetilde{K} = \frac{d}{4a\alpha}$ (respectively, $\widetilde{K} = \frac{d}{4a\alpha} + \frac{(a+c-d)^2}{16\alpha^2}$). □

In [26], D. Perrone investigated three-dimensional contact metric manifolds $(M^3, \eta, \bar{g})$ of constant ξ-sectional curvature. In particular, he characterized such spaces as contact metric manifolds of constant *scalar torsion* $||\tau||$ satisfying $\nabla_\xi \tau = 0$ [26], where the *torsion* $\tau := \mathcal{L}_\xi \bar{g}$ is the Lie derivative of $\bar{g}$ in the direction of the characteristic vector field ξ.

It is also interesting to remark that, among three-dimensional contact metric manifolds satisfying $\nabla_\xi \tau = 2\tau\varphi$, K-contact spaces are the only ones having constant ξ-sectional curvature ([26], Corollary 4.6). When M is compact, $\nabla_\xi \tau = 2\tau\varphi$ is a necessary and sufficient condition for an associated metric $g \in \mathcal{A}$, in order to be a critical point for the functional

$$L(g) = \int_{\bar{M}} \mathrm{Ric}(\xi) dV,$$

where $\mathrm{Ric}(\xi) = \varrho(\xi, \xi)$ and ϱ is the Ricci tensor of $\bar{M}$ [23]. On any contact metric manifold $(M, \eta, \bar{g})$, the torsion τ is related to the tensor h by the formula $\tau = 2\bar{g}(h\varphi\cdot, \cdot)$, from which it follows

$$\nabla_\xi \tau = 2\bar{g}((\nabla_\xi h)\varphi\cdot, \cdot),$$

and so, equations above can be expressed in terms of the tensor h. g-natural contact metric structures on T_1M satisfying these equations were classified in [1]. Taking into account Theorems 7 and 8 of [1] and Theorem 1 above, the following results follow easily.

Proposition 2. *If $(T_1M, \tilde{\eta}, \tilde{G})$ has constant ξ-sectional curvature, then $\tilde{\nabla}_{\tilde{\xi}} \tilde{h} = 0$.*

Corollary 1. *Let $\tilde{G}$ be a Riemannian g-natural metric on T_1M, such that $\tilde{\nabla}_{\tilde{\xi}} \tilde{h} = 2\tilde{h}\tilde{\varphi}$. Then, $(T_1M, \tilde{\eta}, \tilde{G})$ has constant ξ-sectional curvature if and only if it is K-contact.*

3.2. g-natural contact structures of constant φ-sectional curvature

Let $(\bar{M}, \eta, \bar{g}, \xi, \varphi)$ be a contact metric manifold and $Z \in \ker \eta$. The *φ-sectional curvature* determined by Z is the sectional curvature $K(Z, \varphi Z)$ along the plane spanned by Z and φZ. The φ-sectional curvature of a Sasakian manifold determines the curvature completely. A *Sasakian space form* is a Sasakian manifold of constant φ-sectional curvature. We refer to Section 7.3 of [7] for further details and results.

As concerns the standard contact metric structure of the unit tangent sphere bundle, the following result holds:

Theorem 7. [17] *If (M, g) has constant sectional curvature $\bar{c}$ and $\dim M \geq 3$, the standard contact metric structure of T_1M has constant φ-sectional curvature (equal to $(2 \pm \sqrt{5})^2$) if and only if $\bar{c} = 2 \pm \sqrt{5}$.*

Theorem 7 has been generalized for g-natural contact metric structures by Theorem 3. Before proving Theorems 2 and 3, we need to calculate the φ-sectional curvature of $(T_1M, \tilde{\eta}, \tilde{G})$. Note first that, when X is a tangent vector orthogonal to u, then, by (3.5), both X^h and X^{t_G} belong to $\ker \tilde{\eta}$. We have the following

Lemma 1. *Let $(\tilde{\eta}, \tilde{G})$ be a g-natural contact metric structure on T_1M, (x, u) a point of T_1M and X a unit vector tangent to M, orthogonal to u. Then,*

$$\begin{aligned} &K(X^h, \tilde{\varphi}X^h) = K(X^{t_G}, \tilde{\varphi}X^{t_G}) \\ &= -\frac{d}{\alpha}\left[1 - \frac{d}{4(a+c+d)}\right] - \frac{1}{2\alpha}\left[\frac{ad+2b^2}{a+c+d} + \frac{b^4}{\alpha(a+c+d)}\right] g(R_X u, u) \\ &\quad + \frac{a^3}{4\alpha^2} g(R_X^2 u, u) - \frac{a^2(ad+b^2)}{4\alpha^2(a+c+d)}\left[g(R_X u, u)\right]^2, \end{aligned} \tag{3.14}$$

where $R_X u = R(u, X)X$.

