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CR-Warped Product Submanifolds of Nearly Kaehler Manifolds

Bayram Şahin **Rıfat Güneş**

Department of Mathematics, Inonu University
44280, Malatya, Turkey
e-mail: bsahin@inonu.edu.tr rgunes@inonu.edu.tr

Abstract. In this paper, we study warped product CR-submanifolds of nearly Kaehler manifolds. We extend some results of B. Y. Chen [7] on warped product CR-submanifolds of Kaehler manifolds to warped product CR-submanifolds of nearly Kaehler manifolds. We also give an example for such submanifolds in six dimensional sphere.

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1. Introduction

Let $(\bar{M}, g)$ be an almost Hermitian manifold. This means [22] that $\bar{M}$ admits a tensor field J of type (1, 1) on $\bar{M}$ such that, $\forall X, Y \in \Gamma(T\bar{M})$, we have

$$J^2 = -I, \quad g(X, Y) = g(JX, JY). \tag{1.1}$$

An almost Hermitian manifold $\bar{M}$ is called nearly Kaehler manifold if

$$(\bar{\nabla}_X J)X = 0, \ \forall X \in \Gamma(T\bar{M}), \tag{1.2}$$

which is equivalent to

$$(\bar{\nabla}_X J)Y + (\bar{\nabla}_Y J)X = 0, \ \forall X, Y \in \Gamma(T\bar{M}), \tag{1.3}$$

where $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$. T. Fukami and S. Ishihara [12] proved that there exists a nearly Kaehlerian structure on a six dimensional sphere

S^6 by making use of the properties of the Cayley division algebra. This structure is called as the canonical nearly Kaehlerian structure on S^6. In fact, A. Gray [13] showed that S^6 has some almost complex structures, which are different from the canonical almost complex structures, but nonetheless nearly Kaehlerian.

CR-submanifolds of almost Hermitian manifolds were defined by A. Bejancu [3] as a generalization of complex and totally real submanifolds. A CR-submanifold is called *proper* if it is neither complex nor totally real submanifold. The geometry of CR-submanifolds has been studied in several papers since then. We note that the geometry of CR-submanifolds of nearly Kaehler manifolds has been also studied by several authors, [1], [2], [10], [15], [17], [18], [21].

On the other hand, in [4], R. L. Bishop and B. O'Neill introduced a class of warped product manifolds as follows: Let (B, g_1) and (F, g_2) be two Riemannian manifolds, $f : B \to (0, \infty)$ and $\pi : B \times F \to B$, $\eta : B \times F \to F$ the projection maps given by $\pi(p, q) = p$ and $\eta(p, q) = q$ for every $(p, q) \in B \times F$. The warped product $M = B \times F$ is the manifold $B \times F$ equipped with the Riemannian structure such that

$$g(X, Y) = g_1(\pi_* X, \pi_* Y) + (f o\pi)^2 g_2(\eta_* X, \eta_* Y)$$

for every X and Y of M and $*$ is symbol for the tangent map. The function f is called the warping function of the warped product manifold. In particular, if the warping function is constant, then the manifold M is said to be trivial. Let X, Y be vector fields on B and V, W vector fields on F, then from Lemma 7.3 of [4], we have

$$\nabla_X V = \nabla_V X = (\frac{Xf}{f})V \tag{1.4}$$

where ∇ is the Levi-Civita connection on M.

Recently, B. Y. Chen [7], [8] considered warped product CR-submanifolds of Kaehler manifolds and showed that there do not exist warped product CR-submanifolds in the form $M_\perp \times_f M_T$ such that M_T is a holomorphic (complex) submanifold and $M_\perp$ is a totally real submanifold of a Kaehler manifold $\bar{M}$. Then he introduced the notion of CR-warped products of Kaehler manifolds as follows: A submanifold of a Kaehler manifold is called CR-warped product if it is the warped product $M_T \times_f M_\perp$ of a holomorphic submanifold M_T and a totally real submanifold $M_\perp$. He also established a sharp relationship between the warping function f of a warped product CR-submanifold $M_T \times_f M_\perp$ of a Kaehler manifold $\bar{M}$ and the squared norm of the second fundamental form $\| h \|^2$. After the papers of B. Y. Chen, CR-warped product submanifolds have been studied in various manifolds [5], [14], [19], [20].

