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Multiplication Modules and Homogeneous Idealization III

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Abstract. In our recent work we gave a treatment of certain aspects of multiplication modules, projective modules, flat modules and cancellation-like modules via idealization. The purpose of this work is to continue our study and develop the tool of idealization, particularly in the context of closed, divisible injective, and simple modules. We determine when a ring $R(M)$, the idealization of M, is a quasi-Frobenius or a distinguished ring. We also introduce and investigate the concept of M-$\frac{1}{2}$ (weak) cancellation ideals.

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0. Introduction

All rings are commutative with identity and all modules are unital. Let R be a ring and M an R-module. M is a *multiplication* module if every submodule N of M has the form IM for some ideal I of R. Equivalently, $N = [N : M]\,M$, [18]. A submodule K of M is multiplication if and only if $N \cap K = [(N \cap K) : K]\,K = [N : K]\,K$ for all submodules N of M.

Let P be a maximal ideal of R and let $T_P(M) = \{m \in M : (1-p)\,m = 0$ for some $m \in M\}$. Then $T_P(M)$ is a submodule of M. M is called *P-torsion* if $T_P(M) = M$. On the other hand M is called *P-cyclic* provided there exist $m \in M$ and $q \in P$ such that $(1-q)\,M \subseteq Rm$. El-Bast and P. F. Smith, [19, Theorem 1.2], showed that M is multiplication if and only if M is P-torsion or P-cyclic for each maximal ideal P of R. A multiplication module M is locally cyclic and the converse is true if M is finitely generated, [18, Proposition 4]. A submodule N of M is called a *pure* submodule of M if $IN = N \cap IM$ for every ideal I of R, [20]. An ideal I is a pure ideal of R if and only if I is locally either R or zero, [22].

Let N be a submodule of M and I an ideal of R. The *residual* submodule N by I is $[N :_M I] = \{m \in M : Im \subseteq N\}$, [24] and [25]. Obviously, $[N : IM]\, M \subseteq [N :_M I]$. The reverse inclusion is true if M is multiplication. If M is faithful multiplication or projective then $[0 :_M I] = (\mathrm{ann} I)\, M$. See [7] and [2, Lemma 5].

For a ring R and an R-module M, the *idealization* of M is the ring $R_{(+)}M = R\,(M)$ which is formed from the direct sum $R \oplus M$ by defining multiplication of elements (r, m) and (s, n) of $R\,(M)$ by $(r, m)\,(s, n) = (rs, rn + sm)$. The purpose of idealization is to put M inside a commutative ring A so that the structure of M as an R-module is essentially the same as that of M as an A-module, that is an ideal of A. $0_{(+)}M$ is an ideal of $R\,(M)$ satisfying $(0_{(+)}M)^2 = 0$. Every ideal contained in $0_{(+)}M$ has the form $0_{(+)}N$ for some submodule N of M and every ideal contains $0_{(+)}M$ has the form $I_{(+)}M$ for some ideal I of R. Since $R \cong R\,(M)\,/0_{(+)}M, I \to I_{(+)}M$ gives a one-to-one correspondence between the ideals of R and the ideals of $R\,(M)$ containing $0_{(+)}M$. Thus prime (maximal) ideals of $R\,(M)$ have the form $P_{(+)}M$ for some prime (maximal) ideals P of R. *Homogeneous* ideals of $R\,(M)$ have the form $I_{(+)}N$ where I is an ideal of R, N a submodule of M and $IM \subseteq N$. These ideals play a special role in studying properties of $R\,(M)$ and showing how these properties are related to those of R and M. It is shown, [17, Theorem 3.3], that a principal ideal $R\,(M)\,(a, m)$ is homogeneous if and only if $R\,(M)\,(a, m) = Ra_{(+)}(Rm + aM) = R\,(M)\,(a, 0) + R\,(M)\,(0, m)$, and every ideal of $R\,(M)$ is homogeneous if and only if every principal ideal of $R\,(M)$ is homogeneous. A ring $R\,(M)$ is called *homogeneous* if every ideal of $R\,(M)$ is homogeneous, [3] and [4]. It is shown, [17, Corollary 3.4], that if R is an integral domain then $R\,(M)$ is homogeneous if and only if M is divisible. In this case every ideal of $R\,(M)$ is comparable to $0_{(+)}M$. If R is a local ring but not an integral domain, $R\,(M)$ is homogeneous if and only if $M = 0$, [17, Theorem 3.3]. If $I_{(+)}N$ and $J_{(+)}K$ are homogeneous ideals of $R\,(M)$ then

$$\left[I_{(+)}N :_{R(M)} J_{(+)}K\right] = [I : J] \cap [N : K]_{(+)}[N :_M J]$$

is homogeneous, [1, Lemma 1]. In particular, $\mathrm{ann}\,(I_{(+)}N) = (\mathrm{ann} I \cap \mathrm{ann} N)_{(+)}$ $[0 :_M I]$. If M is faithful multiplication or projective then $\mathrm{ann}\,(I_{(+)}IM) = \mathrm{ann} I_{(+)}$ $(\mathrm{ann} I)\, M$.

In our recent work on idealization we investigated the idealization of multiplication modules, projective modules, flat modules, like-cancellation modules, invertible submodules, large and small submodules. We also determined when a ring $R\,(M)$ is a multiplication ring, ZPI-ring, arithmetical ring, Prüfer ring, Bezout ring, (quasi) valuation ring, Marot ring, P-ring, coherent ring, finite conductor ring or generalized GCD ring, see [1]–[5].

In the first part of this paper we investigate the idealization of modules, particularly in the context of closed, divisible, injective and simple modules, continuing and extending some of our results in [1]–[5]. In the second part we show how quasi-Frobenius and distinguished properties of $R(M)$ are related to those of R and M. In the last part we introduce and investigate the concept of M-$\frac{1}{2}$ (weak) cancellation and M-$\frac{1}{2}$ join principal ideals, particularly when M is a multiplication module. In the first and third part of the paper, some of our results of [3] will be improved by weakening the required conditions.

All rings are commutative with identity and all modules are unital. For the basic concepts used, we refer the reader to [20]–[25].

1. Some properties of homogeneous ideals

In this section we explore several properties of a homogeneous ideal of $R(M)$ and show how these properties can be transferred to its components and conversely.

Let R be a ring and N a submodule of an R-module M. Then N is called an *addition complement* (resp. *intersection complement*), briefly adco (resp. inco), of a submodule K of M if $K + N = M$ (resp. $K \cap N = 0$) and N is minimal (resp. maximal) in $K + N = M$ (resp. $K \cap N = 0$), that is for all submodules L of M with $L + K = M$ (resp. $L \cap K = 0$) and $L \subseteq N$ (resp. $N \subseteq L$) then $N = L$. It is shown, [23, Corollary 5.2.2], that for all submodules K and N of M, $K \oplus N = M$ if and only if N is adco and inco of K in M. It is also well-known, [23, Lemmas 5.2.4 and 5.2.5], that for all submodules N, K and L of M if N is adco (resp. inco) of K and K is adco (resp. inco) of L then N is adco (resp. inco) of L. A submodule N of M is *large* in M if for all submodules K of M, $N \cap K = 0$ implies $K = 0$. A submodule X of M is said to be *closed* in M if whenever X is a large submodule of a submodule U of M then $X = U$. It is proved that a submodule X of M is closed in M if and only if X is inco of a submodule of M, [23]. Every submodule of M is a direct summand in M if and only if every submodule of M is closed in M. For properties of adco, inco, large and closed submodules and ideals, see for example [20] and [23].

The following result gives some basic properties of the addition complement, intersection complement and closed ideals and submodules.

Proposition 1. *Let R be a ring, M an R-module, I and J ideals of R and K and N submodules of M.*

(1) *Let M be finitely generated multiplication. If $I + \operatorname{ann}M$ is adco of J then IM is adco of JM and the converse is true if we assume further that M is faithful.*

(2) *Let M be faithful multiplication. If I is inco of J then IM is inco of JM and the converse is true if we assume further that M is finitely generated.*

(3) *Let M be faithful multiplication. If I is closed in R then IM is closed in M and the converse is true if we assume further that M is finitely generated.*

(4) *Let M be finitely generated multiplication. If $[N : M]$ is adco of $[K : M]$ then N is adco of K and the converse is true if we assume further that M is faithful.*

(5) *Let M be faithful multiplication. If $[N : M]$ is inco of $[K : M]$ then N is inco of K and the converse is true if we assume further that M is finitely generated.*

(6) *Let M be faithful multiplication. If $[N : M]$ is closed in R then N is closed in M and the converse is true if we assume further that M is finitely generated.*

Proof. (1) Let M be finitely generated multiplication. Let $I+\mathrm{ann}M$ be adco of J. Then $I+\mathrm{ann}M+J=R$, and hence $IM+JM=M$. Suppose L be a submodule of M such that $L+JM=M$ and $L\subseteq IM$. It follows that $M=([L:M]+J)M$, and by, [22, Theorem 76], we have that

$$R=[L:M]+J+\mathrm{ann}M=[L:M]+J.$$

Since $[L:M]\subseteq[IM:M]=I+\mathrm{ann}M$, we get that $I+\mathrm{ann}M=[L:M]$ and hence $IM=L$. So IM is adco of JM. Conversely, assume M is finitely generated, faithful and multiplication. Let IM be adco of JM. Then $(I+J)\,M=IM+JM=M$, and hence $I+J=R$, [29, Corollary 1 to Theorem 9]. Assume A is an ideal of R such that $A+J=R$ and $A\subseteq I$. Then $AM+JM=M$ and $AM\subseteq IM$. Since IM is adco of $JM, AM=IM$, and hence $A=I$. So I is adco of J.

(2) Suppose M is faithful and multiplication. Let I be inco of J. Then $I\cap J=0$ and by, [19, Corollary 1.7], $IM\cap JM=0$. Let L be a submodule of M such that $L\cap JM=0$ and $IM\subseteq L$. So $[L:M]\cap J\subseteq[L:M]\cap[JM:M]=[(L\cap JM):M]=[0:M]=0$ and $I\subseteq[L:M]$. It follows that $I=[L:M]$ and hence $IM=L$. Then IM is inco of JM. Conversely, let IM be inco of JM. Then $IM\cap JM=0$, and hence $(I\cap J)\,M=0$. This implies that $I\cap J\subseteq[0:M]=0$. Let A be an ideal of R such that $A\cap J=0$ and $I\subseteq A$. Then $AM\cap JM=0$ and $IM\subseteq AM$. Hence $IM=AM$ and hence $I=A$. This gives that I is inco of J.

(3) Let I be closed in R and let IM be large in a submodule L of M. Then I is large in $[L:M]$. For, let $J\subseteq[L:M]$ and $I\cap J=0$. Since M is faithful multiplication, it follows by, [19, Corollary 1.7], that $IM\cap JM=0$. Since $JM\subseteq L$ and IM is large in L, $JM=0$, and hence $J=0$. As I is closed in R, we get that $I=[L:M]$ and hence $L=IM$. So IM is closed in M. For the converse, suppose that IM is closed in M. Let I be large in J. Then IM is large in JM. For, if $L\subseteq JM$ and $IM\cap L=0$, then

$$0=[0:M]=[(IM\cap L):M]=I\cap[L:M]\,.$$

It follows that $[L:M]=0$, and hence $L=0$. This implies that $IM=JM$, and hence $I=J$. So I is closed in R.

The proofs of (4), (5) and (6) follow by (1), (2) and (3) respectively. □

The next two results give some properties of idealization of addition complement, intersection complement and closed ideals and submodules.

Theorem 2. *Let R be a ring, M an R-module and $R(M)$ the idealization of M. Let I be an ideal of R and N a submodule of M.*

(1) *If $0_{(+)}N$ is adco of an ideal H of $R(M)$ then $H=R(M)$.*

(2) *Let M be faithful. If $I_{(+)}M$ is inco of an ideal H of $R(M)$ then $H=0$.*

(3) *N is adco of a submodule K of M if and only $0_{(+)}N$ is adco of $0_{(+)}K$ in $0_{(+)}M$.*

(4) *N is inco of a submodule K of M if and only if $0_{(+)}N$ is inco of $0_{(+)}K$ in $0_{(+)}M$.*

(5) *N is closed in M if and only if $0_{(+)}N$ is closed in $0_{(+)}M$.*

(6) *Let M be faithful multiplication. If $I_{(+)}IM$ is closed in $R(M)$ then I is closed in R. The converse is true if we assume further that M is finitely generated.*

(7) *If $I_{(+)}IM$ is adco of an ideal $J_{(+)}JM$ of $R(M)$ then I is adco of J. The converse is true if $R(M)$ is homogeneous and M finitely generated and faithful.*

(8) *Let M be faithful multiplication. If $I_{(+)}IM$ is inco of an ideal $J_{(+)}JM$ of $R(M)$ then I is inco of J. The converse is true if we assume further that $R(M)$ is homogeneous.*

Proof. (1) If $0_{(+)}N$ is adco of H then $0_{(+)}N + H = R(M)$. Hence $H(0_{(+)}N) = 0_{(+)}N$, and hence $0_{(+)}N \subseteq H$. So $H = R(M)$.

(2) If $I_{(+)}M$ is inco of H then $I_{(+)}M \cap H = 0$, and hence $0_{(+)}M \cap H = 0$. Since M is faithful, $0_{(+)}M$ is large in $R(M)$, [3, Proposition 14]. Hence $H = 0$.

The proofs of (3), (4) and (5) are easy exercises by using the fact that any ideal $H \subseteq 0_{(+)}M$ has the form $0_{(+)}K$ for some submodule K of M.