Proof of Lemma 1. Since X is orthogonal to u, from (3.6) we have

$$\tilde{\varphi}(X^h) = \tfrac{1}{2r\alpha}\left[-bX^h + (a+c)X^{t_G}\right], \quad \tilde{\varphi}(X^{t_G}) = \tfrac{1}{2r\alpha}\left[-aX^h + bX^{t_G}\right]. \tag{3.15}$$

Using (3.15), and taking into account that X is a unit vector, we get

$$K(X^h, \tilde{\varphi}X^h) = K(X^{t_G}, \tilde{\varphi}X^{t_G}) = -\tfrac{1}{\alpha}\tilde{G}(\tilde{R}(X^h, X^{t_G})X^h, X^{t_G}). \tag{3.16}$$

Since X and u are orthogonal, we get $\tilde{R}(X^h, X^{t_G})X^h = Y^h + W^{t_G}$, where we put

$$\begin{aligned} Y = &-\frac{a^2}{2\alpha}(\nabla_X R)(X, u)X - \frac{a^3 b}{4\alpha^2} R(X, u)R_X 5u \\ &+ \frac{ab}{4\alpha^2(a+c+d)}[a(ad+b^2)g(R_X u, u) - \alpha d]R_u X \\ &- \frac{bd}{4\alpha(a+c+d)}\left[ag(R_X u, u) + 2(a+c) + d\right] X \\ &- \frac{b(ad+b^2)}{2\alpha(a+c+d)} g(R_X u, u)X - \frac{bd}{2\alpha}X \\ &+ \frac{a^2 d}{4\alpha^2(a+c+d)}\left[2\alpha g((\nabla_X R)(X, u)X, u) - abg(R_X u, R_u X)\right] u, \end{aligned}$$

$$\begin{aligned} W = &\frac{ab}{2\alpha}(\nabla_X R)(X, u)X + \frac{a^2}{4\alpha} R(X, R_X u)u \\ &+ \frac{ab}{4\alpha^2(a+c+d)}[a(ad+b^2)g(R_X u, u) - \alpha d]R_u X \\ &+ \frac{a^2 b^2}{4\alpha^2} R(X, u)R_X u \\ &+ \frac{b^2 - \alpha}{4\alpha^2(a+c+d)}\left[a(ad+b^2)g(R_X u, u) + \alpha d\right] R_u X \end{aligned} \tag{3.17}$$

$$
\begin{aligned}
&- \frac{ab^2(ad+b^2)}{2\alpha^2(a+c+d)} g(R_X u,u)R_u X \\
&+ \frac{(a+c)d}{4\alpha(a+c+d)}[ag(R_X u,u)+2(a+c)+d]X \\
&+ \frac{b^2}{2\alpha}\left(2-\frac{d}{a+c+d}\right) g(R_X u,u)X + \frac{(a+c)d}{2\alpha}X.
\end{aligned}
$$

Taking into account (2.6) and Proposition 1, we have

$$
\tilde{G}(\tilde{R}(X^h,X^{t_G})X^h,X^{t_G}) = \tilde{G}(Y^h+W^{t_G},gX^{t_G}) = bg(Y,X)+ag(W,X).
$$

After some lengthy but very standard calculations, from (3.2) we then obtain

$$
\begin{aligned}
\tilde{G}(\tilde{R}(X^h,X^{t_G})X^h,X^{t_G}) = & d\left[1-\frac{d}{4(a+c+d)}\right] \\
&+ \left(\frac{ad+2b^2}{2(a+c+d)}+\frac{b^4}{2\alpha(a+c+d)}\right) g(R_X u,u) \qquad (3.18)\\
&- \frac{a^3}{4\alpha} g(R_X^2 u,u) + \frac{a^2(ad+b^2)}{4\alpha(a+c+d)}\left[g(R_X u,u)\right]^2.
\end{aligned}
$$

(3.14) now follows at once from (3.16) and (3.18), which completes the proof of Lemma 1. □

Proof of Theorem 2. Suppose that $(T_1M,\tilde{\eta},\tilde{G})$ has constant φ-sectional curvature $\widetilde{K}$. Note that (3.14) holds for any orthogonal unit vectors u and X. We now use (3.14) in the special case when u is an eigenvector of R_X, that is, $R_X u = \lambda u$. We get

$$
d\left(1-\tfrac{d}{4(a+c+d)}\right)+\tfrac{(ad+2b^2)\alpha+b^4}{2\alpha(a+c+d)}\lambda+\tfrac{a^2}{4(a+c+d)}\lambda^2=-\alpha\widetilde{K}
$$

and so, using (3.7),

$$
\alpha a^2\lambda^2+2\left[\alpha(ad+b^2)+b^4\right]\lambda+\left[\alpha d(4a+4c-3d)+4\alpha^2(a+c+d)\widetilde{K}\right]=0. \qquad (3.19)
$$

Hence, for any unit vector X tangent to M at a point p, the eigenvalues λ of R_X satisfy the second order equation (3.19), having constant coefficients independent from the point. Therefore, (M,g) is a globally Osserman space. Moreover, the Jacobi operator R_X has at most two (constant) nontrivial eigenvalues and so, (M,g) is locally isometric to a two-point homogeneous space. This completes the proof of Theorem 2. □

Note that the converse of Theorem 2 would provide an interesting characterization of two-point homogeneous spaces in terms of their unit tangent sphere bundles. However, the calculations involved are really hard. A partial result, which anyway extends Theorem 7 to an arbitrary g-natural contact metric structure, is given by Theorem 3.

Proof of Theorem 3. Using the fact that (M,g) has constant sectional curvature $\bar{c}$, very long calculations lead to conclude that, for an arbitrary unit vector u tangent to M at x and an arbitrary unit vector $Z = X^h + Y^{t_G}$ of the contact distribution $\ker \tilde{\eta}$ at $(x,u) \in T_1M$, the φ-sectional curvature of the plane generated by Z and $\tilde{\varphi}Z$, is given by

$$
\begin{aligned}
K(Z,\tilde{\varphi}Z) = & \frac{1}{(2r\alpha)^2}\Bigg\{A_1[(a+c)g_x(X,X)-1]g_x(X,X)+A_2 g_x(X,Y)^2 \\
& +A_3 g_x(X,X)g_x(X,Y)+A_4 g_x(X,Y) \\
& +\left[\frac{a^2}{4(a+c+d)}\bar{c}^2+\frac{1}{a+c+d}\left(ad+2b^2+\frac{b^4}{\alpha}\right)\bar{c}\right. \\
& \left.-d\left(1-\frac{d}{4(a+c+d)}\right)\right]\Bigg\},
\end{aligned}
\tag{3.20}
$$