It is known that every Kaehler manifold is a nearly Kaehler but the converse is not true in general. Therefore, the aim of this paper is to extend the concept of CR-warped product submanifolds of Kaehler manifolds to warped product CR-submanifolds of nearly Kaehler manifolds. Our main results improve some results of B. Y. Chen to CR-warped product submanifolds of nearly Kaehler manifolds. First, we prove that there exist no warped product CR-submanifolds in the form $M_\perp \times_f M_T$ such that $M_\perp$ is a totally real submanifold and M_T is a holomorphic submanifold of a nearly Kaehler manifold $\bar{M}$. Then we consider warped product CR-submanifolds in the form $M_T \times_f M_\perp$ in a nearly Kaehler manifold $\bar{M}$ and give an example. We prove a characterization theorem and obtain necessary condition for CR-warped product submanifolds to be a CR-product. We obtain a sharp inequality for the squared norm of the second fundamental form in terms of the warping function for CR-warped product submanifolds of nearly Kaehler manifolds.

We note that although the proof of Theorem 4.1 is same as that of Theorem 5.1 in [7], the proof of preparatory lemmas for the proof of Theorem 4.1 is different.

2. Preliminaries

In this section, we will recall the canonical nearly Kaehlerian structure on a 6-dimensional unit sphere S^6. We will also brief some formulas and definitions which will be useful later. Let C be the Cayley division algebra generated by $\{e_o, e_i, (1 \leq i \leq 7)\}$ over real number field $\mathbf{R}$ and C_+ be the subspace of C consisting of all purely imaginary Cayley numbers. We may identify C_+ with a 7-dimensional Euclidean space $\mathbf{R}^7$ with the canonical inner product $<,>$. A vector cross product for the vectors in $C_+ = \mathbf{R}^7$ is defined by

$$x \wedge y =< x, y > e_o + xy, \;\; x, y \in C_+. \tag{2.1}$$

Then the multiplication table is given by the following:

	1	0	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	$-e_6$
	2	$-e_3$	0	e_1	e_6	$-e_7$	$-e_4$	e_5
	3	e_2	$-e_1$	0	$-e_7$	$-e_6$	$-e_5$	e_4
$e_j \wedge e_k =$	4	$-e_5$	$-e_6$	e_7	0	e_1	e_2	$-e_3$
	5	e_4	e_7	e_6	$-e_1$	0	$-e_3$	$-e_2$
	6	$-e_7$	e_4	$-e_5$	$-e_2$	e_3	0	e_1
	7	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	0

On $S^6(1)$ define a $(1,1)$ tensor field J by putting

$$J_p X = p \wedge X$$

for $p \in S^6(1)$ and $X \in T_p S^6$. The above almost complex structure J together with the induced Riemannian metric $<,>$ on S^6 gives a nearly Kaehlerian structure on S^6 [12]. It is well known that S^6 does not admit any Kaehlerian structures. Let M be a Riemannian manifold isometrically immersed in a nearly Kaehler manifold $\bar{M}$ and denote by the same symbol g the Riemannian metric induced on M. Let $\Gamma(TM)$ be the Lie algebra of vector fields in M and $\Gamma(TM^\perp)$ the set of all vector fields normal to M. Denote by ∇ the Levi-Civita connection of M. Then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.2}$$

and

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{2.3}$$

for any $X, Y \in \Gamma(TM)$ and any $N \in \Gamma(TM^{\perp})$, where $\nabla^{\perp}$ is the connection in the normal bundle $TM^{\perp}$, h is the second fundamental form of M and A_N is the Weingarten endomorphism associated with N. The second fundamental form and the shape operator A are related by

$$g(A_N X, Y) = g(h(X, Y), N). \tag{2.4}$$

Let $\bar{M}$ be a nearly Kaehler manifold with complex structure J and M is a Riemannian manifold isometrically immersed in $\bar{M}$. Then M is called holomorphic (complex) if $J(T_pM) \subset T_pM$, for every $p \in M$, where T_pM denotes the tangent space to M at the point p. M is called totally real if $J(T_pM) \subset T_pM^{\perp}$ for every $p \in M$, where $T_pM^{\perp}$ denotes the normal space to M at the point p. The submanifold M is called a CR-submanifold [3] if there exists a differentiable distribution $D : p \rightarrow D_p \subset T_pM$ such that D is invariant with respect to J and the complementary distribution $D^{\perp}$ is anti-invariant with respect to J. It is clear that holomorphic and totally real submanifolds are CR-submanifolds with $D^{\perp} = \{0\}$ and $D = \{0\}$, respectively. Moreover, every real hypersurface of an almost Hermitian manifold is a CR-submanifold. For a real hypersurface M of an almost Hermitian $\bar{M}$ with a unit normal vector field ξ, the tangent vector $J\xi$ on M is called a characteristic vector field of M. A unit tangent vector X on M is called a principal vector if X is an eigenvector of the shape operator A_{ξ}, the corresponding eigenvalue is called the principal curvature at X. Let M be a CR-submanifold of a nearly Kaehler manifold, then we have

$$TM = D \oplus D^{\perp}.$$

Hence, we also have

$$TM^{\perp} = JD^{\perp} \oplus \nu,$$

where ν denotes the complementary distribution to $JD^{\perp}$ in the normal bundle of M. We note that ν is also invariant with respect to J. For any $X \in TM$ we write

$$JX = TX + FX, \tag{2.5}$$

where $TX \in \Gamma(D)$ and $FX \in \Gamma(JD^{\perp})$. Similarly, for any vector field normal to M, we put

$$JN = BN + CN, \tag{2.6}$$

where $BN \in \Gamma(D^{\perp})$ and $CN \in \Gamma(\nu)$.