(6) Suppose $I_{(+)}IM$ is closed. Let J be an ideal of R such that I is large in J. Then $I_{(+)}IM$ is large in $J_{(+)}JM$. For, let $H \subseteq J_{(+)}JM$ such that $I_{(+)}IM \cap H = 0$. Then $I_{(+)}IM \cap H \cap 0_{(+)}M = 0$. Assume $H \cap 0_{(+)}M = 0_{(+)}K$ for some submodule K of M. Then $0_{(+)}IM \cap K = 0$, and hence $IM \cap K = 0$. Next, $0_{(+)}K = H \cap 0_{(+)}M \subseteq J_{(+)}JM \cap 0_{(+)}M = 0_{(+)}JM$, and hence $K \subseteq JM$. Since I is large in J, IM is large in JM and this implies that $K = 0$ and hence $0 = H \cap 0_{(+)}M$. Since $0_{(+)}M$ is large in $R(M)$, $H = 0$. So $I_{(+)}IM$ is large in $J_{(+)}JM$ gives that $I_{(+)}IM = J_{(+)}JM$, and hence $I = J$. So I is closed in R. Conversely, let I be closed in R and suppose H is an ideal of $R(M)$ such that $I_{(+)}IM$ is large in H. Then $I_{(+)}IM$ is large in $H \cap 0_{(+)}M = 0_{(+)}K$ for some submodule K of M. We show that IM is large in K. Let $L \subseteq K$ with $IM \cap L = 0$. Then $I_{(+)}IM \cap 0_{(+)}L = 0$ where $0_{(+)}L \subseteq 0_{(+)}K$. It follows that $0_{(+)}L = 0$ and hence $L = 0$. As I is closed in R, we get from Proposition 1 that IM is closed in M and hence $IM = K$. This implies that

$$I_{(+)}IM \cap 0_{(+)}M = 0 + IM = 0_{(+)}K = H \cap 0_{(+)}M.$$

Similarly, $I_{(+)}IM$ is large in $H(0_{(+)}M) = 0_{(+)}X$ for some submodule X of M and hence IM is large in X. Since IM is closed in M, $IM = X$, and hence

$$(I_{(+)}IM)(0_{(+)}M) = 0_{(+)}IM = 0_{(+)}X = H(0_{(+)}M).$$

Since M is finitely generated faithful multiplication, $0_{(+)}M$ is finitely generated multiplication, [1, Theorem 2] and [15, Theorem 3.1], and moreover $\operatorname{ann}(0_{(+)}M) = 0_{(+)}M$. So by, [29, Theorem 9], we have that $I_{(+)}IM + 0_{(+)}M = H + 0_{(+)}M$. Using the modular law which states that for all submodules K, L and N of M such that $K \subseteq N$ then $(K+L) \cap N = K + (L \cap N)$, one gets that

$$\begin{aligned} H &= (H + 0_{(+)}M) \cap H = (I_{(+)}IM + 0_{(+)}M) \cap H \\ &= I_{(+)}IM + (H \cap 0_{(+)}M) = I_{(+)}IM + 0_{(+)}IM \\ &= I_{(+)}IM. \end{aligned}$$

Hence $I_{(+)}IM$ is closed in $R(M)$.

(7) Suppose $I_{(+)}IM$ is adco of $J_{(+)}JM$. Then $(I+J)_{(+)}(I+J)\,M = I_{(+)}IM + J_{(+)}JM = R\,(M)$ and hence $I + J = R$. Suppose A is an ideal of R such that $A + J = R$ and $A \subseteq I$. Then $A_{(+)}AM + J_{(+)}JM = A + J_{(+)}(A+J)\,M = R\,(M)$ and $A_{(+)}AM \subseteq I_{(+)}IM$ implies that $A_{(+)}AM = I_{(+)}IM$. Hence $A = I$ and I is adco of J. Suppose now M is finitely generated faithful multiplication and $R\,(M)$ is homogeneous. Let I be adco of J. Then $I + J = R$ and hence $I_{(+)}IM + J_{(+)}JM = R\,(M)$. Suppose $A_{(+)}K$ is an ideal of $R\,(M)$ such that $A_{(+)}K + J_{(+)}JM = R\,(M)$ and $A_{(+)}K \subseteq I_{(+)}IM$. Then $A + J = R$ with $A \subseteq I$ and $K + JM = M$ with $K \subseteq IM$. Since I is adco of J, $A = I$. Since M is finitely generated faithful multiplication, it follows by Proposition 1 that IM is adco of JM and hence $K = IM$. So $A_{(+)}K = I_{(+)}IM$ and this shows that $I_{(+)}IM$ is adco of $J_{(+)}JM$.

(8) If $I_{(+)}IM$ is inco of $J_{(+)}JM$ then $I_{(+)}IM \cap J_{(+)}JM = 0$ implies that $I \cap J = 0$. Suppose B is an ideal of R such that $B \cap J = 0$ and $I \subseteq B$. Since M is faithful multiplication, we have that $B_{(+)}BM \cap J_{(+)}JM = B \cap J_{(+)}(B \cap J)\,M = 0$. As $I_{(+)}IM \subseteq B_{(+)}BM$, we get that $I_{(+)}IM = B_{(+)}BM$, and hence $I = B$. So I is inco of J. Conversely, suppose $R\,(M)$ is homogeneous. Let $B_{(+)}L$ be an ideal of $R\,(M)$ such that $B_{(+)}L \cap J_{(+)}JM = 0$ and $I_{(+)}IM \subseteq B_{(+)}L$. Then $B \cap J = 0$ with $I \subseteq B$ and $L \cap JM = 0$ with $IM \subseteq L$. Since I is inco of J, $I = B$. By Proposition 1, IM is inco of JM and hence $L = IM$. This shows that $B_{(+)}L = I_{(+)}IM$ and hence $I_{(+)}IM$ is inco of $J_{(+)}JM$. □

Proposition 3. *Let R be a ring, M an R-module and $I_{(+)}N$ and $J_{(+)}K$ homogeneous ideals of $R\,(M)$.*

1. *If $I_{(+)}N$ is adco of $J_{(+)}K$ then I is adco of J. If I is adco of J and N is adco of K such that $R\,(M)$ is homogeneous then $I_{(+)}N$ is adco of $J_{(+)}K$.*
2. *If $I_{(+)}N$ is inco of $J_{(+)}K$ then N is inco of K. If I is inco of J and N is inco of K such that $R(M)$ is homogeneous then $I_{(+)}N$ is inco of $J_{(+)}K$.*
3. *If $I_{(+)}N$ is closed in $R\,(M)$ then N is closed in M.*

Proof. (1) Suppose $I_{(+)}N$ is adco of $J_{(+)}K$. Then $I_{(+)}N + J_{(+)}K = R(M)$ which implies that $I + J = R$. Let A be an ideal of R such that $A + J = R$ and $A \subseteq I$. Since $AM \subseteq IM \subseteq N$, $A_{(+)}N$ is a homogeneous ideal of $R\,(M)$ which satisfies that $A_{(+)}N \subseteq I_{(+)}N$ and $A_{(+)}N + J_{(+)}K = R(M)$. This gives that $A_{(+)}N = I_{(+)}N$ and hence $A = I$. So I is adco of J. Now, let $R\,(M)$ be a homogeneous ring, I is adco of J and N is adco of K. Then $I + J = R$ and $N + K = M$, and hence $(I+J)_{(+)}(N+K) = I_{(+)}N + J_{(+)}K = R\,(M)$. Suppose $A_{(+)}L$ be an ideal of $R\,(M)$ such that $A_{(+)}L + J_{(+)}K = R\,(M)$ and $A_{(+)}L \subseteq I_{(+)}N$. It follows that $A + J = R$

with $A \subseteq I$ and $L + K = M$ with $L \subseteq K$. Hence $A = I$ and $L = N$ from which we get that $A_{(+)}L = I_{(+)}N$, and hence $I_{(+)}N$ is adco of $J_{(+)}K$.

(2) Let $I_{(+)}N$ be inco of $J_{(+)}K$. Then $I_{(+)}N \cap J_{(+)}K = 0$ and hence $N \cap K = 0$. Suppose L is a submodule of M such that $L \cap K = 0$ and $N \subseteq L$. Since $IM \subseteq N \subseteq L$, $I_{(+)}L$ is a homogeneous ideal of $R(M)$. Moreover, $I_{(+)}L \cap J_{(+)}K = 0$ and $I_{(+)}N \subseteq I_{(+)}L$. This implies that $I_{(+)}N = I_{(+)}L$ and hence $N = L$. Hence N is inco of K. Assume now $R(M)$ is a homogeneous ring, I inco of J and N inco of K. Then $I \cap J = 0$ and $N \cap K = 0$ which shows that $I_{(+)}N \cap J_{(+)}K = 0$. Let $A_{(+)}L$ be an ideal of $R(M)$ such that $A_{(+)}L \cap J_{(+)}K = 0$ and $I_{(+)}N \subseteq A_{(+)}L$. Hence $A \cap J = 0$ with $I \subseteq A$ and $L \cap K = 0$ with $L \subseteq N$. Therefore $I = A$ and $N = L$ from which we infer that $I_{(+)}N = A_{(+)}L$ and hence $I_{(+)}N$ is inco of $J_{(+)}K$.

(3) Suppose $I_{(+)}N$ is closed in $R(M)$. Then $I_{(+)}N$ is inco of some ideal H of $R(M)$. It follows that $I_{(+)}N \cap H = 0$, and hence $I_{(+)}N \cap H \cap 0_{(+)}M = 0$. Let $H \cap 0_{(+)}M = 0_{(+)}L$ for some submodule L of M. Then $0_{(+)}(N \cap L) = I_{(+)}N \cap 0_{(+)}L = 0$, and hence $N \cap L = 0$. We claim that N is inco of L. Let X be a submodule of M such that $X \cap L = 0$ and $N \subseteq X$. Since $IM \subseteq N \subseteq X, I_{(+)}X$ is a homogeneous ideal of $R(M)$ which satisfies $I_{(+)}X \cap 0_{(+)}L = 0$ and $I_{(+)}N \subseteq I_{(+)}X$. This gives that $I_{(+)}N = I_{(+)}X$ and hence $N = X$. Hence N is closed in M. $\square$

An R-module M is called *finitely cogenerated* if for every non-empty collection of submodules N_λ $(\lambda \in \Lambda)$ of M with $\bigcap_{\lambda\in\Lambda} N_\lambda = 0$, there exists a finite subset Λ' of Λ such that $\bigcap_{\lambda\in\Lambda'} N_\lambda = 0$. M is called *uniform* if the intersection of any two non-zero submodules of M is non-zero, while M has *finite uniform dimension* if it does not contain an infinite direct sum of non-zero submodules, [23]. As a dual of large submodules, a submodule N of M is called *small* in M if for all submodules K of M, $N+K = M$ implies $K = M$. It is shown, [23, Lemma 5.1.4], that a submodule N of M is not small in M if and only if there exists a maximal submodule Q of M with $N \subsetneq Q$.

Compare the next result with [3, Propositions 17 and 18].

Proposition 4. *Let R be a ring, M an R-module and $I_{(+)}N$ a homogeneous ideal of $R(M)$.*

1. *$I_{(+)}N$ is small if and only if I is a small ideal of R.*
2. *Let M be finitely generated and faithful. If N is a small submodule of M then $I_{(+)}N$ is small.*
3. *Let M be faithful multiplication. If I is a large ideal of R then $I_{(+)}N$ is large. The converse is true if we assume further that $I_{(+)}N$ is pure.*
4. *Let M be faithful. If N is a finitely cogenerated (resp. uniform, has finite uniform dimension) submodule of M then $I_{(+)}N$ is finitely cogenerated (resp. uniform, has finite uniform dimension).*

Proof. (1) Suppose $I_{(+)}N$ is not small. There exists a maximal ideal $P_{(+)}M$ of $R(M)$ such that $I_{(+)}N \nsubseteq P_{(+)}M$. Hence $I \nsubseteq P$, and hence I is not small. The statement is reversible.

(2) Suppose M is finitely generated and faithful. Let N be a small submodule of M. Then $[N:M]$ is a small ideal of R. For, let A be an ideal of R such that $[N:M]+A=R$ then $M=[N:M]M+AM\subseteq N+AM\subseteq M$, so that $N+AM=M$. Hence $AM=M$. Since M is finitely generated faithful, it follows by, [22, Theorem 76], that $A=R$, and hence $[N:M]$ is small. It follows by (1) that $[N:M]_{(+)}N$ is a small ideal of $R(M)$. Since $I_{(+)}N\subseteq[N:M]_{(+)}N$, we infer that $I_{(+)}N$ is small. Alternatively, since $[N:M]$ is small we get that $I\subseteq[N:M]$ is small and by (1), $I_{(+)}N$ is small.

(3) Since M is faithful multiplication and I is large, we infer that IM is a large submodule of M, [6, Proposition 12]. Since $IM\subseteq N, N$ is a large submodule of M. Since M is faithful, it follows by, [3, Proposition 14], that $I_{(+)}N$ is large. Conversely, suppose $I_{(+)}N$ is a large and pure ideal of $R(M)$. Let J be an ideal of R such that $I\cap J=0$. Then $IJ=0$, and hence $J\subseteq\operatorname{ann}I$ from which it follows that $J_{(+)}(\operatorname{ann}I)M$ is a homogeneous ideal of $R(M)$. Next

$$\begin{aligned} J_{(+)}(\operatorname{ann}I)M\cap I_{(+)}N &= J\cap I_{(+)}(\operatorname{ann}I)M\cap N \\ &\subseteq 0_{(+)}[0:_M I]\cap N. \end{aligned}$$

Using the fact that for any pure ideal A of R (hence A is locally either R or zero), $A\cap\operatorname{ann}A=0$ is true locally and hence globally, one gets that

$$\begin{aligned} 0 &= I_{(+)}N\cap\operatorname{ann}(I_{(+)}N)=I_{(+)}N\cap(\operatorname{ann}I\cap\operatorname{ann}N)_{(+)}[0:_M I] \\ &= (I\cap\operatorname{ann}I\cap\operatorname{ann}N)_{(+)}N\cap[0:_M I]. \end{aligned}$$

This gives that $J_{(+)}(\operatorname{ann}I)M\cap I_{(+)}N=0$, and hence $J_{(+)}(\operatorname{ann}I)M=0$. So $J=0$ and hence I is a large ideal of R.