where

$$
\left.
\begin{aligned}
A_1 = & \tfrac{a\alpha}{a+c+c}\bar{c}^2+\tfrac{1}{2a(a+c+d)}\left(-8\alpha^2-4ad\alpha-a^2(a+c)d\right. \\
& \left.+2b^4\left(1-\tfrac{b^2}{\alpha}\right)\right)\bar{c}-\tfrac{(a+c)\alpha}{a}\left(1+\tfrac{d}{a+c+d}\right), \\
A_2 = & a^3\left(1-\tfrac{3b^2}{2\alpha}\right)\bar{c}^2+2a\left(2-\tfrac{d}{a+c+d}\right)(b^2-\alpha)\bar{c} \\
& +(a+c)\alpha\left(1+\tfrac{d}{a+c+d}\right), \\
A_3 = & \tfrac{2ab\alpha}{a+c+d}\bar{c}^2-\tfrac{b}{a(a+c+d)}\left(6\alpha^2+\alpha(a(a+c)+4ad)+2\tfrac{b^6}{\alpha}\right)\bar{c} \\
& -\tfrac{2(a+c)b\alpha}{a}\left(1+\tfrac{d}{a+c+d}\right), \\
A_4 = & \tfrac{a^2(ad+b^2)b}{2(a+c+d)\alpha}\bar{c}^2-\tfrac{b\phi}{a+c+d}\bar{c}.
\end{aligned}
\right\}
\tag{3.21}
$$

Suppose now that T_1M has constant φ-sectional curvature $\widetilde{K}$ and so, the value of $K(Z,\tilde{\varphi}Z)$ is the same for all the unit tangent vectors $Z = X^h + Y^{t_G}$. Therefore, in (3.21) we must have $A_i = 0$ for all i, since they are coefficients of terms depending on X and Y. By (3.21) we then have

$$
\left.
\begin{aligned}
2a^2\bar{c}^2+\left(-8\alpha^2-4ad\alpha-a^2(a+c)d+2b^4\left(1-\frac{b^2}{\alpha}\right)\right)\bar{c} & \\
-2(a+c)\alpha\,(a+c+2d) & = 0, \\
a^3\left(1-\frac{3b^2}{2\alpha}\right)\bar{c}^2+2a\left(2-\frac{d}{a+c+d}\right)(b^2-\alpha)\bar{c} & \\
+(a+c)\alpha\left(1+\frac{d}{a+c+d}\right) & = 0, \\
b\left\{2a^2\alpha\bar{c}^2-2\left(6\alpha^2+\alpha(a(a+c)+4ad)+2\frac{b^6}{\alpha}\right)\bar{c}\right. & \\
\left.-4(a+c)\alpha\,(a+c+2d)\right\} & = 0, \\
b\bar{c}\left\{a^2(ad+b^2)\bar{c}-2\alpha\phi\right\} & = 0.
\end{aligned}
\right\}
\tag{3.22}
$$

From the last equation in (3.22), it follows that one of the following cases must occur:

a) $\bar{c}=0$. Taking into account $a+c>0$ and $\alpha>0$, in this case (3.22) reduces to

$$a+c+2d=0.$$

Hence, we have $d=-\frac{a+c}{2}$ and, by (3.7), $b=\pm\sqrt{(a+c)(a-\frac{1}{8})}$. Note that there exists a two-parameters family of g-natural contact structures on the unit tangent sphere bundle of a Euclidean space, for which the φ-sectional curvature is a constant $\widetilde{K}$. Taking into account $d=-\frac{a+c}{2}$ and $b=\pm\sqrt{(a+c)(a-\frac{1}{8})}$, it follows directly from (3.20) that $\widetilde{K}=5$.

b) $\bar{c}\neq 0$. Suppose first that $b=0$. Then, (3.22) becomes

$$\left.\begin{aligned} 2a^2\bar{c}^2+\left(-8\alpha^2-4ad\alpha-a^2(a+c)d\right)\bar{c}-2(a+c)\alpha\,(a+c+2d)=0,\\ a^3\left(1-\frac{3b^2}{2\alpha}\right)\bar{c}^2-2a\alpha\left(2-\frac{d}{a+c+d}\right)\bar{c}\\ +(a+c)\alpha\left(1+\frac{d}{a+c+d}\right)=0. \end{aligned}\right\} \tag{3.23}$$

Since, by (3.7), $a+c+d=4\alpha$, we can rewrite (3.23) in the following way:

$$\left.\begin{aligned} \frac{a}{4}\bar{c}^2-\left(a+c+\frac{5}{8}d\right)\bar{c}-(a+c)^2\left(\frac{a+c+2d}{4\alpha}\right)=0,\\ a^2\bar{c}^2-(a+c-d)\,\bar{c}-(a+c)^2\\ \left(\frac{a+c+2d}{4\alpha}\right)=0. \end{aligned}\right\} \tag{3.24}$$

Subtracting the two equations in (3.24), we then get

$$a(a-\tfrac{1}{4})\bar{c}^2-\tfrac{13}{8}d\bar{c}=0,$$

that is, since $\bar{c}\neq 0$, either $a=\frac{1}{4}$ and $d=0$, or $\bar{c}=\frac{13}{2a(4a-1)}$.