3. Warped products $M_{\perp} \times_f M_T$ in nearly Kaehler manifolds

In this section we consider CR-submanifolds in a nearly Kaehler manifold $\bar{M}$ which are warped products of the form $M_{\perp} \times_f M_T$ such that $M_{\perp}$ is a totally real submanifold and M_T is a holomorphic submanifold of $\bar{M}$.

Theorem 3.1. *Let $\bar{M}$ be a nearly Kaehler manifold. Then there do not exist warped product CR-submanifolds in the form $M = M_{\perp} \times_f M_T$ in $\bar{M}$ such that $M_{\perp}$ is a totally real submanifold and M_T is a holomorphic submanifold of $\bar{M}$.*

Proof. Let us suppose that M be a warped product CR-submanifold in the form $M = M_{\perp} \times_f M_T$ in $\bar{M}$ such that $M_{\perp}$ is a totally real submanifold and M_T is a holomorphic submanifold of $\bar{M}$. Then, from (2.2), we have $g(\nabla_X Z, X) = g(\bar{\nabla}_X Z, X)$ for $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_{\perp})$. Using (1.4), we get $Z(\ln f)g(X, X) = g(\bar{\nabla}_X Z, X)$. Since $D^{\perp}$ and D are orthogonal, we obtain $Z(\ln f)g(X, X) = -g(Z, \bar{\nabla}_X X)$. Then, from (1.1), we derive $Z(\ln f)g(X, X) = -g(JZ, J\bar{\nabla}_X X)$. Hence, we get $Z(\ln f)g(X, X) = -g(JZ, \bar{\nabla}_X JX - (\bar{\nabla}_X J)X)$. Thus, using (1.2) we obtain

$$Z(\ln f)g(X, X) = -g(JZ, \bar{\nabla}_X JX).$$

Then, from (2.2), we have

$$Z(\ln f)g(X, X) = -g(JZ, h(X, JX)) \tag{3.1}$$

for $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_{\perp})$.

On the other hand, from (2.3), we have

$$g(A_{JZ}X, JX) = -g(\bar{\nabla}_X JZ, JX)$$

for $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_{\perp})$. Hence, we get

$$g(A_{JZ}X, JX) = g(JZ, \bar{\nabla}_X JX).$$

Since $\bar{\nabla}$ is torsion free, we have

$$g(A_{JZ}X, JX) = g(JZ, [X, JX] + \bar{\nabla}_{JX} X).$$

Since $[X, JX] \in \Gamma(TM)$ and $JZ \in \Gamma(TM^{\perp})$, we derive

$$g(A_{JZ}X, JX) = g(JZ, \bar{\nabla}_{JX} X).$$

Then, from (1.1), we can write

$$g(A_{JZ}X, JX) = -g(JZ, \bar{\nabla}_{JX} J^2 X).$$

Hence, we have

$$g(A_{JZ}X, JX) = -g(JZ, (\bar{\nabla}_{JX} J)JX + J\bar{\nabla}_{JX} JX).$$

Then, from (1.2), we derive

$$g(A_{JZ}X, JX) = -g(JZ, J\bar{\nabla}_{JX} JX).$$

Thus, using the second equation of (1.1), we arrive at

$$g(A_{JZ}X, JX) = -g(Z, \bar{\nabla}_{JX} JX).$$

Then, we get

$$g(A_{JZ}X, JX) = g(\bar{\nabla}_{JX}Z, JX).$$

Using (2.2) we obtain

$$g(A_{JZ}X, JX) = g(\nabla_{JX}Z, JX).$$

Thus, from (1.4) and (2.4,) we have

$$g(h(X, JX), JZ) = Z(\ln f)g(JX, JX).$$

Using (1.1), we arrive at

$$g(h(X, JX), JZ) = Z(\ln f)g(X, X) \tag{3.2}$$

for $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\perp)$. Thus, from (3.1) and (3.2), we conclude that

$$2Z(\ln f)g(X, X) = 0.$$

Since D is Riemannian, we get

$$Z(\ln f) = 0.$$

This implies that f is constant on $M_\perp$ which shows that M is a usual product. Thus proof is complete.

Remark 1. We note that Theorem 3.1 is a generalization of Theorem 3.1 in [7].