(4) Suppose M is faithful. Let $H_\lambda\,(\lambda\in\Lambda)$ be a non-empty collection of ideals of $R(M)$ that are contained in $I_{(+)}N$ such that $\bigcap_{\lambda\in\Lambda}H_\lambda=0$. Then $\bigcap_{\lambda\in\Lambda}(H_\lambda\cap 0_{(+)}M)=(\bigcap_{\lambda\in\Lambda}H_\lambda)\cap 0_{(+)}M=0$. Assume $H_\lambda\cap 0_{(+)}M=0_{(+)}K_\lambda$ for some submodules K_λ of M. Since $H_\lambda\subseteq I_{(+)}N$, $0_{(+)}K_\lambda\subseteq 0_{(+)}N$ and hence $K_\lambda\subseteq N$. As N is finitely cogenerated, there exists a finite subset Λ' of Λ such that $\bigcap_{\lambda\in\Lambda'}K_\lambda=0$. It follows that

$$0=0+\bigcap_{\lambda\in\Lambda'}K_\lambda=\bigcap_{\lambda\in\Lambda'}0_{(+)}K_\lambda=\left(\bigcap_{\lambda\in\Lambda'}H_\lambda\right)\cap 0_{(+)}M.$$

Since M is faithful, $0_{(+)}M$ is a large ideal of $R(M)$ by [3, Proposition 14] and this shows that $\bigcap_{\lambda\in\Lambda'}H_\lambda=0$. Hence $I_{(+)}N$ is finitely cogenerated. The uniform case is now obvious. Finally, if $I_{(+)}N$ contains a direct sum of subideals H_λ then $0_{(+)}N=I_{(+)}N\cap 0_{(+)}M$ contains a direct sum of subideals $H_\lambda\cap 0_{(+)}M$. Assume $H_\lambda\cap 0_{(+)}M=0_{(+)}K_\lambda$ for some submodules K_λ of M. It follows that $\sum_{\lambda\in\Lambda}K_\lambda$ is a direct sum and contained in N. Since N has finite uniform dimension, all but a finite number of K_λ is zero. If $K_\lambda=0$, then $H_\lambda\cap 0_{(+)}M=0_{(+)}K_\lambda=0$. Since M is faithful, $0_{(+)}M$ is large in $R(M)$ and hence $H_\lambda=0$. Hence $I_{(+)}N$ has finite uniform dimension. $\square$

An R-module M is called *divisible* if for every regular element $r \in R, M = rM$. And M is called *injective* if it is a direct summand of any R-module B that is a submodule of, that is M is injective if for R-modules B with $M \subseteq B, B = M \oplus K$ for some submodule K of B, [23]. A $\mathbb{Z}$-module M is injective if and only if M is divisible. More generally, if R is an integral domain then a torsion-free R-module M is injective if and only if M is divisible, [23]. In this paper we say that a submodule N of M is *weak injective* if it is a direct summand of any submodule of M that is contained in. In particular, if N is weak injective then it is a direct summand of M. Let I be an ideal of R and M a faithful multiplication R-module. If I is weak injective then IM is weak injective. For if N is a submodule of M with $IM \subseteq N$, then $I \subseteq [N : M]$ and hence $[N : M] = I \oplus A$ for some ideal A of R, hence $N = IM \oplus AM$. The converse is true if one assumes further that M is finitely generated. Let $I \subseteq J$ for some ideal J of R. Then $IM \subseteq JM$, and hence $JM = IM \oplus K = IM \oplus [K : M] M$. This implies that $J = I \oplus [K : M]$. So for a submodule N of a faithful multiplication module M, if $[N : M]$ is a weak injective ideal of R then N is weak injective and the converse is true if we suppose further that M is finitely generated.

The next two results show how the divisibility and weak injectivity of a homogeneous ideal of $R(M)$ can be transferred to its components and conversely.

Proposition 5. *Let R be a ring, M an R-module, I an ideal of R and N a submodule of M.*

(1) *If N is divisible then $0_{(+)}N$ is a divisible ideal of $R(M)$ and the converse is true if M is torsion-free.*
(2) *If M is torsion-free and $I_{(+)}IM$ a divisible ideal of $R(M)$ then I is divisible. The converse is true if we assume further that M is divisible.*
(3) *Let $IM \subseteq N$. If M is torsion-free and $I_{(+)}N$ a divisible ideal of $R(M)$ then I is divisible and N divisible. The converse is true if we assume further that M is divisible.*

Proof. (1) Suppose N is divisible. Let (r, m) be a regular element of $R(M)$. Then r is a regular element of R and hence $N = rN$. This gives that $(r, m)(0_{(+)}N) = 0_{(+)}rN = 0_{(+)}N$, and hence $0_{(+)}N$ is divisible. Assume M is torsion-free. Suppose $0_{(+)}N$ is divisible and let $r \in R$ be regular. Then $(r,0)$ is regular in $R(M)$, [1, Lemma 6], and hence $0_{(+)}N = (r,0)(0_{(+)}N) = 0_{(+)}rN$. So $N = rN$, and hence N is divisible.

(2) Suppose M is torsion-free and $I_{(+)}IM$ is divisible. Let $r \in R$ be regular then $(r,0) \in R(M)$ is regular. It follows that $I_{(+)}IM = (r,0)(I_{(+)}IM) = rI_{(+)}rIM$, and hence $I = rI$ which means that I is divisible. Conversely, assume M is torsion-free and divisible and I is divisible. Let $(r, m) \in R(M)$ be regular. It follows by [17, Theorem 3.9] that $R(M)(r,m) = R(M)(r, 0)$ and r is a regular element of R. This implies that $I_{(+)}IM = rI_{(+)}rIM = (r,0)(I_{(+)}IM) = (r,m)(I_{(+)}IM)$. Hence $I_{(+)}IM$ is divisible.

(3) Since $IM \subseteq N, I_{(+)}N$ is a homogeneous ideal of $R(M)$. Suppose M is torsion-free and $I_{(+)}N$ divisible. Let $r \in R$ be regular then $(r,0) \in R(M)$ is regular.

Hence $I_{(+)}N = (r,0)\,(I_{(+)}N) = rI_{(+)}rN$, and hence $I = rI$ and $N = rN$. So I is a divisible ideal of R and N a divisible submodule of M. Conversely, assume M is torsion-free and divisible. Let $(r,m) \in R\,(M)$ be regular then r is regular and by [17, Theorem 3.9], we have that

$$\begin{aligned} R\,(M)\,(r,m)\,(I_{(+)}N) &= (Rr_{(+)}M)\,(I_{(+)}N) = rI_{(+)}(rN + IM) \\ &= rI_{(+)}(rN + rIM) = rI_{(+)}rN = I_{(+)}N. \end{aligned}$$

Hence $I_{(+)}N$ is divisible. □

Proposition 6. *Let R be a ring, M an R-module, I an ideal of R and N a submodule of M.*

(1) *If $0_{(+)}N$ is a weak injective ideal of $R\,(M)$ then $N = 0$.*

(2) *Let $IM \subseteq N$. If $I_{(+)}N$ is a weak injective ideal of $R(M)$ then I is weak injective. Assuming further that M is multiplication then N is weak injective.*

(3) *I is weak injective if and only if $I_{(+)}IM$ is a weak injective ideal of $R\,(M)$.*

Proof. (1) If $0_{(+)}N$ is weak injective then it is a direct summand of $R(M)$ and hence $N = 0$, [4, Proposition 1].

(2) Suppose $I_{(+)}N$ is weak injective. Then $I_{(+)}N$ is a direct summand of $R(M)$ and hence $R\,(M) = I_{(+)}N \oplus H$ for some ideal H of $R\,(M)$. Let $H + 0_{(+)}M = A_{(+)}M$ for some ideal A of R. Then $R\,(M) = I_{(+)}N + A_{(+)}M$ and hence $R = I + A$. Let J be an ideal of R such that $I \subseteq J$. Since $I + A$ is multiplication, we get from, [14, Corollary 1.2] and [29, Proposition 4], that $J = J \cap (I + A) = (J \cap I) + (J \cap A) = I + (J \cap A)$. Since $R\,(M) = I_{(+)}N + H$ is multiplication, it follows again by, [14, Corollary 1.2] and [29, Propostion 4], that

$$\begin{aligned} 0_{(+)}M &= (I_{(+)}N \cap H) + 0_{(+)}M = (I_{(+)}N + 0_{(+)}M) \cap (H + 0_{(+)}M) \\ &= I_{(+)}M \cap A_{(+)}M = (I \cap A)_{(+)}M. \end{aligned}$$

So $I \cap A = 0$ and hence $I \cap (J \cap A) = 0$. This implies that $J = I \oplus (J \cap A)$. Hence I is a weak injective ideal of R. Suppose now M is multiplication. Since $R\,(M) = I_{(+)}N + H$ is a multiplication ideal of $R(M)$, we infer that

$$\begin{aligned} 0_{(+)}M &= (I_{(+)}N + H) \cap 0_{(+)}M \\ &= (I_{(+)}N \cap 0_{(+)}M) + (H \cap 0_{(+)}M) = 0_{(+)}N + (H \cap 0_{(+)}M). \end{aligned}$$

Assume $H \cap 0_{(+)}M = 0_{(+)}L$ for some submodule L of M. Then $M = N + L$. Let K be a submodule of M such that $N \subseteq K$. Hence $K = K \cap (N + L)$. As $M = N + L$ is multiplication, it follows by [14, Corollary 1.2] that $K = (K \cap N) + (K \cap L) = N + (K \cap L)$. On the other hand, we have $I_{(+)}N \cap H = 0$ and hence $I_{(+)}N \cap H \cap 0_{(+)}M = 0$. So $I_{(+)}N \cap 0_{(+)}L = 0$ and hence $N \cap L = 0$ which implies that $N \cap (K \cap L) = 0$. This finally implies that $K = N \oplus (K \cap L)$ and this shows that N is a weak injective submodule of M.

(3) Suppose I is weak injective. Then $R = I \oplus J$ for some ideal J of R. The fact $R = I + J$ implies that $R\,(M) = I_{(+)}IM + J_{(+)}JM$. Let H be an ideal of $R(M)$

such that $I_{(+)}IM \subseteq H$. As $I_{(+)}IM + J_{(+)}JM$ is a multiplication ideal of $R(M)$, we get from, [14, Corollary 1.2], that

$$\begin{aligned} H &= H \cap (I_{(+)}IM + J_{(+)}JM) \\ &= (H \cap I_{(+)}IM) + (H \cap J_{(+)}JM) = (I_{(+)}IM) \cap (H \cap J_{(+)}JM)\,. \end{aligned}$$

Since $I \cap J = 0, IJ = 0$ and hence

$$IM \cap JM = I(IM \cap JM) + J(IM \cap JM) \subseteq IJM = 0.$$

So $I_{(+)}IM \cap J_{(+)}JM = I \cap J_{(+)}IM \cap JM = 0$. This implies that $I_{(+)}IM \cap (H \cap J_{(+)}JM) = 0$, and hence $H = I_{(+)}IM \oplus (H \cap J_{(+)}JM)$. This shows that $I_{(+)}IM$ is a weak injective ideal of $R(M)$. The converse follows by part (2). $\square$

An R-module M is said to be *simple* if $M \neq 0$ and for every submodule N of M either $N = 0$ or $N = M$, [23]. Obviously, simple modules are multiplication. The next result gives some properties of simple ideals and modules.

Proposition 7. *Let R be a ring and M an R-module. Let I be an ideal of R and N a submodule of M.*

(1) *If M is finitely generated faithful multiplication then I is simple if and only IM is a simple submodule of M.*

(2) *If $[N : M]$ is a simple ideal of R and M faithful multiplication then N is simple. The converse is true if we assume further that M is finitely generated.*

(3) *N is simple if and only if $0_{(+)}N$ is a simple ideal of $R(M)$.*

(4) *If $I_{(+)}IM$ is a simple ideal of $R(M)$ then I is simple. The converse is true if M is finitely generated faithful multiplication and every ideal contained in $I_{(+)}IM$ is faithful.*

(5) *Let $IM \subseteq N$. If $I_{(+)}N$ is a simple homogeneous ideal of $R(M)$ and $I \neq 0$ then I is simple. Assuming further that M is faithful then N is simple.*

Proof. (1) Since I is simple and M faithful, $IM \neq 0$. Suppose K is a submodule of M such that $K \subseteq IM$. Then $[K : M] \subseteq [IM : M] = I$. So $[K : M] = 0$, and hence $K = [K : M]\,M = 0$, or $[K : M] = I$. The latter case shows that $K = IM$. Hence IM is simple. Conversely, suppose IM is simple. Then $I \neq 0$. Let $J \subseteq I$ be an ideal of R. Then $JM \subseteq IM$. Hence $JM = 0$, and hence $J = 0$ or $JM = IM$. Since M is finitely generated faithful multiplication, it follows by [29, Corollary to Theorem 9] that $J = I$. So I is simple.

(2) Follows by (1).

(3) Suppose N is simple. Then $0_{(+)}N \neq 0$. Every ideal contained in $0_{(+)}N$ has the form $0_{(+)}K$ for some submodule K of N. Hence $K = 0$ or $K = N$ from which we get that $0_{(+)}K = 0$ or $0_{(+)}K = 0_{(+)}N$. The statement is reversible.

