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Area-stationary Surfaces in Neutral Kähler 4-Manifolds

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Abstract. We study surfaces in TN that are area-stationary with respect to a neutral Kähler metric constructed on TN from a Riemannian metric g on N. We show that holomorphic curves in TN are area-stationary. However, in general, area-stationary surfaces are not holomorphic. We prove this by constructing counter-examples. In the case where g is rotationally symmetric, we find all area-stationary surfaces that arise as graphs of sections of the bundle TN→N and that are rotationally symmetric. When (N,g) is the round 2-sphere, TN can be identified with the space of oriented affine lines in $\mathbb{R}^3$, and we exhibit a two parameter family of area-stationary tori that are neither holomorphic nor Lagrangian.

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One-dimensional submanifolds of neutral Kähler four-manifolds have been studied recently in the context of twistor theory and integrable systems (cf. [2] and references therein). For example, quotienting out by the integral curves of non-null or null Killing vector fields leads to Einstein-Weyl three-manifolds or projective surfaces, respectively.

In the case of two-dimensional submanifolds in neutral Kähler four-manifolds, the objects of study in this paper, the null/non-null dichotomy is insufficient. In

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particular, the metric induced on such a submanifold by the neutral metric can be positive or negative definite, Lorentz or degenerate, with two possible degrees of degeneracy. Moreover, while the geometry of surfaces in positive definite Kähler four-manifolds is well developed, particularly in the Kähler-Einstein case [5], for indefinite metrics many of these results do not hold.

The main purpose of this paper then is two-fold: to investigate the geometric properties of two-dimensional submanifolds of a class of neutral Kähler four-manifolds and, by so doing, to illustrate the differences between the neutral and Hermitian cases.

The particular class of neutral Kähler structures we consider have recently been studied on TN, the total space of the tangent bundle to a Riemannian two-manifold (N,g) [3], [4]. This construction is motivated by the neutral Kähler metric on the space of oriented lines in $\mathbb{R}^3$ and on the space of time-like lines in $\mathbb{R}^{2+1}$. Aside from the signature, these Kähler four-manifolds differ from the more commonly studied Kähler four-manifolds in a number of crucial ways: they are non-compact and are Kähler-Einstein only in the case when (N,g) is flat. They are, however, scalar flat, and the symplectic structure is exact.

In the next section we discuss neutral Kähler metrics and some of their properties. We also outline the construction of the neutral Kähler structure on TN and the geometric structures induced on surfaces in TN. In the following section we derive the equations for area-stationary graphs in TN and show that holomorphic curves are area-stationary. In Section 3 we look at rotationally symmetric graphs and determine all of these that area-stationary. In addition, we give a construction for surfaces on which the induced metric is degenerate at every point.

In the final section, we look at the case of TS^2, which we identify with the space $\mathbb{L}$ of oriented affine lines in $\mathbb{R}^3$. There we construct area-stationary tori that are neither holomorphic nor Lagrangian, and investigate their geometric properties. This two-parameter families of surfaces are analogous to the catenoid in $\mathbb{R}^3$, being the unique rotationally symmetric area-stationary surfaces in $\mathbb{L}$.

1. The neutral Kähler metric on TN

We begin with some general properties of a Kähler surface $(\mathbb{M}, \mathbb{G}, \mathbb{J}, \Omega)$. That is, $\mathbb{M}$ is a real 4-manifold endowed with the following structures. First, there is the metric $\mathbb{G}$, which we do not insist be positive definite – it may also have neutral signature $(++--)$. In order to deal with both cases simultaneously we assume that the metric can be diagonalised pointwise to $(1,1,\epsilon,\epsilon)$, for $\epsilon = \pm 1$.

In addition, we have a complex structure $\mathbb{J}$, which is a mapping $\mathbb{J} : \mathrm{T}_p\mathbb{M} \to \mathrm{T}_p\mathbb{M}$ at each $p \in \mathbb{M}$, which satisfies $\mathbb{J}^2 = -\mathrm{Id}$ and an integrability condition. Finally, there is a symplectic form Ω, which is a closed, non-degenerate 2-form. These structures are required to satisfy the compatibility conditions:

$$\mathbb{G}(\mathbb{J}\cdot, \mathbb{J}\cdot) = \mathbb{G}(\cdot, \cdot) \qquad\qquad \mathbb{G}(\cdot, \cdot) = \Omega(\mathbb{J}\cdot, \cdot).$$

The following calibration identity highlights the difference between the case where $\mathbb{G}$ is positive definite and where it is neutral.