4. Warped products $M_T \times_f M_\perp$ in nearly Kaehler manifolds

Theorem 3.1 shows that there exist no warped product CR-submanifolds in the form $M = M_\perp \times_f M_T$ in $\bar{M}$ such that $M_\perp$ is a totally real submanifold and M_T is a holomorphic submanifold of $\bar{M}$. Therefore, in this section, we consider warped product CR-submanifolds in the form $M = M_T \times_f M_\perp$ such that $M_\perp$ is a totally real submanifold and M_T is a holomorphic submanifold of a nearly Kaehler manifold $\bar{M}$. Follow Chen's definition, we call such warped product CR-submanifolds CR-warped products. Now, we give an example of CR-warped product in S^6.

Example 1. Consider Sekigawa's example [21] of CR-submanifold in S^6, which is the image of the mapping of $S^2 \times S^1$ into S^6:

$$\begin{aligned}
\Psi(y, t) &= \Psi((y_2, y_4, y_6), e^{\sqrt{-1}t}) \\
&= (y_2 \cos t)e_2 - (y_2 \sin t)e_3 + (y_4 \cos 2t)e_4 \\
&\quad + (y_4 \sin 2t)e_5 + (y_6 \cos t)e_6 + (y_6 \sin t)e_7
\end{aligned}$$

where $(y_2, y_4, y_6) \in S^2$ and $e^{\sqrt{-1}t} \in S^1$. Then the tangent bundle TM of submanifold M is given by

$$\begin{aligned}
Z_1 &= -y_2 \sin t\, e_2 - y_2 \cos t e_3 - 2y_4 \sin 2t e_4 + 2y_4 \cos 2t e_5 - y_6 \sin t e_6 - y_6 \cos t e_7 \\
Z_2 &= y_6 \cos t\, e_2 - y_6 \sin t e_3 - y_2 \cos t\, e_6 - y_2 \sin t\, e_7 \\
Z_3 &= y_6 \cos 2t\, e_4 + y_6 \sin 2t\, e_5 - y_4 \cos t\, e_6 - y_4 \sin t\, e_7.
\end{aligned}$$

It is known that M is a CR-submanifold of S^6 such that $D = span\{Z_2, Z_3\}$ and $D^\perp = span\{Z_1\}$. Moreover, we can derive that D is integrable. Denoting the integral manifolds of D and $D^\perp$by M_T and $M_\perp$, respectively, then the induced metric tensor is

$$\begin{aligned} ds^2 &= (y_6^2 + y_2^2)dy_2^2 + y_2\, y_4 dy_2\, dy_4 + (y_6^2 + y_4^2)dy_4^2 + (1 + 3y_4^2)dt^2 \\ &= g_{M_T} + (1 + 3y_4^2)g_{M_\perp}. \end{aligned}$$

Thus it follows that M is a CR-warped product submanifold of S^6 with warping function $f = \sqrt{((1 + 3y_4^2))}$.

Remark 2. We note that H. Hashimoto and K. Mashimo [15] gave a generalization of Sekigawa's example. They also showed that the induced metric of their generalization is a warped product metric (see: [15], Lemma 8). Thus, there are many three dimensional CR-warped product submanifolds in S^6.

Next we give a characterization for CR-warped products in a nearly Kaehler manifolds. We first need the following lemmas.

Lemma 4.1. *Let M be a CR-submanifold of a nearly Kaehler manifold $\bar{M}$. Then, the anti-invariant distribution is integrable if and only if*

$$g(\nabla_Z X, V) = \frac{1}{2} g(A_{JZ}V + A_{JV}Z, JX) \tag{4.1}$$

for $Z, V \in \Gamma(D^\perp)$ and $X \in \Gamma(D)$.

Proof. From (2.2), we have

$$g([Z, V], X) = g(\bar{\nabla}_Z V, X) - g(\nabla_V Z, X)$$

for $Z, V \in \Gamma(D^\perp)$ and $X \in \Gamma(D)$. Then, using (1.1), we get

$$g([Z, V], X) = g(J\bar{\nabla}_Z V, JX) - g(\nabla_V Z, X).$$

Hence, we derive

$$g([Z, V], X) = g(-(\bar{\nabla}_Z J)V + \bar{\nabla}_Z JV, JX) - g(\nabla_V Z, X).$$

Thus, using (1.3), we obtain

$$g([Z, V], X) = g(\bar{\nabla}_V J)Z, JX) + g(\bar{\nabla}_Z JV, JX) - g(\nabla_V Z, X).$$

Hence, we have

$$g([Z, V], X) = g((\bar{\nabla}_V JZ - J\bar{\nabla}_V Z, JX) + g(\bar{\nabla}_Z JV, JX) - g(\nabla_V Z, X).$$

Then, using (2.2) and (2.3), we obtain

$$g([Z, V], X) = -g(A_{JZ}V + A_{JV}Z, JX) - 2g(\nabla_V Z, X).$$

Then, we have

$$g([Z,V],X) = -g(A_{JZ}V + A_{JV}Z, JX) - 2g([V,Z],X) - 2g(\nabla_Z V, X).$$

Hence we get

$$g([V,Z],X) = 2g(\nabla_Z X, V) - g(A_{JZ}V + A_{JV}Z, JX),$$

which proves our assertion.