(4) Suppose $I_{(+)}IM$ is simple. Then $I \neq 0$. Let $J \subseteq I$ be an ideal of R. Then $J_{(+)}JM \subseteq I_{(+)}IM$, and hence $J_{(+)}JM = 0$ or $J_{(+)}JM = I_{(+)}IM$. So $J = 0$ or

$J = I$, and hence I is simple. For the converse, let I be simple. Then $I_{(+)}IM \neq 0$. Let $H \subseteq I_{(+)}IM$ be an ideal of $R(M)$. Since H is faithful by assumption, $H \neq 0$. We show that $H = I_{(+)}IM$. We have $H \cap 0_{(+)}M \subseteq I_{(+)}IM \cap 0_{(+)}M = 0_{(+)}IM$. Since M is faithful, $0_{(+)}M$ is a large ideal of $R(M)$ and hence $H \cap 0_{(+)}M \neq 0$. By (1) and (3), $0_{(+)}IM$ is a simple ideal of $R(M)$. Hence $H \cap 0_{(+)}M = 0_{(+)}IM$. Next, $H + 0_{(+)}M \subseteq I_{(+)}IM + 0_{(+)}M = I_{(+)}M$. Assume $H + 0_{(+)}M = J_{(+)}M$ for some ideal J of R. Then $J \subseteq I$. Since $J_{(+)}JM \subseteq I_{(+)}IM$ is faithful, we infer that J is faithful, [1, Theorem 9]. Hence $J \neq 0$ and therefore $J = I$. So $H + 0_{(+)}M = I_{(+)}M = I_{(+)}IM + 0_{(+)}M$. Using the modular law one gets that

$$
\begin{aligned}
H &= (H + 0_{(+)}M) \cap H = (I_{(+)}IM + 0_{(+)}M) \cap H \\
&= I_{(+)}IM + (H \cap 0_{(+)}M) = I_{(+)}IM + 0_{(+)}IM \\
&= I_{(+)}IM.
\end{aligned}
$$

Hence $I_{(+)}IM$ is a simple ideal of $R(M)$.

(5) Suppose $I_{(+)}N$ is simple. Then $I_{(+)}N \neq 0$. Since $I_{(+)}IM \subseteq I_{(+)}N$ and $I \neq 0$ (and hence $I_{(+)}IM \neq 0$), we get that $I_{(+)}IM = I_{(+)}N$. The fact that I is simple follows by (4). Suppose M is faithful. Since $0_{(+)}N \subseteq I_{(+)}N$, we have either $0_{(+)}N = 0$ (hence $N = 0$) or $I_{(+)}N = 0_{(+)}N$. The case that $N = 0$ implies that $IM = 0$ and hence $I = 0$, a contradiction. Hence $I_{(+)}N = 0_{(+)}N$ and the result follows by (3). $\square$

In the following result we show how purity, multiplicativity or flatness of a homogeneous ideal of $R(M)$ can be transferred to its components. Compare with [2, Theorems 9(3) and 7(2)].

Proposition 8. *Let R be a ring, M an R-module and $I_{(+)}N$ a homogeneous ideal of $R(M)$.*

1. *If $I_{(+)}N$ is pure then I is a pure ideal of R and N a pure submodule of M.*
2. *Let every principal ideal contained in $I_{(+)}N$ be homogeneous. If I is a finitely generated multiplication ideal of R and N a finitely generated multiplication submodule of M then $I_{(+)}N$ is finitely generated multiplication.*
3. *If $I_{(+)}N$ is multiplication then N is a multiplication submodule of M if and only if IM is.*
4. *If $I_{(+)}N$ is flat then I is a flat ideal of R. Assuming further that M is flat then N is a flat submodule of M.*

Proof. (1) Let J be an ideal of R. Since $I_{(+)}N$ is pure, we get that

$$
\begin{aligned}
JI_{(+)}JN &= (J_{(+)}JM)\, I_{(+)}N) = J_{(+))}JM \cap I_{(+)}N \\
&= J \cap I_{(+)}JM \cap N.
\end{aligned}
$$

Hence $JI = J \cap I$ and $JN = JM \cap N$ which shows that I is a pure ideal of R and N a pure submodule of M. Alternatively, let P be a prime ideal of R then $P_{(+)}M$ is a prime ideal of $R(M)$. Hence $I_{P(+)}N_P \cong (I_{(+)}N)_{P(+)M} = 0_{P(+)M}$ or

$R(M)_{P_{(+)}M}$. It follows that $I_P = 0_P$ or $I_P = R_P$ and hence I is a pure ideal of R and $N_P = 0_P$ or $N_P = M_P$. The latter case shows that for any ideal J of R, the equality $JN = JM \cap N$ is true locally and hence globally. So N is a pure submodule of M.

(2) $I_{(+)}N$ is finitely generated, [1, Theorem 9], and hence it is enough to prove the result locally, [18, Proposition 4]. So we may assume $R(M)$ is a local ring and hence R is local. It follows that $I = Ra$ and $N = Rn$ for some $a \in I$ and $n \in N$. Hence $I_{(+)}N = Ra_{(+)}Rn = Ra_{(+)}(Rn + aM) = R(M)(a, n)$, [17, Theorem 3.4]. So $I_{(+)}N$ is multiplication.

(3) Suppose $I_{(+)}N$ is multiplication. We first show that I is multiplication. Let $J \subseteq I$ be an ideal of R. Then $JM \subseteq IM \subseteq N$ and hence $J_{(+)}N \subseteq I_{(+)}N$ is a homogeneous ideal of $R(M)$. It follows that $J_{(+)}N = H(I_{(+)}N)$ for some ideal H of $R(M)$, and hence

$$\begin{aligned} J_{(+)}N &= J_{(+)}N + 0_{(+)}IM = H(I_{(+)}N) + 0_{(+)}IM \\ &= (H + 0_{(+)}M)(I_{(+)}N). \end{aligned}$$

Assume $H + 0_{(+)}M = A_{(+)}M$ for some ideal A of R. Then $J_{(+)}N = AI_{(+)}(AN + IM)$. So $J = AI$ and hence I is multiplication. Assume now that IM is multiplication. Let P be a maximal ideal of R such that $N_P \neq 0_P$. Then $P_{(+)}M$ is a maximal ideal of $R(M)$ and $(I_{(+)}N)_{P_{(+)}M} \neq 0_{P_{(+)}M}$. Hence $I_{(+)}N$ is $P_{(+)}M$-principal and by, [1, Theorem 9], we have that I is P-principal. From, [1, Proposition 10], we get that $I = I[IM : N]$ and hence $I_P = I_P[IM : N]_P$. Since I_P is principal, we infer from, [22, Theorem 76], that $R_P = [IM : N]_P + \operatorname{ann}(I_P)$. As I is multiplication and P-principal, we get that $I_P \neq 0_P$, that is, $\operatorname{ann}(I_P) \neq R_P$ and hence $R_P = [IM : N]_P$. There exists $p \in P$ such that $(1 - p) \in [IM : N]$ and hence $(1 - p)N \subseteq IM$. The fact that N is multiplication follows by, [29, Corollary to Lemma 1]. Conversely, assume that N is multiplication. Let P be a maximal ideal of R such that $(IM)_P \neq 0_P$. Hence $I_P \neq 0_P$ and as we have seen above we have $p \in P$ such that $(1 - p)N \subseteq IM$. The fact that IM is multiplication follows by [13, Corollary 2] and [14, Proposition 4.1].

(4) Suppose $I_{(+)}N$ is a flat ideal of $R(M)$. The fact that I is a flat ideal of R follows by [2, Theorem 7(2)]. Since $I \cong I_{(+)}N/0_{(+)}N$ is a flat R-module, we infer from, [25, Example 1, p. 54] and [20, Corollary 11.21], that $0_{(+)}N$ is a pure R-submodule of $I_{(+)}N$. Consider $R(M)$ as a direct sum of R and M and since M is a flat R-module, one gets that $R(M)$ is a flat R-module. The map $f : R \to R(M)$ defined by $f(a) = (a, 0)$ is a homomorphism of rings and hence $R(M)$ is a flat R-Algebra, [25, pp. 45–46]. Now, $I_{(+)}N$ is flat over $R(M)$ and $R(M)$ is a flat R-Algebra, so by the transitivity of flatness one obtains that $I_{(+)}N$ is flat over R, see for example [25, p. 46]. Using the fact that pure submodules of flat modules are flat, [20], we infer that $0_{(+)}N$ is a flat module over R. Let $\sum_{i=1}^{k} r_i n_i = 0$, where $r_i \in R$ and $n_i \in N$. Then $\sum_{i=1}^{k} r_i(0, n_i) = (0, 0)$. Since $0_{(+)}N$ is flat over R, it follows by, [25, Theorem 7.6], that there exist $(0, h_j) \in 0_{(+)}N$ and $a_{ij} \in R, 1 \leq j \leq \ell$, such

that

$$0=\sum_{i=1}^{k} r_i a_{ij},\ (0,n_i)=\sum_{j=1}^{\ell} a_{ij}\,(0,h_j)\,.$$

This implies that $0=\sum_{i=1}^{k} r_i a_{ij}$ for all j and $n_i=\sum_{j=1}^{\ell} a_{ij}h_j$ for all i and this shows that N is a flat R-module, [25, Theorem 7.6]. □

Let R be a ring and K, N submodules of an R-module M. Then K divides N, denoted $K|N$ if $N = IK$ for some ideal I of R. If $K|N$ then $N \subseteq K$ and the converse is true if M is multiplication. The common divisor by every common divisor of K and N (if it does exist) is denoted by GCD (K, N). Similarly, we define LCM (K, N) as a submodule of M which is a common multiple of K and N which divides every common multiple of K and N (if such exists). The existence and arithmetic properties of these in the case of finitely generated faithful multiplication modules (ideals), finitely generated projective modules (ideals) and invertible submodules (ideals) are discussed in [9]–[12]. The next theorem gives some relationships between the gcd and lcm of a homogeneous ideal and those ones of its components.

Theorem 9. *Let R be a ring and M an R-module.*

(1) *For all submodules K and N of M, if M is multiplication and* $\gcd(0_{(+)}K, 0_{(+)}N)$ *exists then* GCD(K, N) *exists. The converse is true if M is a faithful prime module and in both cases*

$$\gcd(0_{(+)}K, 0_{(+)}N) = 0_{(+)}\mathrm{GCD}\,(K, N)\,.$$

(2) *For all submodules K and N of M,* $\mathrm{lcm}\,(0_{(+)}K, 0_{(+)}N)$ *exists if and only if* LCM (K, N) *exists and in this case*

$$\mathrm{lcm}(0_{(+)}K, 0_{(+)}N) = 0_{(+)}\mathrm{LCM}\,(K, N)\,.$$

(3) *Let M be divisible and torsion-free. For all invertible ideals I and J of R,* $\gcd(I, J)$ *exists if and only if* $\gcd(I_{(+)}IM, J_{(+)}JM)$ *and in this case*

$$\gcd(I_{(+)}IM, J_{(+)}JM) = \gcd(I,J)_{(+)}\gcd(I,J)\,M.$$

(4) *Let M be divisible and multiplication. For all homogeneous ideals $I_{(+)}N$ and $J_{(+)}K$ of $R(M)$, if I and J are finitely generated faithful multiplication ideals of R and* $\mathrm{lcm}\,(I, J)$ *exists then* $\mathrm{lcm}\,(I_{(+)}N, J_{(+)}K)$ *exists. The converse is true if we assume further that M is faithful, and in both cases*

$$\mathrm{lcm}\,(I_{(+)}N, J_{(+)}K) = \mathrm{lcm}\,(I, J)\,_{(+)}\mathrm{LCM}\,(N, K)\,.$$

Proof. (1) Suppose M is multiplication and let $\gcd(0_{(+)}K, 0_{(+)}N) = H$. Since $K = IM$ for some ideal I of R, $0_{(+)}K = 0_{(+)}IM = (I_{(+)}M)\,(0_{(+)}M)$ and hence $0_{(+)}M|0_{(+)}K$. Similarly, $0_{(+)}M|0_{(+)}N$, and hence $0_{(+)}M|H$. So $H \subseteq 0_{(+)}M$ and this means that $H = 0_{(+)}G$ for some submodule G of M. Now, $0_{(+)}G|0_{(+)}K$ implies that

$0_{(+)}K = H'\left(0_{(+)}G\right) = \left(H' + 0_{(+)}M\right)\left(0_{(+)}G\right)$ for some ideal H' of $R\left(M\right)$. Assume $H' + 0_{(+)}M = A_{(+)}M$ for some ideal A of R. Then $0_{(+)}K = 0_{(+)}AG$, and hence $K = AG$, that is $G|K$. Similarly, $G|N$. Suppose G' is any submodule of M such that $G'|K$ and $G'|N$. Then $K = BG'$ and $N = CG'$ for some ideals B and C of R. This implies that $0_{(+)}K = 0_{(+)}BG' = \left(B_{(+)}M\right)\left(0_{(+)}G'\right)$ and $0_{(+)}N = \left(C_{(+)}M\right)\left(0_{(+)}G'\right)$. So $0_{(+)}G'|0_{(+)}K$ and $0_{(+)}G'|0_{(+)}N$, and this gives that $0_{(+)}G'|0_{(+)}G$. If $0_{(+)}G = H''\left(0_{(+)}G'\right)$ for some ideal H'' of $R\left(M\right)$, then $0_{(+)}G = \left(H'' + 0_{(+)}M\right)\left(0_{(+)}G'\right) = \left(D_{(+)}M\right)\left(0_{(+)}G'\right)$ for some ideal D of R. Hence $G = DG'$ and hence $G'|G$. So $G = \mathrm{GCD}(K,N)$ and hence $\gcd\left(0_{(+)}K, 0_{(+)}N\right) = 0_{(+)}\mathrm{GCD}\left(K,N\right)$. Conversely, let $\mathrm{GCD}\left(K,N\right) = G$. Then $G|K$ and $G|N$ and hence $0_{(+)}G|0_{(+)}K$ and $0_{(+)}G|0_{(+)}N$. Assume H is an ideal of $R\left(M\right)$ such that $H|0_{(+)}K$ and $H|0_{(+)}N$. Then $0_{(+)}K \subseteq H$ and hence $\mathrm{ann}H \subseteq \mathrm{ann}\left(0_{(+)}K\right) = \mathrm{ann}K_{(+)}M$. Since M is faithful and prime, we infer that $\mathrm{ann}H \subseteq 0_{(+)}M$ and hence $\mathrm{ann}H = 0_{(+)}L$ for some submodule L of M. This implies that $H\left(0_{(+)}L\right) = 0$ and hence $H \subseteq \mathrm{ann}\left(0_{(+)}L\right) = \mathrm{ann}L_{(+)}M = 0_{(+)}M$. So $H = 0_{(+)}G'$ for some submodule G' of M. As $0_{(+)}G'|0_{(+)}K$ and $0_{(+)}G'|0_{(+)}N$, we get that $G'|K$ and $G'|N$. Hence $G'|G$ and this implies that $0_{(+)}G'|0_{(+)}G$. So $0_{(+)}G = \gcd\left(0_{(+)}K, 0_{(+)}N\right)$.