If $a=\frac{1}{4}$ and $d=0$ (which also includes the case of the standard contact metric structure of T_1M), then, by (3.24) we find $\widetilde{K}=(2\pm\sqrt{5})^2$. On the other hand, if $a\neq\frac{1}{4}$, we get a contradiction. In fact, using $\bar{c}=\frac{13}{2a(4a-1)}$ in the first equation of (3.24), we find $a=-\frac{53}{57}<0$, which can not occur.

Next, we shall assume $b\neq 0$. Through some very long calculations, we eventually find that this case does not occur. In fact, the fourth equation in (3.22) implies $a^2(ad+b^2)\bar{c}=2\alpha\phi$. Note that $ad+b^2\neq 0$, because $\alpha\phi>0$ by (1.1). Therefore, we obtain

$$\bar{c}=\frac{2\alpha\phi}{a^2(ad+b^2)}. \tag{3.25}$$

We use (3.25) in the first and third equations of (3.22) and we get

$$\left.\begin{aligned} &\frac{\alpha^2\phi^2}{a^3(ad+b^2)^2}-\frac{\phi}{4a^3(ad+b^2)}\left[8\alpha^2+4ad\alpha+a^2(a+c)d-2b^4\left(1-\frac{b^2}{\alpha}\right)\right]\\ &\qquad\qquad\qquad\qquad\qquad -\frac{(a+c)\alpha}{a}\left(1+\frac{d}{4\alpha}\right)=0,\\ &\frac{\alpha^2\phi^2}{a^3(ad+b^2)^2}-\frac{\phi}{4a^3(ad+b^2)}\left[6\alpha^2+a\alpha(a+c+4d)g+\frac{2b^6}{\alpha}\right]\\ &\qquad\qquad\qquad\qquad\qquad -\frac{(a+c)\alpha}{a}\left(1+\frac{d}{4\alpha}\right)=0, \end{aligned}\right\} \tag{3.26}$$

so that, subtracting the second equation of (3.26) from the first one, we easily obtain

$$\phi = 2b^2. \tag{3.27}$$

Therefore, taking into account (3.7), (3.25) and (3.26), we find

$$\begin{aligned} b^2 &= \tfrac{4a^2(a+c)}{3+4a}, \quad d = \tfrac{(a+c)(12a^2+8a-3)}{3+4a},\\ \alpha &= \tfrac{3a(a+c)}{3+4a}, \quad \bar{c} = \tfrac{16(a+c)}{(4a+3)(4a^2+4a-1)}. \end{aligned} \tag{3.28}$$

Next, comparing the first two equations in (3.22), we find

$$\begin{aligned} \left(\frac{a^2}{4}-a^3+\frac{3a^3b^2}{2\alpha}\right)\bar{c}^2 = \frac{1}{8\alpha}\Bigg[&8\alpha^2+4ad\alpha+a^2(a+c)d-2b^4\left(1-\frac{b^2}{\alpha}\right)\\ &+4a\,(8\alpha-2d)\,(b^2-\alpha)\Bigg]\,\bar{c}. \end{aligned} \tag{3.29}$$

Taking into account (3.28), after some calculations (3.29) becomes

$$a^2\left(2a^2-a+\tfrac{1}{4}\right)\bar{c} = \tfrac{a(a+c)}{72(4a+3)}\left(-2176a^3+5172a^2+1416a-333\right), \tag{3.30}$$

where we also used the fact that $\bar{c}\neq 0$. Since $\left(2a^2-a+\frac{1}{4}\right)\neq 0$, starting from (3.28) and (3.30) we eventually get

$$-8706a^5+11984a^4+28528a^3-3144a^2-1596a+45=0. \tag{3.31}$$

Next, using (3.28) and (3.31), from the first equation of (3.26) we obtain

$$\begin{aligned} &22079868576000a^4+28745654903552a^3-3956388519552a^2\\ &\qquad\qquad -1621058224320a-70597445616=0. \end{aligned} \tag{3.32}$$

Applying the Descartes rule for the sign of the roots of the polynomials in (3.31) and (3.32), we can deduce that the polynomial in (3.31) admits at most three positive roots, while the one in (3.32) admits at most one positive root. We checked that positive solutions of (3.31) belong to $]0,\frac{1}{10}[\,\cup\,]\frac{7}{24},\frac{1}{3}[\,\cup\,]2,+\infty[$, while the positive solution of (3.32) belongs to $]\frac{1}{4},\frac{7}{24}[$. Therefore, (3.31) and (3.32) are never satisfied simultaneously and so, this case can not occur $\square$

3.3. g-natural contact structures of T_1M whose tensor l annihilates the vertical distribution

The $(1,1)$-tensor field l on $\bar{M}$, defined by $l(X) = R(X, \xi, \xi)$ for all $X \in \mathfrak{X}(M)$, naturally appears in the study of the geometry of $(\bar{M}, \eta, g)$. For example, K-contact spaces are characterized by the equation $l = -\varphi^2$. If $l = 0$, then sectional curvatures of all planes containing ξ are equal to zero. We may refer to [24] for these and further results on l. Note that there are many contact metric manifolds satisfying $l = 0$ ([7], p. 153).