Theorem 1. [4] *Let $p \in \mathbb{M}$ and $v_1, v_2 \in T_p\mathbb{M}$ span a plane. Then*

$$\Omega(v_1, v_2)^2 + \epsilon\varsigma^2(v_1, v_2) = \det \mathbb{G}(v_i, v_j),$$

where $\varsigma^2(v_1, v_2) \geq 0$ with equality iff $\{v_1, v_2\}$ spans a complex plane.

In the Hermitian case, $\epsilon = 1$ and the above equality implies Wirtinger's inequality: the symplectic area is bounded above by the metric area.

Given a surface Σ in $\mathbb{M}$, we say that Σ is *holomorphic* if $\mathbb{J}$ preserves the tangent space of Σ, while it is *Lagrangian* if the symplectic form pulled back to Σ vanishes. A further consequence of the above theorem is that, in the positive definite case, a surface cannot be both holomorphic and Lagrangian. In the neutral case, however, this is not true: a surface can be both holomorphic and Lagrangian, the only requirement being that the metric must be maximally degenerate along such a surface. We call such surfaces *totally null* surfaces and the full details of this are included in Proposition 3 below.

We turn now to the construction of a neutral Kähler structure on TN – further details can be found in [3], [4]. Given a Riemannian 2-manifold (N,g,j) we construct a canonical Kähler structure ($\mathbb{J}$,Ω,$\mathbb{G}$) on the tangent bundle TN as follows. The Levi-Civita connection associated with g splits the tangent bundle TTN$\cong$TN$\oplus$TN and the almost complex structure is defined to be $\mathbb{J} = \mathrm{j} \oplus \mathrm{j}$. This turns out to satisfy the appropriate integrability condition and so we have a complex structure on TN.

To define the symplectic form, consider the metric g as a mapping from TN to T*N and pull back the canonical symplectic 2-form Ω^* on T*N to a symplectic 2-form Ω on TN. Finally, the metric is defined as above by $\mathbb{G}(\cdot, \cdot) = \Omega(\mathbb{J}\cdot, \cdot)$. The triple ($\mathbb{J}$, Ω, $\mathbb{G}$) determines a Kähler structure on TN.

Proposition 1. [3] *Let $(\mathrm{TN}, \mathbb{J}, \Omega, \mathbb{G})$ be the Kähler surface, as above. Then the metric $\mathbb{G}$ has neutral signature (++−−) and is scalar-flat. Moreover, $\mathbb{G}$ is Kähler-Einstein iff g is flat, and $\mathbb{G}$ is conformally flat iff g is of constant curvature.*

Choose holomorphic coordinates ξ on N so that $ds^2 = e^{2u} d\xi d\bar{\xi}$ for $u = u(\xi, \bar{\xi})$, and corresponding coordinates (ξ,η) on TN by identifying

$$(\xi, \eta) \leftrightarrow \eta \frac{\partial}{\partial \xi} + \bar{\eta} \frac{\partial}{\partial \bar{\xi}} \in \mathrm{T}_\xi \mathrm{N}.$$

These coordinates turn out to be holomorphic with respect to the above complex structure on TN and the symplectic 2-form has the following expression:

$$\Omega = 2\mathbb{R}e \left(e^{2u} d\eta \wedge d\bar{\xi} + \eta \partial(e^{2u}) d\xi \wedge d\bar{\xi} \right).$$

Here we have introduced the notation ∂ for differentiation with respect to ξ-notation that we will use throughout this paper. The symplectic 2-form is globally exact $\Omega = d\Theta$, where $\Theta = \eta e^{2u} d\bar{\xi} + \bar{\eta} e^{2u} d\xi$. Thus, for a closed surface Σ in TN

$$\int_\Sigma \Omega = 0.$$

On the other hand, the Kähler metric $\mathbb{G}$ is given in holomorphic coordinates by

$$\mathbb{G} = 2\mathbb{I}\mathrm{m}\left(e^{2u}d\bar{\eta}d\xi - \eta\partial(e^{2u})d\xi d\bar{\xi}\right).$$

We now consider surfaces in TN which arise as the graph of a local section of the bundle TN→N, that is, a map $\xi \to (\xi, \eta = F(\xi, \bar{\xi}))$. For such a surface introduce the following notation for the complex slopes:

$$\sigma = -\partial\bar{F} \qquad\qquad \rho = e^{-2u}\partial\left(Fe^{2u}\right),$$

and let $\lambda = \mathbb{I}\mathrm{m}\,\rho$.