Lemma 4.2. *Let M be a CR-submanifold of a nearly Kaehler manifold $\bar{M}$. Then, we have*

$$g(\nabla_X Y, Z) = g(JY, A_{JZ}X) - g(JY, \nabla_Z JX) + g(Y, \nabla_Z X) \tag{4.2}$$

for $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.

Proof. Using (2.2) and (1.1), we have $g(\nabla_X Y, Z) = -g(JY, J\bar{\nabla}_X Z)$ for $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$. Hence, we get $g(\nabla_X Y, Z) = -g(JY, \bar{\nabla}_X JZ - (\bar{\nabla}_X J)Z)$. Then, from (1.3), we obtain

$$g(\nabla_X Y, Z) = -g(JY, \bar{\nabla}_X JZ) - g(JY, (\bar{\nabla}_Z J)X).$$

Hence, we get

$$g(\nabla_X Y, Z) = -g(JY, \bar{\nabla}_X JZ) - g(JY, \bar{\nabla}_Z JX - J\bar{\nabla}_Z X).$$

Thus, using (1.1), (2.2) and (2.3) we obtain (4.2).

Lemma 4.3. *Let M be a CR-warped product of a nearly Kaehler manifold $\bar{M}$. Then, we have*

$$g(h(X, JY), JZ) = 0 \tag{4.3}$$

and

$$g(h(X,Z), JV)) = -JX(\ln f)g(Z,V) \tag{4.4}$$

for $X, Y \in \Gamma(TM_T)$ and $Z, V \in \Gamma(TM_{\perp})$.

Proof. From (1.4), we get $g(\nabla_X X, Z) = X(\ln f)g(X,Z) = 0$ for $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_{\perp})$. Then, using (2.2) and (1.1), we have $g(J\bar{\nabla}_X X, JZ) = 0$. Hence

$$g(-(\bar{\nabla}_X J)X + \bar{\nabla}_X JX, JZ) = 0.$$

Thus, (1.2) implies

$$g(\bar{\nabla}_X JX, JZ) = 0.$$

Using (2.2), we get

$$g(h(X, JX), JZ) = 0.$$

Then, by polarization for h, we derive

$$g(h(X, JY), JZ) + g(h(Y, JX), Z) = 0.$$

Now, since $M = M_T \times_f M_\perp$ is a warped product manifold, D is integrable. Then, from a result of [10], it follows that $h(X, JY) = h(JX, Y)$ for $X, Y \in \Gamma(D)$. Hence, we obtain (4.3).

On the other hand, from (2.3), we get $g(A_{JZ}X, V) = -g(\bar{\nabla}_X JZ, V)$ for $X \in \Gamma(D)$ and $Z, V \in \Gamma(D^\perp)$. Hence, we have

$$g(A_{JZ}X, V) = -g((\bar{\nabla}_X J)Z + J\bar{\nabla}_X Z, V).$$

Using (1.2), we obtain

$$g(A_{JZ}X, V) = g((\bar{\nabla}_Z J)X, V) - g(J\bar{\nabla}_X Z, V).$$

By direct computations, from (1.1), we derive

$$g(A_{JZ}X, V) = g(\bar{\nabla}_Z JX - J\bar{\nabla}_Z X, V) + g(\bar{\nabla}_X Z, JV).$$

Using, again (1.1), we have

$$g(A_{JZ}X, V) = g(\bar{\nabla}_Z JX, V) + g(\bar{\nabla}_Z X, JV) + g(\bar{\nabla}_X Z, JV).$$

Thus, from (2.2), we get

$$g(A_{JZ}X, V) = g(\nabla_Z JX, V) + g(h(Z, X), JV) + g(h(X, Z), JV).$$

Since h is symmetric, (1.4) and (2.4) imply that

$$g(h(V, X), JZ) = JX(\ln f)g(Z, V) + 2g(h(X, Z), JV) \tag{4.5}$$

for $X \in \Gamma(D)$ and $Z, V \in \Gamma(D^\perp)$. Interchanging the role of Z and V, we obtain

$$g(h(Z, X), JV) = JX(\ln f)g(Z, V) + 2g(h(X, V), JZ) \tag{4.6}$$

Thus from (4.5) and (4.6) we get

$$g(h(X, Z), JV) = -JX(\ln f)g(Z, V).$$

Now, we are ready to prove a characterization theorem for CR-warped products in nearly Kaehler manifolds. But we first recall that we have the following result of S. Hiepko [16], (cf. [11], Remark 2.1):

Let D_1 be a vector subbundle in the tangent bundle of a Riemannian manifold M and D_2 be its normal bundle. Suppose that the two distributions are involutive. We denote the integral manifolds of D_1 and D_2 by M_1 and M_2, respectively. Then M is locally isometric to warped product $M_1 \times_f M_2$ if the integral manifold M_1 is totally geodesic and the integral manifold M_2 is an extrinsic sphere, i. e, M_2 is a totally umbilical submanifold with parallel mean curvature vector.