D. Blair [5] proved that T_1M, equipped with its standard contact metric structure $(\eta, \bar{g})$, satisfies $lU = 0$ for all vertical vector field U on T_1M if and only if the base manifold (M, g) is flat. Moreover, in this case ξ is a *nullity vector field*, that is, $R(Z, W)\xi = 0$ for all $Z, W \in \mathfrak{X}(T_1M)$. We extend these results to any g-natural contact metric structure $(\tilde{\eta}, \tilde{G})$ over T_1M, proving the following

Theorem 8. *Let $\tilde{G}$ be a Riemannian g-natural metric on T_1M. $(T_1M, \tilde{\eta}, \tilde{G})$ satisfies $lU = \tilde{R}(U, \tilde{\xi})\tilde{\xi} = 0$ for all vertical vector fields U on T_1M if and only if $d = 0$ and the base manifold (M, g) is flat. Moreover, in this case $\tilde{R}(Z, W)\tilde{\xi} = 0$ for all vector fields Z, W on T_1M.*

Proof. Assume first that $(T_1M, \tilde{\eta}, \tilde{G})$ satisfies $lU = 0$ for all vertical vector fields U on T_1M. Then, in particular,

$$\tilde{R}(Y^{t_G}, \tilde{\xi}_{(x,u)})\tilde{\xi}_{(x,u)} = 0, \tag{3.33}$$

for all Y orthogonal to u, at any point $(x, u) \in T_1M$. Using formula (ii) of Proposition 1 to express $\tilde{R}(Y^{t_G}, \tilde{\xi}_{(x,u)})\tilde{\xi}_{(x,u)}$, (3.33) implies

$$\begin{aligned} &\left\{-\tfrac{a^2}{2\alpha}(\nabla_u R)(Y, u)u - \tfrac{ab}{2\alpha}R_u Y + \tfrac{bd}{2\alpha}Y\right\}^h \\ &+\left\{\tfrac{ab}{2\alpha}(\nabla_u R)(Y, u)u - \tfrac{a^2}{4\alpha}R_u^2 Y + \tfrac{ad+2b^2}{2\alpha}R_u Y - \tfrac{d(4(a+c)+d)}{4\alpha}Y\right\}^{t_G} = 0. \end{aligned} \tag{3.34}$$

In (3.34), the tangential part is the tangential lift of a vector Z orthogonal to u. Hence, $Z^{t_G} = Z^v$ and the horizontal and vertical parts of (3.34) both vanish. So, we get

$$\left.\begin{aligned} &-\tfrac{a^2}{2\alpha}(\nabla_u R)(Y, u)u - \tfrac{ab}{2\alpha}R_u Y + \tfrac{bd}{2\alpha}Y = 0, \\ &\tfrac{ab}{2\alpha}(\nabla_u R)(Y, u)u - \tfrac{a^2}{4\alpha}R_u^2 Y + \tfrac{ad+2b^2}{2\alpha}R_u Y - \tfrac{d(4(a+c)+d)}{4\alpha}Y = 0. \end{aligned}\right\} \tag{3.35}$$

From the first formula in (3.35), taking into account $a > 0$, we obtain

$$(\nabla_u R)(Y, u)u = \tfrac{2b}{a^2}(dY - aR_u Y). \tag{3.36}$$

Using (3.36) in the second formula of (3.35), we easily get

$$a^3 R_u^2 Y - 2a^2 d R_u Y + d(ad + 4\alpha)Y = 0, \tag{3.37}$$

for all Y orthogonal to u. (3.37) implies at once that R_u has at most two distinct (constant) eigenvalues. Therefore, (M,g) is globally Osserman and, since there are at most two distinct eigenvalues for R_u, (M,g) is locally isometric to a two-point homogeneous space [22]. In particular, (M,g) is locally symmetric. So, $\nabla R = 0$ and (3.36) implies either $b = 0$ or $R_u Y = \frac{d}{a} Y$, for all Y orthogonal to u. We treat these two cases separately.

If $R_u Y = \frac{d}{a} Y$ for all Y orthogonal to u, then (M,g) has constant sectional curvature $\tilde{c} = \frac{d}{a}$. From (3.37) we then get at once $4\alpha d = 0$. By (1.1), $\alpha > 0$ and so, $d = 0$. Moreover, (M,g) has constant sectional curvature $\tilde{c} = \frac{d}{a} = 0$, that is, it is flat.

Next, assume $b = 0$. If (M,g) has constant sectional curvature, we find a special case of the previous one. In fact, we find at once that $d = 0$ and (M,g) is flat. Thus, in the sequel we assume (M,g) is locally isometric to a two-point homogeneous space of non-constant sectional curvature, and we prove that this case can not occur. Since $b = 0$, by (1.1) we have $\alpha = a(a+c)$ and (3.37) becomes

$$a^2 R_u^2 Y - 2ad R_u Y + d(d + 4(a+c))Y = 0,$$

for all Y orthogonal to u. Hence, the eigenvalues λ of R_u satisfy

$$a^2 \lambda^2 - 2ad\lambda + d(d + 4(a+c)) = 0. \tag{3.38}$$

Since (M,g) is locally isometric to a two-point homogeneous space of non-constant sectional curvature, R_u has two (constant) eigenvalues λ_1, λ_2 with $\lambda_2 = 4\lambda_1$ or conversely [15]. We now calculate the eigenvalues of R_u from (3.38) and we obtain

$$\lambda_1 = \tfrac{d - 2\sqrt{-d(a+c)}}{a}, \quad \lambda_2 = \tfrac{d + 2\sqrt{-d(a+c)}}{a}. \tag{3.39}$$

Imposing that $\lambda_2 = 4\lambda_1$ or conversely, we easily find that either $d = 0$ or $d = -\frac{100}{9}(a+c)$. If $d = 0$, then (3.39) implies $\lambda_1 = \lambda_2 = 0$, that is, (M,g) is flat, against the assumption that (M,g) has not constant sectional curvature. On the other hand, if $d = -\frac{100}{9}(a+c)$, then (3.7) gives $\frac{1}{r^2} = a + c + d = -\frac{91}{9}(a+c) < 0$, which is a contradiction.