Proposition 2. [3] *A graph of a local section is holomorphic iff $\sigma = 0$ and is Lagrangian iff $\lambda = 0$.*

Turning to the metric on a graph, the following makes explicit the identity contained in Theorem 1:

Proposition 3. [4] *The metric (and its inverse) induced on the graph of a section by the Kähler metric is given in coordinates $(\xi, \bar{\xi})$ by*

$$\mathbb{G} = e^{2u}\begin{bmatrix} i\sigma & -\lambda \\ -\lambda & -i\bar{\sigma} \end{bmatrix} \qquad\qquad \mathbb{G}^{-1} = \frac{e^{-2u}}{\lambda^2 - \sigma\bar{\sigma}}\begin{bmatrix} i\bar{\sigma} & -\lambda \\ -\lambda & -i\sigma \end{bmatrix}.$$

In particular, the determinant of the induced metric is $|\mathbb{G}| = (\lambda^2 - \sigma\bar{\sigma})e^{4u}$. Thus, the metric is Lorentz iff $\lambda^2 < \sigma\bar{\sigma}$, Riemannian iff $\lambda^2 > \sigma\bar{\sigma}$ and degenerate if $\lambda^2 = \sigma\bar{\sigma}$. The metric induced on a holomorphic, Lagrangian graph is identically zero, and we call such a surface totally null.

This has the following corollary:

Corollary 1. *The metric induced on a closed surface in* TN *cannot be positive (or negative) definite everywhere.*

Proof. Since the symplectic form is exact, as noted previously, its integral over any closed surface is zero. Thus, the symplectic form must vanish somewhere on the surface. By Theorem 1, at such points the determinant of the induced metric is either zero or negative. That is, the metric must be either degenerate or Lorentz at these points. □

2. Area-stationary graphs

The area form of the induced metric is $|\mathbb{G}|^{\frac{1}{2}}d\xi \wedge d\bar{\xi}$, and the following proposition deals with stationary values of the area functional:

Proposition 4. *Let $\Sigma \hookrightarrow$ TN be a surface which is given by the graph of a function $\xi \to (\xi, \eta = F(\xi, \bar{\xi}))$. Assume that the metric is not degenerate on Σ. Then Σ is area-stationary iff*

$$i\partial\left(\frac{\lambda}{\sqrt{|\lambda^2 - \sigma\bar{\sigma}|}}\right) - e^{-2u}\bar{\partial}\left(\frac{\sigma e^{2u}}{\sqrt{|\lambda^2 - \sigma\bar{\sigma}|}}\right) = 0. \qquad (2.1)$$

Proof. From Proposition 3 the area functional evaluated on a graph Σ is

$$A(\Sigma) = \int_\Sigma d\mathrm{vol}(\mathbb{G}) = \int_\Sigma |\mathbb{G}|^{\frac{1}{2}} d\xi d\bar{\xi} = \int_\Sigma |\lambda^2 - \sigma\bar{\sigma}|^{\frac{1}{2}} e^{2u}\, d\xi d\bar{\xi}.$$

Varying the graph F we have

$$\delta A(F) = \tfrac{1}{2}\int_\Sigma |\lambda^2 - \sigma\bar{\sigma}|^{-\frac{1}{2}} \delta(\lambda^2 - \sigma\bar{\sigma}) e^{2u}\, d\xi d\bar{\xi}.$$

Now

$$\begin{aligned}\delta(\lambda^2 - \sigma\bar{\sigma}) =& 2\lambda\delta\lambda - \sigma\delta\bar{\sigma} - \bar{\sigma}\delta\sigma \\ =& -i\lambda e^{-2u}\left(\partial(\delta F e^{2u}) - \bar{\partial}(\delta\bar{F} e^{2u})\right) + \sigma\bar{\partial}(\delta F) + \bar{\sigma}\partial(\delta\bar{F}),\end{aligned}$$

and so

$$\delta A(F) = \frac{1}{2}\int_\Sigma \left[-i\lambda e^{-2u}\left(\partial(\delta F e^{2u}) - \bar{\partial}(\delta\bar{F} e^{2u})\right) + \sigma\bar{\partial}(\delta F) + \bar{\sigma}\partial(\delta\bar{F})\right] \frac{e^{2u}\, d\xi d\bar{\xi}}{|\lambda^2 - \sigma\bar{\sigma}|^{\frac{1}{2}}}.$$