(2) Suppose $\mathrm{lcm}\left(0_{(+)}K, 0_{(+)}N\right) = H$. Then $0_{(+)}K|H$ and $0_{(+)}N|H$. So $H \subseteq 0_{(+)}(K \cap N)$, and hence $H = 0_{(+)}L$ for some submodule $L \subseteq K \cap N$. So $0_{(+)}K|0_{(+)}L$ and $0_{(+)}N|0_{(+)}L$. This implies that $K|L$ and $N|L$. Suppose L' is any submodule of M such that $K|L'$ and $N|L'$. Then $0_{(+)}K|0_{(+)}L'$ and $0_{(+)}N|0_{(+)}L'$. So $0_{(+)}L|0_{(+)}L'$ and hence $L|L'$ and $L = \mathrm{LCM}\left(K,N\right)$. This also shows that $\mathrm{lcm}\left(0_{(+)}K, 0_{(+)}N\right) = 0_{(+)}L = 0_{(+)}\mathrm{LCM}\left(K,N\right)$. The converse is now obvious.

(3) For all ideals of A of R, if $A_{(+)}AM$ is an invertible ideal of $R\left(M\right)$ then A is invertible and the converse is true if M is torsion-free, [4, Theorem 2]. Suppose M is divisible and torsion-free. Then for any ideal A of R, if $A_v = \left(A^{-1}\right)^{-1}$ is invertible then

$$\begin{aligned}\left(A_{(+)}AM\right)_v &= \left(\left(A_{(+)}AM\right)^{-1}\right)^{-1} = \left(A^{-1}{}_{(+)}M\right)^{-1} = A_{v(+)}M\\ &= A_{v(+)}A_vM,\end{aligned}$$

[21, Theorem 25.10] and [17, Theorem 3.9]. Suppose $\gcd\left(I,J\right)$ exists. It follows by, [12, Theorem 2.1], that $\gcd\left(I,J\right) = \left(I+J\right)_v$. Since $\left(I+J\right)_v$ is invertible and M divisible, we infer that $\left(I+J\right)_{v(+)}\left(I+J\right)_v M$ is an invertible ideal of $R\left(M\right)$, see [4, Theorem 5]. Moreover,

$$\left(I_{(+)}IM + J_{(+)}JM\right)_v = \left(\left(I+J\right)_{(+)}\left(I+J\right)M\right)_v = \left(I+J\right)_v + \left(I+J\right)_v M.$$

Hence $\left(I_{(+)}IM + J_{(+)}JM\right)_v$ is an invertible of $R\left(M\right)$ and hence $\gcd(I_{(+)}\mathrm{IM}, J_{(+)}JM)$ exists and

$$\gcd\left(I_{(+)}IM, J_{(+)}JM\right) = \left(I+J\right)_{v\,(+)}\left(I+J\right)_v M = \gcd\left(I,J\right)_{(+)}\gcd\left(I,J\right)M.$$

Conversely, suppose $\gcd\left(I_{(+)}IM, J_{(+)}JM\right)$ exists. Then $\left(\left(I+J\right)_{(+)}\left(I+J\right)M\right)_v = \left(I+J\right)_{v(+)}M$ is an invertible ideal of $R\left(M\right)$. It follows by [4, Theorem 2] that $\left(I+J\right)_v$ is an invertible ideal of R and $\gcd\left(I,J\right)$ exists and is $\left(I_{(+)}J\right)_v$. Since

$(I+J)_v$ is invertible and M divisible $(I+J)_{v(+)}M = (I+J)_{v(+)}(I+J)_v M = (I_{(+)}IM + J_{(+)}JM)_v$. Hence $\gcd(I_{(+)}IM, J_{(+)}JM) = \gcd(I,J)_{(+)}\gcd(I,J)\, M$.

(4) Suppose M is divisible, I and J are finitely generated faithful multiplication ideals of R. It follows by, [4, Theorem 5], that $I_{(+)}N$ and $J_{(+)}K$ are finitely generated multiplication ideals of $R(M)$. Since M is multiplication, we infer from, [1, Theorem 9] and [3, Theorem 3], that N and K are multiplication submodules of M. If $\mathrm{lcm}(I,J)$ exists then $\mathrm{lcm}(I,J) = I \cap J$ is a finitely generated faithful multiplication ideal of R, [11, Lemma 5]. It follows that $(I \cap J_{(+)}(N \cap K))$ is a finitely generated multiplication ideal of $R(M)$ and consequently $K \cap N$ is a multiplication submodule of M, [4, Theorem 5] and [1, Theorem 9]. Using the fact that for all submodules X and Y of M if $X \cap Y$ is multiplication then $\mathrm{LCM}(X,Y)$ exists and $\mathrm{LCM}(X,Y) = X \cap Y$, we get that $\mathrm{lcm}(I_{(+)}N, J_{(+)}K) = I_{(+)}N \cap J_{(+)}K = (I \cap J) + (N \cap K) = \mathrm{lcm}(I,J)_{(+)}\mathrm{LCM}(N,K)$. For the converse, suppose M is faithful multiplication. Then $I_{(+)}N$ and $J_{(+)}K$ are finitely generated faithful multiplication ideals of $R(M)$. Suppose $\mathrm{lcm}(I_{(+)}N, J_{(+)}K)$ exists. It follows by [11, Lemma 5] that $\mathrm{lcm}(I_{(+)}N, J_{(+)}K) = (I \cap J)_{(+)}(N \cap K)$ is a finitely generated faithful multiplication ideal of $R(M)$. Hence $I \cap J$ is multiplication and hence $\mathrm{lcm}(I,J)$ exists and is $I \cap J$. Since M is multiplication, $N \cap K$ is multiplication and hence $\mathrm{lcm}(N,K)$ exists and is $N \cap K$. Hence

$$\mathrm{lcm}(I_{(+)}N, J_{(+)}K) = \mathrm{lcm}(I,J)_{(+)}\mathrm{LCM}(N,K).$$

This completes the proof of the theorem. □

2. Annihilator conditions via idealization

In [2] we discussed two annihilator conditions, namely (a.c.) and property (A) via idealization. A ring R satisfies *Property* (A) if each finitely generated ideal $I \subseteq Z(R)$ has a nonzero annihilator, where $Z(R)$ is the set of zero divisors of R. A condition closely intertwined with Property (A) is the *annihilator condition*; abbreviated as (a.c.): A ring R satisfies (a.c.) if for each pair of elements a and b in R, there exists c in R such that $\mathrm{ann}(Ra + Rb) = \mathrm{ann}(c)$. More generally, we say that an R-module M satisfies (a.c.) if for each pair of elements m and n in M, there exists k in M such that $\mathrm{ann}(Rm + Rn) = \mathrm{ann}(k)$, [21]. It is proved that if M is flat and $R(M)$ has property (A) then so too has R and the converse is true if M is finitely generated. On the other hand, let $R(M)$ be homogeneous. If M is flat and each of R and M satisfies the (a.c.), then so too does $R(M)$ and the converse is true if M is finitely generated faithful and multiplication, [2, Theorem 16]. In this paper we investigate another two annihilator conditions on a ring R. Recall that R is a *quasi-Frobenius* ring, shortly QF ring, if R is Artinian and $\mathrm{ann}(\mathrm{ann}I) = I$ for each ideal I of R. And an R-module M is called a QF module if M is Artinian and $[0 :_M \mathrm{ann}N] = N$ for each submodule N of M, [23, p. 336]. A ring R is called *distinguished* if $\mathrm{ann}I \neq 0$ for each proper ideal I of R and M is distinguished if $[0 :_M I] \neq 0$ for each proper ideal I of R, [26]. A ring R is QF if and only if every ideal of R is projective, equivalently, every ideal of R is injective, [23, Theorem 13.6.1]. Since QF rings are Artinian rings, we first give necessary and sufficient conditions for a ring $R(M)$ to be Artinian. The proof of the next result is given in [17, Theorem 4.8] however we give here different proof which may be of some interest.

Proposition 10. *Let R be a ring and M an R-module. Then $R(M)$ is Artinian (resp. Noetherian) if and only if R is Artinian (resp. Noetherian) and M is Artinian (resp. Noetherian).*

Proof. Suppose $R(M)$ is Artinian. Let $I_1 \supseteq I_2 \supseteq \cdots$ be a descending chain of ideals of R. Then $I_{1(+)}I_1M \supseteq I_{2(+)}I_2M \supseteq \cdots$ is a descending chain of ideals of $R(M)$ and hence there exists a positive integer k such that $I_{k(+)}I_kM = I_{k+1(+)}I_{k+1}M = \cdots$. This implies that $I_k = I_{k+1} = \cdots$, and hence R is Artinian. Next, let $N_1 \supseteq N_2 \supseteq \cdots$ be a descending chain of submodules of M then $0_{(+)}N_1 \supseteq 0_{(+)}N_2 \supseteq \cdots$ is a descending chain of ideals of $R(M)$. There exists a positive integer l such that $0_{(+)}N_l = 0_{(+)}N_{l+1} = \cdots$, and hence $N_l = N_{l+1} = \cdots$. Hence M is Artinian. Conversely, suppose R is an Artinian ring and M an Artinian module. Let $H_1 \supseteq H_2 \supseteq \cdots$ be a descending chain of ideals of $R(M)$. Then $H_1 + 0_{(+)}M \supseteq H_2 + 0_{(+)}M \supseteq \cdots$ and $H_1 \cap 0_{(+)}M \supseteq H_2 \cap 0_{(+)}M \supseteq \cdots$ are descending chains of ideals of $R(M)$. Assume $H_i + 0_{(+)}M = I_{i(+)}M$ and $H_i \cap 0_{(+)}M = 0_{(+)}N_i$ for some ideals I_i of R and some submodules N_i of M. This gives that $I_1 \supseteq I_2 \supseteq \cdots$ is a descending chain of ideals of R and $N_1 \supseteq N_2 \supseteq \cdots$ is a descending chain of submodules of M. Hence there exist positive integers k and l such that $I_k = I_{k+1} = \cdots$ and $N_l = N_{l+1} = \cdots$. It follows that $H_k + 0_{(+)}M = H_{k+1} + 0_{(+)}M = \cdots$, and $H_l \cap 0_{(+)}M = H_{l+1} \cap 0_{(+)}M = \cdots$. If $k < l$, then $H_k + 0_{(+)}M = H_{k+1} + 0_{(+)}M$ and $H_k \cap 0_{(+)}M = H_{k+1} \cap 0_{(+)}M$. So by using the modular law, one gets that

$$\begin{aligned} H_k &= (H_k + 0_{(+)}M) \cap H_k = (H_{k+1} + 0_{(+)}M) \cap H_k \\ &= H_{k+1} + (H_k \cap 0_{(+)}M) = H_{k+1} + (H_{k+1} \cap 0_{(+)}M) \\ &= H_{k+1}. \end{aligned}$$

Similarly, if $k \geq l$ then $H_l = H_{l+1} = \cdots$ and this shows that $R(M)$ is an Artinian ring. The Noetherian case is now obvious by replacing the descending chain of ideals (submodules) by ascending chain of ideals (submodules). Alternatively, let $R(M)$ be Noetherian. If I is an ideal of R, then $I_{(+)}IM$ is a finitely generated ideal of $R(M)$. It follows by [1, Theorem 7] that I is finitely generated. If N is a submodule of M then $0_{(+)}N$ is a finitely generated ideal of $R(M)$, and hence N is finitely generated [1, Theorem 2] and [14, Theorem 3.1]. So R is a Noetherian ring and M a Noetherian module. Conversely, suppose R is a Noetherian ring and M a Noetherian module. Suppose H is an ideal of $R(M)$, and let $H + 0_{(+)}M = I_{(+)}M$ for some ideal I of R and $H \cap 0_{(+)}M = 0_{(+)}N$ for some submodule N of M. Since I is finitely generated and M Noetherian (hence finitely generated), $I_{(+)}M = H + 0_{(+)}M$ is finitely generated. Also N is finitely generated since M is Noetherian. So $0_{(+)}N = H \cap 0_{(+)}M$ is finitely generated. The fact that H is finitely generated follows by [25, Example 2.3, p. 13]. Hence $R(M)$ is Noetherian. $\square$

We next determine when $R(M)$ is a QF ring or a distinguished ring.