Conversely, assume now that $d = 0$ and (M,g) is flat. Then, (3.34) gives at once that $\tilde{R}(\tilde{\xi}, Y^{t_G})\tilde{\xi} = 0$ for all Y orthogonal to u, that is, $lU = 0$ for any vertical vector U.

Finally, we assume $d = 0$ and (M,g) is flat and use Proposition 1 to calculate $\tilde{R}(Z,W)\tilde{\xi}$ for any vector fields Z, W on T_1M. Because of (3.4), equations (i) and (ii) in Proposition 1 give at once $\tilde{R}(X^h, Y^h)\tilde{\xi} = \tilde{R}(X^h, Y^{t_G})\tilde{\xi} = 0$ for all X and $Y \in \mathfrak{X}(M)$ (satisfying Convention 1). Next, by equations (ii) and (iii) in Proposition 1, we find

$$\tilde{G}(\tilde{R}(X^{t_G}, Y^{t_G})\tilde{\xi}, Z^h) = -\tilde{G}(X^{t_G}, \tilde{R}(\tilde{\xi}, Y^{t_G})Z^h) = 0,$$
$$\tilde{G}(\tilde{R}(X^{t_G}, Y^{t_G})\tilde{\xi}, Z^{t_G}) = -\tilde{G}(\tilde{\xi}, \tilde{R}(X^{t_G}, Y^{t_G})Z^{t_G}) = 0,$$

for all X, Y, Z vector fields on T_1M (satisfying Convention 1).

Therefore, $\tilde{R}(Z,W)\tilde{\xi} = 0$ for all $Z, W \in \mathfrak{X}(T_1M)$ □

3.4. g-natural contact structures for which T_1M is a (k,μ)-space

A (k,μ)-*space* $(\bar{M},\eta,\bar{g})$ is a contact metric manifold whose characteristic vector field ξ belongs to the so-called (k,μ)-*nullity distribution*, that is, satisfies

$$R(Z,W)\xi = k(\eta(W)Z - \eta(Z)W) + \mu(\eta(W)hZ - \eta(Z)hW), \tag{3.40}$$

for some real constants k and μ and for all vector fields Z, W on $\bar{M}$. We can refer to [7] for a survey on (k,μ)-spaces. Here we just recall that they generalize Sasakian manifolds, and that non-Sasakian (k,μ)-spaces have been completely classified [9]. Note that on any (k,μ)-space we have $k \leq 1$, and $k = 1$ if and only if $(\bar{M},\eta,\bar{g})$ is Sasakian. Moreover, any (k,μ)-space is a strongly pseudo-convex CR-manifold [7]. Recall that a *strongly pseudo-convex CR-structure* on a manifold $\bar{M}$ is a contact form η together with an integrable complex structure J on the contact subbundle $D := \ker\eta$ (i.e., a bundle map $J : D \to D$ such that $J^2 = -I$), such that the associated *Levi form* L_η, defined by

$$L_\eta(X,Y) = -d\eta(X,JY), \quad X, Y \in D,$$

is definite positive. In this case, $(\bar{M},\eta,J)$ is called a *strongly pseudo-convex manifold.*

The following result was proved in [1]:

Theorem 9. [1] *Let $\tilde{G}$ be a Riemannian g-natural metric on T_1M. $(T_1M,\tilde{\eta},\tilde{G})$ gives rise to a strongly pseudo-convex CR-structure if and only if the base manifold (M,g) has constant sectional curvature.*

It was proved in [8] that T_1M, equipped with its standard contact metric structure $(\eta,\bar{g})$, is a (k,μ)-space if and only if the base manifold (M,g) has constant curvature $\bar{c}$. In this case, $k = \bar{c}(2-\bar{c})$ and $\mu = -2\bar{c}$.

Proof of Theorem 4. Assume first that $(T_1M,\tilde{\eta},\tilde{G})$ is a (k,μ)-space. Then, $(T_1M,\tilde{\eta},\tilde{G})$ gives rise to a strongly pseudo-convex CR-structure and so, by Theorem 9, the base manifold (M,g) has constant sectional curvature $\bar{c}$.

To prove the converse, we use Proposition 1 to calculate $\tilde{R}(X^h,Y^h)\tilde{\xi}_{(x,u)}$, $\tilde{R}(X^h,Y^{t_G})\tilde{\xi}_{(x,u)}$ and $\tilde{R}(X^{t_G},Y^{t_G})\tilde{\xi}_{(x,u)}$ for all $(x,u) \in T_1M$ and $X, Y \in M_x$, and we see if there exist some values of k and μ for which (3.40) is satisfied. When both X and Y are orthogonal to u, standard calculations show that $\tilde{R}(X^h,Y^h)\xi_{(x,u)} = \tilde{R}(X^h,Y^{t_G})\xi_{(x,u)} = \tilde{R}(X^{t_G},Y^{t_G})\xi_{(x,u)} = 0$ and so, (3.40) is trivially satisfied. Since we only need to consider tangential lifts of vectors orthogonal to u (see Convention 2), it will be enough to consider $\tilde{R}(X^h,\tilde{\xi})\tilde{\xi}$ and $\tilde{R}(X^{t_G},\tilde{\xi})\tilde{\xi}$. Moreover, because of the symmetries of the curvature tensor, we clearly can assume X^h orthogonal to $\tilde{\xi}_{(x,u)}$, that is, X orthogonal to u.