Integrating by parts we get

$$\begin{aligned}\delta A(F) =& \frac{1}{2}\int_\Sigma \left(ie^{2u}\partial\left(\frac{\lambda}{|\lambda^2 - \sigma\bar{\sigma}|^{\frac{1}{2}}}\right) - \bar{\partial}\left(\frac{\sigma e^{2u}}{|\lambda^2 - \sigma\bar{\sigma}|^{\frac{1}{2}}})\right)\right) \delta F\, d\xi d\bar{\xi} \\ &+ \frac{1}{2}\int_\Sigma \left(-ie^{2u}\bar{\partial}\left(\frac{\lambda}{|\lambda^2 - \sigma\bar{\sigma}|^{\frac{1}{2}}}\right) - \partial\left(\frac{\bar{\sigma} e^{2u}}{|\lambda^2 - \sigma\bar{\sigma}|^{\frac{1}{2}}})\right)\right) \delta\bar{F}\, d\xi d\bar{\xi}.\end{aligned}$$

A graph F is area-stationary if $\delta A(F) = 0$ for all δF, and so the result follows. □

The previous proposition has the following corollary:

Corollary 2. *Holomorphic graphs in* TN *are area-stationary.*

Proof. Suppose that the graph of the section is holomorphic. Then $\sigma = 0$ and we see that

$$i\partial\left(\frac{\lambda}{\sqrt{|\lambda^2 - \sigma\bar{\sigma}|}}\right) - e^{-2u}\bar{\partial}\left(\frac{\sigma e^{2u}}{\sqrt{|\lambda^2 - \sigma\bar{\sigma}|}}\right) = i\partial\left(\frac{\lambda}{|\lambda|}\right) = 0,$$

and so by the previous proposition it is area-stationary. □

3. Rotationally symmetric area-stationary graphs

Let (N,g) be a Riemannian two-manifold.

Definition 1. *The metric* g *is rotationally symmetric if there exists a conformal coordinate system* $(\xi, \bar{\xi})$ *such that* $\mathrm{g} = e^{2u} d\xi d\bar{\xi}$ *for* $u = u(|\xi|)$.

In other words, the metric is invariant under $\xi \to \xi e^{iC}$. Such an isometry of (N,g) lifts to an isometry $(\xi,\eta) \to (\xi e^{iC}, \eta e^{iC})$ of the Kähler metric on TN by the derivative map [4].

Definition 2. *Let* g *be rotationally symmetric. A surface in* TN *is rotationally symmetric if it is invariant under the induced isometry of* TN. *A graph* $\xi \to (\xi, \eta = F(\xi,\bar\xi))$ *is rotationally symmetric iff* $F(\xi,\bar\xi) = G(R)e^{i\theta}$ *for some complex-valued function* G*, where* $\xi = Re^{i\theta}$.

The following theorem characterises area-stationary graphs in TN that are rotationally symmetric:

Theorem 2. *Let (N,g) be a rotationally symmetric Riemannian two-manifold and* $\mathbb{G}$ *be the associated neutral Kähler metric on* TN. *A rotationally symmetric surface which is given by the graph of a local section* $\xi \to (\xi, \eta = F(\xi,\bar\xi))$ *is area-stationary with respect to* $\mathbb{G}$ *iff*

$$F = \left[A_1R + B_1R^{-1}e^{-2u} \pm i\left[A_2R^2 + B_2e^{-2u} - B_1^2R^{-2}e^{-4u}\right]^{\frac{1}{2}}\right]e^{i\theta},$$

for $A_1, A_2, B_1, B_2 \in \mathbb{R}$, $A_2 \neq 0$, *where* $\xi = Re^{i\theta}$ *and* $g = e^{2u}d\xi d\bar\xi$ *for* $u = u(R)$.

Proof. Let $F = (H \pm i\Psi^{\frac{1}{2}})e^{i\theta}$ for real functions $H = H(R)$ and $\Psi = \Psi(R)$. Substituting this in equation (2.1) we get a pair of coupled non-linear 2nd order ordinary differential equations for H and Ψ, which can be written

$$\ddot\Psi + p_1\dot\Psi + q_1\Psi = L_1 \qquad \ddot\Psi + p_2\dot\Psi + q_2\Psi = L_2, \tag{3.1}$$

where a dot represents differentiation with respect to R and

$$p_1 = -\frac{1 + R^2(\ddot u - 2\dot u^2)}{R(1 + R\dot u)} \qquad q_1 = -\frac{2(\dot u - R(\ddot u - 2\dot u^2))}{R(1 + R\dot u)}$$