Theorem 4.1. *A proper CR-submanifold of a nearly Kaehler manifold $\bar{M}$ is locally CR-warped product if and only if*

$$A_{JZ}X = -(JX)(\mu)Z, g(\nabla_Z X, Y) = 0 \tag{4.7}$$

for some function μ *such that* $W(\mu) = 0$, $W \in \Gamma(D^{\perp})$ *and*

$$2g(\nabla_Z X, V) = g(A_{JZ}V + A_{JV}Z, JX), \tag{4.8}$$

where $X, Y \in \Gamma(D)$ *and* $Z, V \in \Gamma(D^{\perp})$.

Proof. If M is a CR-warped product, then, M_T is totally geodesic in M. Thus, $g(\nabla_X Y, Z) = 0$ for $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$. Hence, we obtain $g(Y, \nabla_X Z) = 0$. Then from (1.4) we get $g(Y, \nabla_Z X) = 0$ which is the second equation of (4.7). On the other hand, from (4.3), we have $g(h(X, JY), JZ) = 0$ for $X, Y \in \Gamma(D)$ and $Z, V \in \Gamma(D^{\perp})$. Then, using (2.4), we get $g(A_{JZ}X, JY) = 0$. This implies that $A_{JZ}X \in \Gamma(D^{\perp})$. On the other hand, from (2.4) and (4.4) we have

$$g(A_{JZ}X, V) = -JX(\ln f)g(Z, V)$$

for $X \in \Gamma(D)$ and $Z, V \in \Gamma(D^{\perp})$. Moreover, f is a function on M_T, we have $W(\ln f) = 0$ for $W \in \Gamma(D^{\perp})$. This proves the first equation of (4.7). Furthermore, since A is self-adjoint, from (4.4), we have

$$g(A_{JZ}V, JX) = X(\ln f)g(Z, V).$$

Interchanging the role of Z and V, we get

$$g(A_{JV}Z, JX) = X(\ln f)g(Z, V).$$

Thus, from above two equations, we obtain

$$g(A_{JZ}V + A_{JV}Z, JX) = 2X(\ln f)g(Z, V).$$

Then, using (1.4), we arrive at

$$g(A_{JZ}V + A_{JV}Z, JX) = 2(\nabla_Z X, V)$$

which implies (4.8). Conversely, suppose that M is a CR-submanifold of a nearly Kaehler manifold $\bar{M}$ satisfying (4.7) and (4.8). Then, from (4.2) and (4.7), it follows that D is totally geodesic. (4.8) implies that $D^{\perp}$ is integrable. Let $M_{\perp}$ be a leaf of $D^{\perp}$. We now denote the second fundamental form of $M_{\perp}$ in M by h_2. From (4.8), we also have

$$-2g(\nabla_Z V, X) = g(A_{JZ}V + A_{JV}Z, JX)$$

for $X \in \Gamma(D)$ and $Z, V \in \Gamma(D^{\perp})$. Then, since A is self-adjoint, (4.7) implies that

$$-2g(\nabla_Z V, X) = X(\mu)g(Z, V) + X(\mu)g(V, Z).$$

Thus, we obtain

$$g(h_2(Z, V), X) = g(\nabla_Z V, X) = -X(\mu)g(Z, V).$$

This shows that $M_\perp$ is totally umbilical in M. Moreover, by direct computations, we get

$$\begin{aligned} g(\nabla_Z \operatorname{grad} \mu, \mathrm{X}) &= g(\nabla_Z \operatorname{grad} \mu, \mathrm{X}) \\ &= [Zg(\operatorname{grad} \mu, \mathrm{X}) - \mathrm{g}(\operatorname{grad} \mu, \nabla_{\mathrm{Z}}\mathrm{X})] \\ &= [Z(X(\mu)) - [Z, X]\mu - g(\operatorname{grad} \mu, \nabla_{\mathrm{X}}\mathrm{Z})] \\ &= [[Z, X]\mu + X(Z(\mu)) - [Z, X]\mu - g(\operatorname{grad} \mu, \nabla_{\mathrm{X}}\mathrm{Z})] \\ &= [X(Z(\mu)) - g(\operatorname{grad} \mu, \nabla_{\mathrm{X}}\mathrm{Z})]. \end{aligned}$$