Theorem 11. *Let R be a ring and M an R-module.*

(1) *If $R(M)$ is a QF ring then M is a QF module. Assuming further that M is projective then R is a QF ring.*

(2) *Let M be faithful. If R is a QF ring and M a QF module then $R(M)$ is a QF ring.*

(3) *If $R(M)$ is a distinguished ring then R is a distinguished ring or M a distinguished module. The converse is true if M is faithful.*

(4) *Let M be faithful multiplication. If $R(M)$ is a distinguished ring then R is a distinguished ring and M a distinguished module.*

Proof. (1) If $R(M)$ is a QF ring then $R(M)$ is Artinian, so by Proposition 10, R is Artinian and M Artinian. Suppose N is a submodule of M. Then

$$\begin{aligned} 0_{(+)}N &= \operatorname{ann}(\operatorname{ann}(0_{(+)}N)) = \operatorname{ann}(\operatorname{ann}N_{(+)}M) \\ &= \operatorname{ann}(\operatorname{ann}N) \cap \operatorname{ann}M_{(+)}[0 :_M \operatorname{ann}N], \end{aligned}$$

and hence $N = [0 :_M \operatorname{ann}N]$. This gives that M is a QF module. Suppose now M is projective and I an ideal of R. It follows by, [2, Lemma 5], that

$$I_{(+)}IM = \operatorname{ann}(\operatorname{ann}(I_{(+)}IM)) = \operatorname{ann}(\operatorname{ann}I)_{(+)}\operatorname{ann}(\operatorname{ann}(I))M.$$

So $I = \operatorname{ann}(\operatorname{ann}I)$, and R is a QF ring.

(2) Suppose M is faithful. Since R is QF (hence Artinian) and M is a QF (hence Artinian) R-module, we infer from Proposition 10 that $R(M)$ is Artinian. Let H be an ideal of $R(M)$. Assume $H + 0_{(+)}M = I_{(+)}M$ and $H \cap 0_{(+)}M = 0_{(+)}N$ for some ideal I of R and some submodule N of M. Then

$$\begin{aligned} H + 0_{(+)}M &= I_{(+)}M = \operatorname{ann}(\operatorname{ann}I)_{(+)}M \\ &\supseteq \operatorname{ann}(\operatorname{ann}I) \cap \operatorname{ann}M_{(+)}[0 :_M \operatorname{ann}I] + 0_{(+)}M \\ &= \operatorname{ann}(\operatorname{ann}(I_{(+)}M)) + 0_{(+)}M \\ &= \operatorname{ann}(\operatorname{ann}(H + 0_{(+)}M) + 0_{(+)}M \\ &= \operatorname{ann}(\operatorname{ann}H \cap 0_{(+)}M) + 0_{(+)}M \\ &\supseteq \operatorname{ann}(\operatorname{ann}H) + \operatorname{ann}(0_{(+)}M) + 0_{(+)}M \\ &= \operatorname{ann}(\operatorname{ann}H) + 0_{(+)}M \supseteq H + 0_{(+)}M. \end{aligned}$$

On the other hand,

$$\begin{aligned} H \cap 0_{(+)}M &= 0_{(+)}N = 0_{(+)}[0 :_M \operatorname{ann}N] \\ &= \operatorname{ann}(\operatorname{ann}(0_{(+)}N)) = \operatorname{ann}(\operatorname{ann}(H \cap 0_{(+)}M)) \\ &\supseteq \operatorname{ann}(\operatorname{ann}H + 0_{(+)}M) \\ &= \operatorname{ann}(\operatorname{ann}H) \cap 0_{(+)}M \supseteq H \cap 0_{(+)}M. \end{aligned}$$

The application of the modular law shows that $H = \operatorname{ann}(\operatorname{ann}H)$ and hence $R(M)$ is a QF ring.

(3) Suppose $R\,(M)$ is distinguished. Let I be a proper ideal of R. Then $I_{(+)}M$ is a proper ideal of $R(M)$ and hence $0 \neq \mathrm{ann}\,(I_{(+)}M) = \mathrm{ann}I \cap \mathrm{ann}M_{(+)}[0 :_M I]$. If $\mathrm{ann}I \cap \mathrm{ann}M \neq 0$, then $\mathrm{ann}I \neq 0$ and hence R is distinguished. If $[0 :_M I] \neq 0$, then obviously M is distinguished. For the converse, assume R is distinguished and M is faithful and let H be a proper ideal of $R\,(M)$. Assume $H{+}0_{(+)}M = I_{(+)}M$ for some ideal I of R. Then

$$\begin{aligned}\mathrm{ann}H &\supseteq \mathrm{ann}\,(I_{(+)}M) = \mathrm{ann}I \cap \mathrm{ann}M_{(+)}\,[0 :_M I]\\ &= 0_{(+)}\,[0 :_M I] \supseteq 0_{(+)}\,(\mathrm{ann}I)\,M.\end{aligned}$$

Since M is faithful and $\mathrm{ann}I \neq 0$, $(\mathrm{ann}I)\,M \neq 0$. So $\mathrm{ann}H \neq 0$ and $R\,(M)$ is distinguished. Suppose now that M is distinguished. Then $\mathrm{ann}H \supseteq 0_{(+)}[0 :_M I] \neq 0$ and clearly $R\,(M)$ is distinguished.

(4) Suppose M is faithful multiplication. Let I be a proper ideal of R. Then $0 \neq \mathrm{ann}\,(I_{(+)}IM) = 0_{(+)}(\mathrm{ann}I)\,M$. Hence $(\mathrm{ann}I)\,M \neq 0$, and hence $\mathrm{ann}I \neq 0$. So R is distinguished. Moreover, $[0 :_M I]{\neq}0$ and this shows that M is distinguished. □

An R-module M is called *finitely annihilated* if $\mathrm{ann}M = \mathrm{ann}N$ for some finitely generated submodule N of M, [30]. The next result gives necessary and sufficient conditions for a homogeneous ideal of $R(M)$ to be finitely annihilated.

Proposition 12. *Let R be a ring, M an R-module, I an ideal of R and N a submodule of M.*

(1) *N is finitely annihilated if and only if $0_{(+)}N$ is a finitely annihilated ideal of $R\,(M)$.*
(2) *Suppose M is finitely generated faithful multiplication and $IM \subseteq N$. If $I_{(+)}N$ is finitely annihilated then I is finitely annihilated and N finitely annihilated.*
(3) *Suppose M is faithful multiplication or projective. If I is finitely annihilated then $I_{(+)}IM$ is a finitely annihilated ideal of $R\,(M)$. The converse is true if M is finitely generated faithful and multiplication.*
(4) *Suppose M is finitely generated faithful multiplication and $IM \subseteq N$. If $I_{(+)}N$ is finitely generated and I finitely annihilated then $I_{(+)}N$ is a finitely annihilated ideal of $R\,(M)$.*
(5) *Suppose M is faithful multiplication and $IM \subseteq N$. If I is finitely annihilated and $\mathrm{ann}I \subseteq \mathrm{ann}N$ then $I_{(+)}N$ is a finitely annihilated ideal of $R\,(M)$.*
(6) *Suppose R has the property that $\mathrm{ann}A = \mathrm{ann}(\mathrm{ann}A)$ for each ideal A of R and M a faithful multiplication module such that $IM \subseteq N$. If I is a finitely annihilated ideal of $R\,(M)$ then $I_{(+)}N$ is finitely annihilated.*

Proof. (1) Suppose $0_{(+)}N$ is finitely annihilated. There exists a finitely generated ideal $H \subseteq 0_{(+)}N$ of $R\,(M)$ such that $\mathrm{ann}H = \mathrm{ann}(0_{(+)}N)$. Assume $H = 0_{(+)}K$ for some finitely generated submodule K of N, [15, Theorem 3.1]. Then $\mathrm{ann}K_{(+)}M = \mathrm{ann}H = \mathrm{ann}\,(0_{(+)}N) = \mathrm{ann}N_{(+)}M$, and hence $\mathrm{ann}K = \mathrm{ann}N$. So N is finitely annihilated. The statement is reversible.

(2) Suppose M is finitely generated faithful multiplication. Let $I_{(+)}N$ be finitely annihilated. There exists a finitely generated ideal H of $R(M)$ that is contained in $I_{(+)}N$ with $\operatorname{ann}H = \operatorname{ann}(I_{(+)}N)$. Then $\operatorname{ann}H = \operatorname{ann}N_{(+)}(\operatorname{ann}I)\,M$, and hence

$$\operatorname{ann}(H + 0_{(+)}M) = \operatorname{ann}H \cap \operatorname{ann}(0_{(+)}M) = \operatorname{ann}H \cap 0_{(+)}M = 0_{(+)}(\operatorname{ann}I)\,M.$$

Assume $H + 0_{(+)}M = J_{(+)}M$ for some ideal J of R. Then

$$0_{(+)}(\operatorname{ann}J)M = \operatorname{ann}(J_{(+)}M) = 0_{(+)}(\operatorname{ann}I)\,M.$$

This gives that $(\operatorname{ann}J)M = (\operatorname{ann}I)M$ and hence $\operatorname{ann}J = \operatorname{ann}I$. Since $H+0_{(+)}M = J_{(+)}M$ is finitely generated, it follows by, [1, Theorem 9], that J is finitely generated, and this shows that I is finitely annihilated. Alternatively, suppose $H = \sum_{i=1}^{n} R(M)(a_i, m_i)$. It follows that

$$\operatorname{ann}N_{(+)}(\operatorname{ann}I)\,M = \operatorname{ann}(I_{(+)}N) = \operatorname{ann}H = \bigcap_{i=1}^{n}\operatorname{ann}(a_i, m_i).$$

For all i,

$$\begin{aligned}
\operatorname{ann}(a_i, m_i) &\supseteq \operatorname{ann}(Ra_{i(+)}(Rm_i + a_iM)) \\
&= \operatorname{ann}(a_i) \cap \operatorname{ann}(Rm_i + a_iM)_{(+)}\operatorname{ann}(a_i)\,M \\
&= \operatorname{ann}(Rm_i + a_iM)_{(+)}\operatorname{ann}(a_i)\,M.
\end{aligned}$$

Since M is faithful multiplication, it follows by [19, Corollary 1.7], that

$$\begin{aligned}
\operatorname{ann}H &\supseteq \bigcap_{i=1}^{n}\operatorname{ann}(Rm_i + a_iM)_{(+)}\left(\bigcap_{i=1}^{n}\operatorname{ann}(a_i)\right)M \\
&= \operatorname{ann}\left(\sum_{i=1}^{n}(Rm_i + a_iM)\right)_{(+)}\left(\sum_{i=1}^{n}\operatorname{ann}(a_i)\right)M.
\end{aligned}$$

Since M is finitely generated faithful multiplication, we infer from [29, Corollary to Theorem 9] that $\operatorname{ann}I \supseteq \operatorname{ann}\sum_{i=1}^{n}Ra_i \supseteq \operatorname{ann}I$, so that $\operatorname{ann}I = \operatorname{ann}\sum_{i=1}^{n}Ra_i$, and I is finitely annihilated. Also $\operatorname{ann}N \supseteq \operatorname{ann}\left(\sum_{i=1}^{n}(Rm_i + a_iM)\right) \supseteq \operatorname{ann}N$, so that $\operatorname{ann}N = \operatorname{ann}\left(\sum_{i=1}^{n}Rm_i + a_iM\right)$, and N is finitely annihilated.

(3) Assume M is faithful multiplication or projective. Let $J \subseteq I$ be finitely generated ideal of R such that $\operatorname{ann}I = \operatorname{ann}J$. It follows by, [1, Theorem 7], that $J_{(+)}JM$ is a finitely generated ideal of $R(M)$ that is contained in $I_{(+)}IM$. Moreover,

$$\begin{aligned}
\operatorname{ann}(I_{(+)}IM) &= \operatorname{ann}I_{(+)}(\operatorname{ann}I)\,M = \operatorname{ann}J_{(+)}(\operatorname{ann}J)\,M \\
&= \operatorname{ann}(J_{(+)}JM).
\end{aligned}$$

Hence $I_{(+)}IM$ is finitely annihilated. The converse follows by (2).

(4) Since M is finitely generated and $I_{(+)}N$ is finitely generated, it follows by, [1, Therorem 9], that N is finitely generated. Let $J \subseteq I$ be a finitely generated ideal of R such that $\operatorname{ann}I = \operatorname{ann}J$. Since $JM \subseteq IM \subseteq N, J_{(+)}N$ is a homogeneous ideal of $R(M)$ that is finitely generated. As

$$\begin{aligned}\operatorname{ann}(I_{(+)}N) &= \operatorname{ann}N_{(+)}(\operatorname{ann}I)\,M = \operatorname{ann}N_{(+)}(\operatorname{ann}J)\,M\\ &= \operatorname{ann}(J_{(+)}N),\end{aligned}$$

we infer that $I_{(+)}N$ is finitely annihilated.

(5) Using the same arguments of parts (3) and (4) and the fact that $\operatorname{ann}I \subseteq \operatorname{ann}N$, there exists a finitely generated ideal $J_{(+)}JM$ that is contained in $I_{(+)}N$ with the property $\operatorname{ann}(I_{(+)}N) = \operatorname{ann}(J_{(+)}JM)$. So $I_{(+)}N$ is finitely annihilated.

(6) Assume that $J \subseteq I$ is a finitely generated ideal of R such that $\operatorname{ann}I = \operatorname{ann}J$ and $K \subseteq N$ is a finitely generated submodule of M such that $\operatorname{ann}K = \operatorname{ann}N$. Since M is faithful and multiplication, we infer that

$$\operatorname{ann}(I_{(+)}N) = \operatorname{ann}N_{(+)}(\operatorname{ann}I)\,M = \operatorname{ann}K_{(+)}(\operatorname{ann}J)\,M.$$

Since $\operatorname{ann}K = \operatorname{ann}N \subseteq \operatorname{ann}I = \operatorname{ann}J$, we have that $(\operatorname{ann}[K:M])J = \operatorname{ann}(K)J = 0$, and hence $J \subseteq \operatorname{ann}(\operatorname{ann}([K:M])) = [K:M]$. This gives that $JM \subseteq K$ and hence $J_{(+)}K$ is a finitely generated homogeneous ideal of $R(M)$ that is contained in $I_{(+)}N$ and $\operatorname{ann}(I_{(+)}N) = \operatorname{ann}(J_{(+)}K)$. So $I_{(+)}N$ is finitely annihilated. □

Let R be a ring and M an R-module. Define

$$T(M)=\{m\in M : \operatorname{ann}(m) \neq 0\} \text{ and } \operatorname{Si}(M)=\{m\in M : \operatorname{ann}(m) \text{ is large in } R\}.$$

$T(M)$ is called the *torsion* submodule of M and $\operatorname{Si}(M)$ the *singular* submodule of M, [23]. If R is an integral domain then $T(M)$ is closed in M, [23, p. 139].