$$L_1 = \frac{R\dot H - H}{R^2(1 + R\dot u)^2}\left[R^2(1 + R\dot u)\ddot H - (1 + 2R\dot u + R^2\ddot u)(R\dot H - H)\right],$$

and

$$p_2 = -\frac{2R\ddot H}{R\dot H - H} - \frac{3 + 4R\dot u^2 - R^2(\ddot u - 2\dot u^2)}{R(1 + R\dot u)}$$

$$q_2 = -\frac{4R\ddot H}{R\dot H - H} - \frac{2(3\dot u + R(6\dot u^2 - \ddot u) - 2R^2(\ddot u - 2\dot u^2)\dot u)}{R(1 + R\dot u)}$$

$$L_2 = -\frac{2(R\dot H - H)^2}{R^2}.$$

To solve these equations proceed as follows: first solve the homogenous version of the first equation in (3.1) for Ψ and then use variation of parameters to solve the inhomogeneous equation for $\Psi = \Psi(R, \dot u, \ddot u, H, \dot H, \ddot H)$. Then substitute this in the second equation of (3.1) and solve for $H = H(R)$.

We now carry this out in detail. Start by noting that $\Psi = R^2$ is a solution of the homogenous version of the first equation of (3.1). Now the other linearly independent solution of the homogenous equation can be found by recourse to the following lemma:

Lemma 1. [1] *Let $\Psi = \Psi_1$ be a solution of the 2nd order linear homogenous ordinary differential equation $\ddot{\Psi} + p(R)\dot{\Psi} + q(R)\Psi = 0$. Then the other linearly independent solution is*

$$\Psi_2 = \Psi_1 \int \Psi_1^{-2} e^{-P} dR \qquad \textit{where} \qquad P(R) = \int p(R) dR.$$

In our case, $\Psi_1 = R^2$ and

$$p(R) = p_1 = -\frac{1 + R^2(\ddot{u} - 2\dot{u}^2)}{R(1 + R\dot{u})} = -\frac{d}{dR} \ln \left[R(1 + R\dot{u}) e^{-2u} \right],$$

so that the second solution is

$$\Psi_2 = R^2 \int R^{-3}(1 + R\dot{u}) e^{-2u} dR = -\tfrac{1}{2} e^{-2u}.$$

Thus the homogenous solution to first equation of (3.1) is

$$\Psi = A_2 R^2 + B_2 e^{-2u},$$

for real constants A_2 and B_2.

To solve the full equation we now use variation of parameters:

$$\Psi = R^2(A_2 - I_1) + e^{-2u}(B_2 + I_2),$$

where

$$I_1 = \int \frac{L_1}{2R(1 + R\dot{u})} dR \qquad I_2 = \int \frac{R L_1}{2(1 + R\dot{u}) e^{-2u}} dR.$$

The first of these can be completely integrated

$$I_1 = -\left[\frac{R\dot{H} - H}{2R(1 + R\dot{u})} \right]^2,$$

while the second can be integrated by parts to

$$I_2 = -\left[\frac{R\dot{H} - H}{2(1 + R\dot{u})} \right]^2 e^{2u} + \int \frac{(R\dot{H} - H)^2}{2R(1 + R\dot{u})} e^{2u} dR.$$

Thus the solution of the first equation of (3.1) is

$$\Psi = A_2 R^2 + B_2 e^{-2u} + e^{-2u} \int \frac{(R\dot{H} - H)^2}{2R(1 + R\dot{u})} e^{2u} dR. \tag{3.2}$$

Substituting this in the second equation of (3.1) yields the following:

$$A_2(1 + R\dot{u}) \left[R^2(1 + R\dot{u})\ddot{H} + [1 + 2R\dot{u} - R^2(\ddot{u} - 2\dot{u}^2)] \left[R\dot{H} - H \right] \right] = 0.$$

If $A_2 = 0$ we find that the surface is degenerate, (cf. Proposition 5 below). Also, since $1 + R\dot{u}$ is not identically zero, we must solve

$$R^2(1 + R\dot{u})\ddot{H} + [1 + 2R\dot{u} - R^2(\ddot{u} - 2\dot{u}^2)]\left[R\dot{H} - H\right] = 0.$$

This has one solution given by $H_1 = R$ and we apply Lemma 1 to find the second solution:

$$H_2 = R\int R^{-2}e^{-P}dR,$$

where

$$P = \int \frac{1 + 2R\dot{u} - R^2(\ddot{u} - 2\dot{u}^2)}{R(1 + R\dot{u})}dR = -\ln\left(R^{-1}(1 + R\dot{u})e^{-2u}\right).$$