Since $Z(\mu) = 0$, we obtain

$$g(\nabla_Z \operatorname{grad} \mu, \mathrm{X}) = -\mathrm{g}(\operatorname{grad} \mu, \nabla_{\mathrm{X}}\mathrm{Z}).$$

On the other hand, since $\operatorname{grad} \mu \in \Gamma(\mathrm{TM_T})$ and M_T is totally geodesic in M, it follows that $\nabla_X Z \in \Gamma(TM_\perp)$ for $X \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$. Hence $g(\nabla_Z \operatorname{grad} \mu, \mathrm{X}) = 0$. Then the spherical condition is also fulfilled, that is $M_\perp$ is an extrinsic sphere in M. Thus we conclude that M is a warped product and proof is complete.

We recall that a CR-submanifold is called mixed geodesic [3] if $h(X, Z) = 0$ for $X \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$. Using (4.4) we have the following:

Corollary 4.1. *A mixed totally geodesic CR-warped product $M = M_T \times_f M_\perp$ in a nearly Kaehler manifold $\bar{M}$ is a CR-product.*

In particular, if $JD^\perp = TM^\perp$, i. e., M is an anti-holomorphic submanifold of $\bar{M}$, we have:

Corollary 4.2. *An anti-holomorphic warped product submanifold $M = M_T \times_f M_\perp$ in a nearly Kaehler manifold $\bar{M}$ is a CR-product if and only if M is mixed totally geodesic.*

In the rest of this section, we obtain an inequality for the squared norm of the second fundamental form in terms of the warping function for CR-warped products in a nearly Kaehler manifold. We note that this inequality was proved in [7] for Kaehler case.

Lemma 4.4. *Let M be a CR-submanifold of a nearly Kaehler manifold $\bar{M}$. Then, we have:*

$$g(h(Z, Z), JW) = g(h(Z, W), JZ) \tag{4.9}$$

for $Z, W \in \Gamma(D^\perp)$.

Proof. From (2.2), we have $g(h(Z, Z), JW) = g(\bar{\nabla}_Z Z, JW)$ for $Z, W \in \Gamma(D^\perp)$. Using (1.1), we get $g(h(Z, Z), JW) = -g(J\bar{\nabla}_Z Z, W)$. Hence, we derive

$$g(h(Z, Z), JW) = -g(-(\bar{\nabla}_Z J)Z + \bar{\nabla}_Z JZ, W).$$

Thus, using (1.2) and (2.3), we obtain

$$g(h(Z,Z),JW)=g(A_{JZ}Z,W).$$

Then, from (2.4), we have (4.9).

Theorem 4.2. *Let $M=M_T\times_f M_\perp$ be a CR-warped product in a nearly Kaehler manifold $\bar{M}$. Then we have*

(1) *The squared norm of the second fundamental form of M satisfies*

$$\| h \|^2 \geq 2p \| \nabla(\ln f) \|^2, \tag{4.10}$$

where $\nabla \ln f$ is the gradient of $\ln f$ and p is the dimension of $M_\perp$.

(2) *If the equality sign of (4.10) holds identically, then M_T is a totally geodesic submanifold and $M_\perp$ is a totally umbilical submanifold of $\bar{M}$. Moreover, M is a minimal submanifold in $\bar{M}$*

(3) *When M is anti-holomorphic and $p>1$, the equality sign of (4.10) holds identically if and only if $M_\perp$ is totally umbilical submanifold of $\bar{M}$.*

(4) *If M is real hypersurface of $\bar{M}$ (that is $p=1$), then the equality sign of (4.10) holds identically if and only if the characteristic vector field $J\xi$ of M is a principal vector field with zero as its principal curvature. Also in this case, the equality sign of (4.10) holds identically if and only if M is a minimal hypersurface in $\bar{M}$.*

Proof. We take an orthonormal frame $\{e_1, e_{2q}, e_{2q+1}, \ldots, e_m, e_1^*=J(e_{2q+1}), \ldots, e_p^*, e_1, \ldots, e_{2r}\}$ of $\bar{M}$ along M such that $\{e_1,\ldots,e_{2q}\}$, $\{e_{2q+1},\ldots,e_m\}$ and $\{e_1,\ldots, e_{2r}\}$ are bases of D, $D^\perp$ and ν, respectively. Since

$$\| h \|^2 = \| h(D,D) \|^2 + \| h(D^\perp,D^\perp) \|^2 + 2 \| h(D^\perp,D) \|^2 .$$