The next result characterizes the torsion and singular subideals of homogeneous ideals of $R(M)$.

Proposition 13. *Let R be a ring, M an R-module and $I_{(+)}N$ a homogeneous ideal of $R(M)$.*

1. *If M is torsion-free then $T(I_{(+)}N) = T(I)_{(+)}N$.*
2. *If M is faithful multiplication and $R(M)$ a homogeneous ring then* $\operatorname{Si}(I_{(+)}N) = \operatorname{Si}(I)_{(+)}N$.

Proof. (1) Let $(a,m) \in T(I_{(+)}N)$. Then $\operatorname{ann}(a,m) \neq 0$. Since M is torsion-free, we infer that $\operatorname{ann}(a) \neq 0$. Hence $a \in T(I)$ and hence $(a,m) \in T(I)_{(+)}N$. So $T(I_{(+)}N) \subseteq T(I)_{(+)}N$. For the reverse inclusion, if $(a,m) \in T(I)_{(+)}N$, then $a \in T(I)$. Hence $\operatorname{ann}(a) \neq 0$, and hence $\operatorname{ann}(a,m) \neq 0$. So $(a,m) \in T(I_{(+)}N)$ and hence $T(I)_{(+)}N \subseteq T(I_{(+)}N)$. This shows that $T(I_{(+)}N) = T(I)_{(+)}N$.

(2) Suppose $(a, m) \in \mathrm{Si}\,(I_{(+)}N)$. Then $\mathrm{ann}\,(a,m)$ is a large ideal of $R\,(M)$. Since $R\,(M)$ is homogeneous, it follows by, [17, Theorem 3.3], that

$$\begin{aligned}\mathrm{ann}\,(a,m) &= \mathrm{ann}\,(Ra_{(+)}Rm + aM)\\ &= \mathrm{ann}\,(a) \cap \mathrm{ann}\,(m)\,_{(+)}\mathrm{ann}\,(a)\,M.\end{aligned}$$

By [3, Proposition 17], $\mathrm{ann}\,(a)\,M$ is a large submodule of M. Since M is faithful multiplication, we infer from [6, Proposition 12] that $\mathrm{ann}\,(a)$ is a large ideal of R. Hence $a \in \mathrm{Si}(I)$ and hence $(a,m) \in \mathrm{Si}\,(I)_{(+)}N$. If $(a,m) \in \mathrm{Si}\,(I)_{(+)}N$ then $a \in \mathrm{Si}\,(I)$ and hence $\mathrm{ann}\,(a)$ is a large ideal of R. It follows that $\mathrm{ann}\,(a)\,M$ is a large submodule of M, [6, Proposition 12]. [3, Proposition 14] shows that $\mathrm{ann}\,(a,m) = \mathrm{ann}\,(a) \cap \mathrm{ann}\,(m)\,_{(+)}\mathrm{ann}\,(a)\,M$ is a large ideal of $R\,(M)$. So $(a,m) \in \mathrm{Si}\,(I_{(+)}N)$ and hence $\mathrm{Si}(I_{(+)}N) = \mathrm{Si}\,(I)_{(+)}N$. □

3. Cancellation modules

Generalizing the case for ideals, an R-module M is defined to be *cancellation* (resp. *weak cancellation*) if $IM = JM$ for ideals I and J of R then $I = J$ (resp. $I + \mathrm{ann}M = J + \mathrm{ann}M$), [15] and [28]. Examples of cancellation modules include invertible ideals, free modules [28] and finitely generated faithful multiplication modules, [29, Corollary to Theorem 9]. If M is a finitely generated faithful multiplication R-module (hence cancellation), then it is easily verified that $[IN : M] = I\,[N : M]$ for all ideals I of R and all submodules N of M. Anderson, [15], defined an R-module M to be *restricted cancellation* if $IM = JM \neq 0$ for some ideals I and J of R then $I = J$. Every cancellation module is restricted cancellation but the converse is not true in general, [15]. An R-module M is a cancellation module if and only if it is a faithful weak cancellation module. It is also shown, [15, Theorem 2.5], that M is restricted cancellation if and only if it is weak cancellation and $\mathrm{ann}M$ is comparable to every ideal of R. A submodule N of M is called *join principal* if $[(IN + K) : N] = I + [K : N]$ for all submodules K of M and all ideals I of R. Setting $K = 0, N$ becomes weak cancellation. A submodule N is join principal if and only if each of its homomorphic images is weak cancellation, [15, Theorem 2.2]. An R-module M is called $\frac{1}{2}$ *cancellation* (resp. $\frac{1}{2}$ *weak cancellation*) if for all ideals I of R, $IM = M$ implies $I = R$ (resp. $I + \mathrm{ann}M = R$), [5] and [27]. M is a $\frac{1}{2}$ cancellation module if and only if $M \neq PM$ for each maximal ideal P of R. In [5], we introduced and investigated the concept of $\frac{1}{2}$ join principal submodules. A submodule N of M is $\frac{1}{2}$ *join principal* if for all ideals I of R and all submodules K of M, $N = IN + K$ implies $R = I + [K : N]$.

Let R be a ring and M an R-module. An ideal I of R is called *M-cancellation* (resp. *M-weak cancellation*) if for all submodules K and N of M, $IK = IN$ implies $K = N$ (resp. $K + [0 :_M I] = N + [0 :_M I]$). Equivalently, $[IN :_M I] = N$ (resp. $[IN :_M I] = N + [0 :_M I]$) for all submodules N of M. I is said to be an *M-restricted cancellation* ideal if for all submodules K and N of M, if $IK = IN \neq 0$ then $K = N$. An ideal I of R is called *M-join principal* if $[(IK + N) :_M I] = K + [N :_M I]$ for all submodules K and N of M. Setting $N = 0$, I becomes an M-weak cancellation ideal, [7]. Every M-cancellation ideal is M-restricted cancellation and every M-restricted cancellation is M-weak cancellation, and the three concepts coincide if both I and M are faithful. It is proved that an ideal I of R is M-restricted cancellation if and only if I is M-weak cancellation and $[0:_M I]$ is comparable to every submodule N of M, [7, Proposition 1.5]. Several facts on M-cancellation properties are given in [7]. Motivated by the terminology of $\frac{1}{2}$ join principal and $\frac{1}{2}$ weak (cancellation) modules, we introduce the following definitions. Let R be a ring and M an R-module. An ideal I of R is called an M-$\frac{1}{2}$ *join principal* ideal of R if for all submodules K and N of M, $IM = IK + N$ implies $M = K + [N :_M I]$. It is easily verified that I is M-$\frac{1}{2}$ join principal if $IM \subseteq IK + N$ then $M = K + [N :_M I]$. Setting $N = 0$, we define I to be M-$\frac{1}{2}$ *weak cancellation*. In particular, I is called an M-$\frac{1}{2}$ *cancellation* ideal of R if for all submodules K of M, $IM = IK$ implies $M = K$. If N is a proper pure submodule of M then $[N : M]$ is not M-$\frac{1}{2}$ cancellation. For, if N is pure in M, then $JN = JM \cap N$ for all ideals J of R. Take $J = [N : M]$ then $[N : M]\, N = [N : M]\, M \cap N = [N : M]\, M$. M-cancellation ideals are M-$\frac{1}{2}$ cancellation but not conversely. For a prime p, the Prüfer p-group $M = \mathbb{Z}_{p^\infty}$ is a faithful but not multiplication $\mathbb{Z}$-module. The only non-zero submodules of M are $\mathbb{Z}_p \subseteq \mathbb{Z}_{p^2} \subseteq \cdots \mathbb{Z}_{p^\infty}$. For each $k > 1$, $p\mathbb{Z}_{p^k} = \mathbb{Z}_{p^{k-1}}$. Hence $p^2\mathbb{Z}_{p^2} = p^2\mathbb{Z}_p = 0$, and hence $p^2\mathbb{Z}$ is not M-cancellation. On the other hand, for each $k > 1$, $p^2\mathbb{Z}_{p^\infty} \neq p^2\mathbb{Z}_{p^k}$, and hence $p^2\mathbb{Z}$ is an M-$\frac{1}{2}$ cancellation ideal of R. Otherwise, if $p^2\mathbb{Z}_{p^\infty} = p^2\mathbb{Z}_{p^k}$, then $p^k\mathbb{Z}_{p^\infty} = 0$, and hence $p^k\mathbb{Z} \subseteq [0 : \mathbb{Z}_{p^\infty}] = 0$, a contradiction. Neither M-$\frac{1}{2}$ cancellation ideals are $\frac{1}{2}$ cancellation nor $\frac{1}{2}$ cancellation ideals are M-$\frac{1}{2}$ cancellation. To explain the last statement, recall the ideal associated with an R-module M, $\theta(M) = \sum\limits_{m \in M} [Rm : M]$, which proved useful in studying multiplication modules. A finitely generated module M is multiplication if and only if $\theta(M) = R$. If M is multiplication then $M = \theta(M)\, M$ from which it follows that for any submodule N of M, $N = \theta(M)\, N$. If M is a faithful multiplication module then $\theta(M)$ is a pure ideal of R, that is multiplication and idempotent, [16, Theorem 2.6]. Let M be a faithful multiplication R-module but not finitely generated. Then $\theta(M)$ is M-$\frac{1}{2}$ cancellation because for each submodule N of M, if $\theta(M)\, M = \theta(M)\, N$, then $M = N$, [16, Lemma 1.1]. But $\theta(M)$ is not $\frac{1}{2}$ cancellation, since otherwise $\theta(M) = (\theta(M))^2$ implies $\theta(M) = R$ and hence M is finitely generated. Let $R = \mathbb{Z}/4\mathbb{Z}$ and $I = 2\mathbb{Z}/4\mathbb{Z}$. Then I is a $\frac{1}{2}$ cancellation ideal of R, [5]. Consider $M = \mathbb{Z}/2\mathbb{Z}$ as an R-module, then $IM = 0$ and this shows that I is not an M-$\frac{1}{2}$ cancellation ideal of R.

The following two results give some properties of $M - \frac{1}{2}$ (weak) cancellation ideals. First let us recall the *trace* ideal of an R-module M, $\mathrm{Tr}(M) = \sum\limits_{f \in Hom(M,R)} f(M)$. If M is a projective module then $M = \mathrm{Tr}(M)\, M$, $\mathrm{ann} M = \mathrm{ann}\mathrm{Tr}(M)$ and $\mathrm{Tr}(M)$ is a pure ideal of R, [20, Proposition 3.30]. For any submodule N of M, $N = \mathrm{Tr}(M)\, N$, [8, Theorem 3.1]. If M is faithful multiplication then $\mathrm{Tr}(M) = \theta(M)$, [16, Theorem 2.3].

Proposition 14. *Let R be a ring, M an R-module and I an ideal of R.*

(1) *I is M-$\frac{1}{2}$ cancellation if and only if $IM \neq IQ$ for each maximal submodule Q of M.*

(2) *Let M be faithful multiplication or projective. If I is a faithful M-$\frac{1}{2}$ weak cancellation ideal then I is M-$\frac{1}{2}$ cancellation.*

(3) *Let M be finitely generated faithful multiplication. If I is projective or faithful multiplication and contains an M-$\frac{1}{2}$ cancellation ideal of R then I is M-$\frac{1}{2}$ cancellation.*

(4) *If I is a pure ideal of R and contains an M-$\frac{1}{2}$ cancellation ideal of R then I is M-$\frac{1}{2}$ cancellation.*

Proof. (1) Obviously, if I is M-$\frac{1}{2}$ cancellation then $IM \neq IQ$ for each maximal submodule Q of M. Conversely, suppose I is not M-$\frac{1}{2}$ cancellation. There exists a proper submodule N of M with $IM = IN$, and hence there exists a maximal submodule Q of M with $N \subseteq Q$. So $IM = IN \subseteq IQ \subseteq IM$, so that $IM = IQ$, a contradiction.

(2) Let N be a submodule of M with $IM = IN$. Then $M = N + [0 :_M I]$. Since M is faithful multiplication or projective, $[0 :_M I] = (\text{ann} I)\, M = 0$. So $M = N$, and I is M-$\frac{1}{2}$ cancellation.

(3) Let $J \subseteq I$ be M-$\frac{1}{2}$ cancellation and N a submodule of M with $IM = IN$. Since M is finitely generated faithful and multiplication, $I = I\,[N : M]$, and hence $\text{Tr}\,(I) = \text{Tr}\,(I)\,[N : M]$. This implies that $\text{Tr}\,(I)\, M = \text{Tr}\,(I)\, N$. If I is projective, then $J = J\text{Tr}\,(I)$, [8, Theorem 3.1], and hence $JM = JN$ from which it follows that $M = N$, and I is M-$\frac{1}{2}$ cancellation. Alternatively, if I is not M-$\frac{1}{2}$ cancellation then $IM = IQ$ for some maximal submodule Q of M. Hence $\text{Tr}\,(I)\, M = \text{Tr}\,(I)\, Q$ and hence $JM = J\text{Tr}\,(I)\, M = J\text{Tr}\,(I)\, Q = JQ$, a contradiction. If I is faithful and multiplication then $\text{Tr}\,(I) = \theta\,(I)$ and $J = J\theta\,(I)$, [16, Lemma 1.1]. The result is now clear.