From (i) of Proposition 1, taking into account (3.4) and the fact that (M,g) has constant sectional curvature $\bar{c}$, we have

$$\begin{aligned}\tilde{R}(X^h,\tilde{\xi}_{(x,u)})\tilde{\xi}_{(x,u)} = {} & \frac{r^2}{4\alpha}\left[-3a^2\bar{c}^2 + (4\alpha + 2ad)\bar{c} + d^2\right]X^h \\ & + \frac{r^2}{\alpha}\left[ab\bar{c}^2 - bd\bar{c}\right]X^{t_G}\end{aligned} \tag{3.41}$$

for any vector X orthogonal to u. On the other hand, from (3.8) we get at once

$$\begin{aligned} k[\tilde{\eta}(\tilde{\xi}_{(x,u)})X^h - \tilde{\eta}(X^h)\tilde{\xi}_{(x,u)}] + \mu[\tilde{\eta}(\tilde{\xi}_{(x,u)})hX^h - \tilde{\eta}(X^h)h\tilde{\xi}_{(x,u)}] \\ = \left[k + \frac{a\bar{c} - (a+c)}{4\alpha}\mu\right] X^h - \frac{b\bar{c}}{2\alpha}\mu X^{t_G}. \end{aligned} \tag{3.42}$$

Comparing (3.41) with (3.42) and taking into account (3.7), we get

$$\begin{aligned} &\tfrac{1}{16\alpha^2}\left\{-3a^2\bar{c}^2 + (4\alpha + 2ad)\bar{c} + d^2 - 16\alpha^2 k - 4\alpha[a\bar{c} - (a+c)]\mu\right\} X^h \\ &+ \tfrac{1}{4\alpha^2}\left(ab\bar{c}^2 - bd\bar{c} + 2b\bar{c}\mu\right) X^{t_G} = 0. \end{aligned} \tag{3.43}$$

Since X is orthogonal to u, $X^{t_G} = X^v$ and so, the horizontal and vertical parts of (3.43) both vanish, that is,

$$\left.\begin{aligned} -3a^2\bar{c}^2 + (4\alpha + 2ad)\bar{c} + d^2 = 16\alpha^2 k + 4\alpha[a\bar{c} - (a+c)]\mu, \\ 5ab\bar{c}^2 - bd\bar{c} = -2\alpha b\bar{c}\mu. \end{aligned}\right\}. \tag{3.44}$$

In the same way, from (ii) of Proposition 1 and from (3.8), we respectively get

$$\begin{aligned} \tilde{R}(X^{t_G}, \tilde{\xi}_{(x,u)})\tilde{\xi}_{(x,u)} = {} & \frac{r^2}{\alpha}\left[ab\bar{c} - bd\right] X^h \\ & + \frac{r^2}{4\alpha}\left[a^2\bar{c}^2 - 2(ad + 2b^2)\bar{c} + d(4(a+c) + d)\right] X^{t_G} \end{aligned} \tag{3.45}$$

and

$$\begin{aligned} k[\tilde{\eta}(\tilde{\xi}_{(x,u)})X^{t_G} - \tilde{\eta}(X^{t_G})\tilde{\xi}_{(x,u)}] + \mu[\tilde{\eta}(\tilde{\xi}_{(x,u)})hX^{t_G} - \tilde{\eta}(X^{t_G})h\tilde{\xi}_{(x,u)}] \\ = -\frac{b}{2\alpha}\mu X^h + \left[k - \frac{a\bar{c} - (a+c)}{4\alpha}\mu\right] X^{t_G}. \end{aligned} \tag{3.46}$$

Comparing (3.45) and (3.46), taking into account (3.7) we find

$$\left.\begin{aligned} ab\bar{c} - bd = -2\alpha b\mu, \\ a^2\bar{c}^2 - 2(2b^2 - ad)\bar{c} + d[4(a+c) + d] = 16\alpha^2 k - 4\alpha[a\bar{c} - (a+c)]\mu. \end{aligned}\right\} \tag{3.47}$$

Therefore, $(T_1M, \tilde{\eta}, \tilde{G})$ is a (k,μ)-space if both (3.44) and (3.47) hold. The first equation of (3.47) implies that either $b = 0$ or $\mu = \frac{1}{2\alpha}(d - a\bar{c})$. When $\mu = \frac{1}{2\alpha}(d - a\bar{c})$, from (3.47) we also get

$$k = \tfrac{1}{16\alpha^2}\left[-a^2\bar{c}^2 + 2(\alpha - b^2)\bar{c} + d(2(a+c) + d)\right]$$

and so, (1.2) is satisfied. If $b = 0$, from (3.44) and (3.47) we find again (1.2), unless $a\bar{c} - (a+c) = 0$, that is, $\bar{c} = \frac{a+c}{a}$. In this last case, $k = 1$ and μ is undetermined. In fact, by Theorem 2 of [1], we have that $b = 0$ and $\bar{c} = \frac{a+c}{a}$ if and only if $(T_1M, \tilde{\eta}, \tilde{G})$ is Sasakian. $\square$

In [9], E. Boeckx showed that non-Sasakian (k,μ)-spaces are determined, up to isometries, by the value of the invariant

$$I_{(k,\mu)} = \frac{1-\mu/2}{\sqrt{1-k^2}}.$$

Using (1.2), we can determine $I_{(k,\mu)}$ for all g-natural contact metric structures corresponding to non-Sasakian (k,μ)-spaces. Taking into account Theorem 4, standard calculations lead to the following

Theorem 10. *Let (M,g) be a Riemannian manifold of constant sectional curvature $\bar{c}$, and (k,μ) any real pair with $k<1$. There exists a g-natural contact metric structure $(\tilde{\eta},\tilde{G})$ such that $(T_1M,\tilde{\eta},\tilde{G})$ is a (k,μ)-space if and only if*

(i) *either $I_{(k,\mu)} > -1$ and $(I^2_{(k,\mu)}-1)\bar{c} > 0$, or*

(ii) *$I_{(k,\mu)} = 1$ and $\bar{c} = 0$.*

In particular, all non-Sasakian (k,μ)-spaces such that $I_{(k,\mu)} > -1$, can be realized as g-natural contact metric structures on a Riemannian manifold (M,g) of (suitable) constant sectional curvature.