We have

$$\begin{aligned} \| h \|^2 &= \sum_{k=m+1}^{n}\sum_{i,j=1}^{2q} g(h(e_i,e_j),\bar{e}_k)^2 + \sum_{k=m+1}^{n}\sum_{s,l=1}^{p} g(h(e_s,e_l),\bar{e}_k)^2 \\ &\quad +2\sum_{k=m+1}^{n}\sum_{i=1}^{2q}\sum_{s=1}^{p} g(h(e_i,e_s),\bar{e}_k)^2 \end{aligned}$$

where $\{\bar{e}_k\}_{k=1}^n$ is a basis of $TM^\perp$. Then, from the decomposition of normal bundle of a CR-submanifold of a nearly Kaehler manifold, we get

$$\begin{aligned} \| h \|^2 &= \sum_{k=m+1}^{n}\sum_{i,j=1}^{2q} g(h(e_i,e_j),\bar{e}_k)^2 + \sum_{k=m+1}^{n}\sum_{s,l=1}^{p} g(h(e_s,e_l),\bar{e}_k)^2 \\ &\quad +2\sum_{t,s=1}^{p}\sum_{i=1}^{2q} g(h(e_i,e_s),Je_t)^2 + 2\sum_{l=1}^{2r}\sum_{i=1}^{2q}\sum_{s=1}^{p} g(h(e_i,e_s),Je_l)^2. \end{aligned}$$

Then, from (4.4) we have

$$\begin{aligned}\| h \|^2 &= \sum_{k=m+1}^{n}\sum_{i,j=1}^{2q} g(h(e_i,e_j),\bar{e}_k)^2 + \sum_{k=m+1}^{n}\sum_{s,l=1}^{p} g(h(e_s,e_l),\bar{e}_k)^2 \\ &\quad +2p \| \nabla(\ln f) \|^2 +2\sum_{l=1}^{2r}\sum_{i=1}^{2q}\sum_{s=1}^{p} g(h(e_i,e_s),Je_l)^2.\end{aligned}$$

This proves (4.10). If the equality case of (4.10) holds identically, we obtain

$$h(D,D)=0,\ h(D^{\perp},\ D^{\perp})=0,\ h(D,D^{\perp})\subset JD^{\perp}. \tag{4.11}$$

Since M_T is totally geodesic in M, the first equation in (4.11) implies that M_T is totally geodesic in $\bar{M}$. On the other hand, using (2.2) and (1.4) we get

$$h_2(Z,V)=-(\nabla\ln f)g(Z,V) \tag{4.12}$$

for $Z,V\in\Gamma(M_{\perp})$. Then, (4.12) and the second equation (4.11) imply that $M_{\perp}$ is totally umbilical in $\bar{M}$. (4.11) also implies that M is minimal in $\bar{M}$.

Now suppose that M is an anti-holomorphic warped product in $\bar{M}$. Then from (4.3) we get

$$h(D,D)=0. \tag{4.13}$$

Thus, if $M_{\perp}$ is totally umbilical in $\bar{M}$, we can write

$$\bar{h}(Z,V)=g(Z,V)\bar{H}, \tag{4.14}$$

for $Z,V\in\Gamma(D^{\perp})$, where $\bar{H}$ is a normal vector field to $M_{\perp}$. Since

$$\bar{h}(Z,V)=h_2(Z,V)+h(Z,V),$$

(4.14) implies that there is a normal vector field N such that

$$h(Z,V)=g(Z,V)N.$$

Hence, for each unit vector field $V\in\Gamma(D^{\perp})$ and each unit vector field $Z\in\Gamma(D^{\perp})$, perpendicular to V, we derive

$$g(N,JV)=g(h(Z,Z),JV).$$

Then using (4.9) we get

$$g(N,JV)=g(h(Z,V),JZ)=g(Z,V)g(N,JZ)=0. \tag{4.15}$$

Since M is anti-holomorphic, (4.15) implies that either $p=1$ or

$$h(D^{\perp},D^{\perp})=0. \tag{4.16}$$

Then, from (4.4), (4.13) and (4.16), it follows that the equality case holds, when $p>1$.

If $p = 1$, then M is a real hypersurface in $\bar{M}$. Then, the characteristic vector field $J\xi$ is a principal vector field with zero as its principal curvature if and only if (4.16) holds. Thus, in this case, we also have equality case. If the characteristic vector field $J\xi$ is a principal vector field with zero as its principal curvature, from (4.11) we also know that the condition (4.16) holds if and only if M is minimal in $\bar{M}$. The converse is clear. Thus proof is complete.

Remark 3. Thus we show that Theorem 5.1 of [7] is also valid for warped product CR-submanifolds in the form $M_T \times_f M_\perp$ in a nearly Kaehler manifold $\bar{M}$ such that M_T is a holomorphic submanifold and $M_\perp$ is a totally real submanifold of $\bar{M}$. It is also clear that Theorem 4.2 is a generalization of Theorem 5.1 in [7].

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