Thus

$$H_2 = R\int R^{-3}(1 + R\dot{u})e^{-2u}dR = -\tfrac{1}{2}R^{-1}e^{-2u},$$

and the complete solution is

$$H = A_1R + B_1R^{-1}e^{-2u}.$$

Substituting this back in equation (3.2) we find that

$$\Psi = A_2R^2 + B_2e^{-2u} - B_1^2R^{-2}e^{-4u},$$

which completes the theorem. □

The following deals with the case $A_2 = 0$:

Proposition 5. *Let (N,g) be a rotationally symmetric Riemannian two-manifold and $\mathbb{G}$ be the associated neutral Kähler metric on* TN. *Then the induced metric is degenerate on the graph $\xi \to (\xi, \eta = F(\xi, \bar{\xi}))$ with:*

$$F = \left[H(R) \pm i\left[B_2e^{-2u} + e^{-2u}\int \frac{(R\dot{H} - H)^2}{2R(1 + R\dot{u})}e^{2u}dR\right]^{\frac{1}{2}}\right]e^{i\theta},$$

for any real differentiable function H.

Proof. The slopes λ and σ can be readily computed for this surface and it is then found that $\lambda^2 = \sigma\bar{\sigma}$. □

Remark. In presence of any Killing vector field on (N,g) we expect that a corresponding invariant area-stationary graph-type surface in TN exists and is given by a similar construction.

4. The space of oriented affine lines in $\mathbb{R}^3$

The four-manifold TS^2 can be identified with the space $\mathbb{L}$ of oriented affine lines in $\mathbb{R}^3$, and the neutral Kähler metric $\mathbb{G}$, as constructed above with g equal to the round metric on S^2, is invariant under the action induced on $\mathbb{L}$ by the Euclidean isometry group acting on $\mathbb{R}^3$ [3].

A surface Σ in $\mathbb{L}$ is a two-parameter family of oriented lines (or line congruence) in $\mathbb{R}^3$, which is the graph of a section of the bundle $\mathbb{L} \to S^2$ iff it can be parameterised by the direction ξ of the oriented lines. Moreover, a surface in $\mathbb{L}$ is Lagrangian iff the associated line congruence is orthogonal to a surface in $\mathbb{R}^3$.

For the round metric $e^{2u} = 4(1+\xi\bar{\xi})^{-2}$ and the above construction of area-stationary line congruences in $\mathbb{L}$ simplifies to:

Theorem 3. *A rotationally symmetric surface in $\mathbb{L}$ which is given by the graph of a local section $\xi \to (\xi, \eta = F(\xi,\bar{\xi}))$ is area-stationary with respect to $\mathbb{G}$ iff*

$$F = \left[A_1R + B_1R^{-1}(1+R^2)^2 \pm i\left[A_2R^2 + B_2(1+R^2)^2 - B_1^2R^{-2}(1+R^2)^4\right]^{\frac{1}{2}}\right] e^{i\theta},$$

for $A_1, A_2, B_1, B_2 \in \mathbb{R}$, $A_2 \neq 0$, where $\xi = Re^{i\theta}$ and $g = e^{2u}d\xi d\bar{\xi}$ for $u = u(R)$.

To find closed area-stationary surfaces, we must have $B_1 = 0$, since otherwise the surface is asymptotic to the fibre of the bundle $\mathbb{L} \to S^2$ at $R = 0$. In addition, by a translation we can set $A_1 = 0$ and the surface is determined by

$$F = \pm i\left[B_2 + C_2R^2 + B_2R^4\right]^{\frac{1}{2}} e^{i\theta},$$

for $C_2 \in \mathbb{R}$ such that $-2B_2 \leq C_2$, and $B_2 \geq 0$.

This can be extended through $R = 0$ and $R \to \infty$ and yields a two parameter family of area-stationary tori for $C_2 \neq 2B_2$. Under the projection map $\mathbb{L} \to S^2$ these tori double cover the sphere, except at the north and south pole, where the inverse image of each of these points is a circle.

The induced metric is positive definite on the upper part and negative definite on the lower part of these tori, or vice versa, depending on the sign of $C_2 - 2B_2$. The inner and outer meridian circles (given by $R = 1$) are totally null: the surface is both Lagrangian and holomorphic at these points.

Finally for $C_2 = 2B_2$

$$F = \pm i\sqrt{B_2}\left(1+R^2\right) e^{i\theta},$$

is a torus on which the induced metric is degenerate.

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