3.5. Locally symmetric g-natural contact structures

Locally symmetric spaces are one of the main topics in Riemannian geometry. In the framework of contact metric geometry, local symmetry has been extensively investigated, obtaining many rigidity results. As concerns the unit tangent sphere bundle, Blair proved the following

Theorem 11. [6] *$(T_1M,\eta,\bar{g})$ is locally symmetric if and only if either (M,g) is flat or it is a surface of constant sectional curvature* 1.

Theorem 11 has been extended by replacing local symmetry by semi-symmetry ([10], [14]). Recently, Boeckx and Cho [11] showed definitively the rigidity of the hypothesis of local symmetry in contact Riemannian geometry, by proving the following

Theorem 12. [11] *A locally symmetric contact metric space is either Sasakian and of constant curvature* 1, *or locally isometric to the unit tangent sphere bundle of a Euclidean space with its standard contact metric structure.*

Proof of Theorem 5. Because of Theorem 12, it is enough to show that $(T_1M,\tilde{\eta},\tilde{G})$ can not be a Sasakian space of constant curvature 1. As the authors proved in [1], $(T_1M,\tilde{\eta},\tilde{G})$ is Sasakian if and only if $b=0$ and (M,g) has constant sectional curvature $\bar{c}=\frac{a+c}{a}>0$. Moreover, $(T_1M,\tilde{\eta},\tilde{G})$ has now constant sectional curvature 1 and so,

$$\tilde{R}(V,W)Z = \tilde{G}(W,Z)V - \tilde{G}(V,Z)W, \tag{3.48}$$

for all V,W,Z tangent vectors to T_1M at the same point of T_1M. Consider now a point (x,u) of T_1M and two unit vectors X,Y orthogonal to u and to each other.

We calculate $\tilde{R}(X^{t_G}, Y^{t_G})Y^{t_G}$ both using (3.48) and formula (iii) of Proposition 1, and we easily obtain

$$aX^{t_G} = \tilde{R}(X^{t_G}, Y^{t_G})Y^{t_G} = \frac{(a+c)(\alpha+\phi)+\alpha d}{2\alpha(a+c+d)}, \tag{3.49}$$

where we used the fact that $b = 0$. Taking into account (3.7), from (3.49) we obtain $a = 1$ and so, again by (3.7), $d = 3(c+1)$.

Next, from (3.48) and (ii) of Proposition 1 we also get

$$\begin{aligned} 0 = \tilde{R}(X^h, X^{t_G})Y^h = \Big\{ & \frac{a^2}{4\alpha}R(X, R(X,u)Y)u + \frac{a(a+c)}{2\alpha}R(X,Y)X \\ & + \frac{(a+c)d}{2\alpha}Y - \frac{a^2 d}{4\alpha(a+c+d)}g(R(X,u)Y,u)R_u X \\ & + \frac{ad(a+c)}{4\alpha(a+c+d)}g(R(X,u)Y,u)X \Big\}^{t_G}. \end{aligned} \tag{3.50}$$

Since (M, g) has constant sectional curvature $\bar{c}$, (3.50) easily yields

$$\frac{(a+c)(d-a-c)}{2\alpha} Y^{t_G} = 0, \tag{3.51}$$

for any Y (orthogonal to u and X). Therefore, since $a + c > 0$, we necessarily have $d - a - c = 0$, that is, $d = a + c = 1 + c$. On the other hand, we already found $d = 3(c+1)$ and so, $a + c = 1 + c = 0$, which contradicts (1.1). $\square$

D. Perrone [25] proved that a locally symmetric contact metric manifold $(\bar{M}, \eta, g, \xi, \varphi)$ satisfies $\nabla_\xi h = 0$. g-natural contact metric structures satisfying $\tilde{\nabla}_{\tilde{\xi}}\tilde{h} = 0$ have been classified in [1]. Taking into account the proof of Theorem 8 of [1] and Theorem 5 above, we prove the following

Corollary 2. *A g-natural contact metric structure $(\tilde{\eta}, \tilde{G})$ on T_1M satisfies $\tilde{\nabla}_{\tilde{\xi}}\tilde{h} = 0$ but is not locally symmetric if and only if*

- *either (M, g) is flat and $d = 0$ but $\tilde{G} \neq \bar{g}$, or*
- *(M, g) has constant curvature $\bar{c} > 0$ and $\tilde{G} = a.\widetilde{g^s} + (\bar{c}-1)a.\widetilde{g^v} + \bar{c}a(4a-1).\widetilde{k^v}$, or*
- *(M, g) is locally isometric to a compact rank-one symmetric space (of non-constant sectional curvature and Jacobi eigenvalues $(p, 4p)$ with $p > 0$), and either $\tilde{G} = a.\widetilde{g^s} + (p-1)a.\widetilde{g^v}4pa.\widetilde{k^v}$ or $\tilde{G} = a.\widetilde{g^s} + (4p-1)a.\widetilde{g^v} + pa.\widetilde{k^v}$.*

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