(4) Let N be a submodule of M with $IM = IN$. Suppose $J \subseteq I$ is M-$\frac{1}{2}$ cancellation and I is pure, then $J = JI$ and hence $JM = JN$. So $M = N$, and hence I is M-$\frac{1}{2}$ cancellation. □

Proposition 15. *Let R be a ring, M an R-module and I an ideal of R.*

(1) *Let M be finitely generated, faithful and multiplication. Then I is $\frac{1}{2}$ join principal (resp. $\frac{1}{2}$ weak cancellation) if and only if I is M-$\frac{1}{2}$ join principal (resp. M-$\frac{1}{2}$ weak cancellation).*

(2) *If M is finitely generated faithful multiplication and I is $\frac{1}{2}$ cancellation then I is M-$\frac{1}{2}$ cancellation. The converse is true if M is $\frac{1}{2}$ cancellation.*

(3) *If M is multiplication and IM is a $\frac{1}{2}$ join principal (resp. $\frac{1}{2}$ weak cancellation) submodule of M then I is M-$\frac{1}{2}$ join principal (resp. M-$\frac{1}{2}$ weak cancellation). The converse is true if we assume further that M is finitely generated.*

(4) *If M is multiplication and IM is a $\frac{1}{2}$ cancellation submodule of M then I is M-$\frac{1}{2}$ cancellation. The converse is true if M is $\frac{1}{2}$ cancellation.*

Proof. (1) Suppose I is $\frac{1}{2}$ join principal. Let K and N be submodules of M such that $IM = IK + N$. Then $IM = (I\,[K : M] + [N : M])\,M$, and hence $I = I\,[K : M] + [N : M]$. This implies that $R = [K : M] + [[N : M] : I] = [K : M] + [N : IM]$, and hence $M = [K : M]\,M + [N : IM]\,M = K + [N :_M I]$. So I is M-$\frac{1}{2}$ join principal. Conversely, let I be M-$\frac{1}{2}$ join principal and let A and B be ideals of R with $I = AI + B$. Then $IM = I\,(AM) + BM$, and hence $M = AM + [BM :_M I] = (A + [BM : IM])\,M$. It follows that $R = A + [B : I]$, and I is $\frac{1}{2}$ join principal. The $\frac{1}{2}$ weak cancellation case is obvious by taking $N = 0$ and $B = 0$ respectively.

(2) Obvious.

(3) We only discuss the $\frac{1}{2}$ join principal case. Let M be multiplication and IM a $\frac{1}{2}$ join principal submodule of M. Let K and N be submodules of M with $IM = IK + N$. Then $IM = [K : M]\,IM + N$, and hence $R = [K : M] + [N : IM]$. So $M = K + [N :_M I]$, and I is an M-$\frac{1}{2}$ join principal ideal of R. Conversely, let M be finitely generated multiplication and I an M-$\frac{1}{2}$ join principal. Suppose J is an ideal of R and K a submodule of M with $IM = J\,(IM) + K$. Then $M = JM + [K :_M I] = (J + [K : IM])\,M$, and hence $R = J + [K : IM] + \text{ann}M = J + [K : IM]$. This shows that IM is $\frac{1}{2}$ join principal.

(4) Obvious. □

The following result is now straightforward.

Corollary 16. *Let R be a ring and M a finitely generated faithful multiplication R-module. The following statements are equivalent for a submodule N of M.*

(1) *N is $\frac{1}{2}$ join principal (resp. $\frac{1}{2}$ weak cancellation, $\frac{1}{2}$ cancellation).*

(2) *$[N : M]$ is a $\frac{1}{2}$ join principal (resp. $\frac{1}{2}$ weak cancellation, $\frac{1}{2}$ cancellation) ideal of R.*

(3) *$[N : M]$ is an M-$\frac{1}{2}$ join principal (resp. M-$\frac{1}{2}$ weak cancellation, M-$\frac{1}{2}$ cancellation) ideal of R.*

The next theorem gives some relationships between M-join principal (resp. M-cancellation) and M-$\frac{1}{2}$ join principal (resp.M-$\frac{1}{2}$cancellation) submodules (ideals) and their products.

Proposition 17. *Let R be a ring, M an R-module and I, J ideals of R.*

(1) *If I is M-join principal and J is M-$\frac{1}{2}$ join principal (resp. M-$\frac{1}{2}$ weak cancellation) then IJ is M-$\frac{1}{2}$ join principal (resp. M-$\frac{1}{2}$ weak cancellation).*

(2) *If I is M-cancellation and J is M-$\frac{1}{2}$ weak cancellation (resp. M-$\frac{1}{2}$ cancellation) then IJ is M-$\frac{1}{2}$ weak cancellation (resp. M-$\frac{1}{2}$ cancellation).*

(3) *If I is M-cancellation and IJ is M-$\frac{1}{2}$ join principal (resp. M-$\frac{1}{2}$ weak cancellation) then J is M-$\frac{1}{2}$ join principal (resp. M-$\frac{1}{2}$ weak cancellation).*

Proof. (1) We only do the join principal case. Let K and N be submodules of M such that $IJM = IJK + N$. Since I is M-join principal, we infer that $JM \subseteq [IJM :_M I] = [(IJK + N) :_M I] = JK + [N :_M I]$. But J is M-$\frac{1}{2}$ join principal. Thus $M = K + [[N :_M I] :_M J] = K + [N :_M IJ]$, and hence IJ is M-$\frac{1}{2}$ join principal.

(2) Suppose N is a submodule of M such that $IJM = IJN$. Then $JM = JN$. Since J is M-$\frac{1}{2}$ weak cancellation, $M = N + [0 :_M J] \subseteq N + [0 :_M IJ]$, so that $M = N + [0 :_M IJ]$ and IJ is M-$\frac{1}{2}$ weak cancellation. The $\frac{1}{2}$ cancellation case is now obvious.

(3) Suppose K and N are submodules of M such that $JM = JK + N$. Then $IJM = IJK + IN$, and hence

$$M = K + [IN :_M IJ] = K + [[IN :_M I] :_M J] = K + [N :_M J],$$

and J is an M-$\frac{1}{2}$ join principal ideal of R. The $\frac{1}{2}$ weak cancellation modules case follows by taking $N = 0$. □

The following result shows how properties such as M-$\frac{1}{2}$ join principal (resp. $M - \frac{1}{2}$ cancellation) are related to $0_{(+)}M$-$\frac{1}{2}$ join principal (resp. $0_{(+)}M - \frac{1}{2}$ cancellation).

Proposition 18. *Let R be a ring, M an R-module and I an ideal of R.*

(1) *I is M-$\frac{1}{2}$ join principal ideal if and only if $I_{(+)}IM$ is an $0_{(+)}M$-$\frac{1}{2}$ join principal ideal of $R(M)$.*

(2) *I is M-$\frac{1}{2}$ weak cancellation ideal if and only if $I_{(+)}IM$ is an $0_{(+)}M$-$\frac{1}{2}$ weak cancellation ideal of $R(M)$.*

(3) *I is M-$\frac{1}{2}$ cancellation ideal if and only if $I_{(+)}IM$ is an $0_{(+)}M$-$\frac{1}{2}$ cancellation ideal of $R(M)$.*

Proof. (1) Suppose I is $M - \frac{1}{2}$ join principal. Let K and N be submodules of M and let $(0_{(+)}M)(I_{(+)}IM) = (0_{(+)}K)(I_{(+)}IM) + 0_{(+)}N$. Then $0_{(+)}IM = 0_{(+)}(IK + N)$, and hence $IM = IK + N$. It follows that $M = K + [N :_M I]$, and this gives that

$$\begin{aligned} 0_{(+)}M &= 0_{(+)}(K + [N :_M I]) = 0_{(+)}K + 0_{(+)}[N :_M I] \\ &= 0_{(+)}K + [0_{(+)}N :_{0(+)M} I_{(+)}IM], \end{aligned}$$

and $I_{(+)}IM$ is an M-$\frac{1}{2}$ join principal ideal of $R(M)$. The converse is now clear.

(2) The M-$\frac{1}{2}$ weak cancellation case follows from (1) by letting $N = 0$.

(3) Clear. □

The next result gives some conditions under which the ideal $I_{(+)}IM$ becomes weak cancellation (resp. restricted cancellation). Compare with [3, Proposition 10 and Theorem 11].

Proposition 19. *Let R be a ring, M an R-module and I an ideal of R.*

(1) *If* $I_{(+)}IM$ *is a cancellation ideal of* $R(M)$ *then* I *is cancellation and the converse is true if* M *is torsion-free.*

(2) *If* $I_{(+)}IM$ *is a weak cancellation ideal of* $R(M)$ *then* I *is weak cancellation and the converse is true if* M *is finitely generated, faithful and multiplication.*

(3) *If* $I_{(+)}IM$ *is a restricted cancellation ideal of* $R(M)$ *then* I *is restricted cancellation and* $IM \neq 0$ *implies* I *is faithful. The converse is true if* M *is finitely generated, faithful and multiplication.*

Proof. (1) [5, Proposition 11].

(2) Suppose $I_{(+)}IM$ is a weak cancellation ideal of $R(M)$. By [15, Theorem 2.4], $I_{(+)}IM$ is a cancellation $R(M)/\mathrm{ann}(I_{(+)}IM) \cong R(M)/\mathrm{ann}I_{(+)}[0:_M I] \cong R/\mathrm{ann}I_{(+)}M/[0:_M I]$-module. It follows by (1) that I is a cancellation $R/\mathrm{ann}I$-module and hence I is a weak cancellation ideal of R, [15, Theorem 2.4]. Conversely, suppose M is finitely generated faithful multiplication and I is weak cancellation. Then I is a cancellation $R/\mathrm{ann}I$-module. Since M is a multiplication R-module, it is easily verified that $M/(\mathrm{ann}I)M$ is a multiplication $R/\mathrm{ann}I$-module. Moreover, $[0:_{R/\mathrm{ann}I} M/(\mathrm{ann}I)M] = [(\mathrm{ann}I)M : M]/\mathrm{ann}I$. Since M is finitely generated faithful and multiplication (hence cancellation), $[0:_{R/\mathrm{ann}I} M/(\mathrm{ann}I)M] = \mathrm{ann}I/\mathrm{ann}I = 0$. That is $M/(\mathrm{ann}I)M$ is a faithful multiplication $R/\mathrm{ann}I$-module, and hence it is a torsion free $R/\mathrm{ann}I$-module, [19, Lemma 4.1]. It follows by (1) that $I_{(+)}IM$ is a cancellation $R/\mathrm{ann}I_{(+)}M/(\mathrm{ann}I)M \cong R(M)/\mathrm{ann}(I_{(+)}IM)$-module. This implies that $I_{(+)}IM$ is a weak cancellation ideal of $R(M)$.

(3) Suppose $I_{(+)}IM$ is a restricted cancellation ideal of $R(M)$. Then $I_{(+)}IM$ is weak cancellation and by (2), I is a weak cancellation ideal of R. Let A be an ideal of R then $A_{(+)}AM \subseteq \mathrm{ann}(I_{(+)}IM) = \mathrm{ann}I_{(+)}[0:_M I]$ from which we get $A \subseteq \mathrm{ann}I$ or $\mathrm{ann}(I_{(+)}IM) \subseteq A_{(+)}AM$ and this case gives that $\mathrm{ann}I \subseteq A$. By, [15, Theorem 2.5], I is restricted cancellation. Now suppose $IM \neq 0$. Then $0 \neq (I_{(+)}IM)(0_{(+)}M) = (I_{(+)}IM)(\mathrm{ann}I_{(+)}M)$, and hence $\mathrm{ann}I = 0$. Conversely, suppose M is finitely generated, faithful and multiplication. Let I be restricted cancellation. Then I is weak cancellation and by (2), $I_{(+)}IM$ is weak cancellation. If $0 \neq IM$, then I is faithful. Since M is faithful multiplication, $I_{(+)}IM$ is faithful and hence $I_{(+)}IM$ is restricted cancellation (in fact, $I_{(+)}IM$ is cancellation). If $IM = 0$, then $M = [0:_M I] = (\mathrm{ann}I)M$. Since M is finitely generated, faithful and multiplication (hence cancellation), $R = \mathrm{ann}I$, that is $I = 0$, and this case shows that $I_{(+)}IM = 0$ is again a restricted cancellation ideal of $R(M)$. □

We close by a result that gives necessary and sufficient conditions for a homogeneous ideal of $R(M)$ to be join principal.

Theorem 20. *Let* R *be a ring,* M *an* R*-module and* $I_{(+)}N$ *a homogeneous ideal of* $R(M)$*.*

(1) *If* $I_{(+)}N$ *is join principal then* I *is a weak cancellation ideal of* R *and* N/IM *is a weak cancellation* R*-submodule of* M/IM*.*

(2) *If $I_{(+)}N$ is join principal then I is a join principal ideal of R. The converse is true if $H = [H :_{R(M)} 0_{(+)}M]$ for any ideal H of $R(M)$.*

(3) *Let M be cancellation. If $I_{(+)}N$ is weak cancellation then N is a weak cancellation submodule of M.*

(4) *Let M be cancellation. If $I_{(+)}N$ is cancellation then N is a cancellation submodule of M.*

(5) *Suppose $I_{(+)}N$ is restricted cancellation. If $I^2 \neq 0$ then I is a restricted cancellation ideal of R and if we assume further that M is restricted cancellation then N is a restricted cancellation submodule of M. If $I^2 = 0$ then $I_{(+)}N$ is a nilpotent ideal of $R(M)$ and if R is reduced or N faithful then I is restricted cancellation and N is restricted cancellation.*

Proof. (1) [15, Theorem 2.2] says that if $I_{(+)}N$ is a join principal ideal of $R(M)$ then $I_{(+)}N/0_{(+)}N \cong I$ is a weak cancellation ideal of R, and

$$\begin{aligned} I_{(+)}N/I_{(+)}IM &= I_{(+)}IM + 0_{(+)}N/I_{(+)}IM \\ &\cong 0_{(+)}N/I_{(+)}IM \cap 0_{(+)}N \\ &= 0_{(+)}N/0_{(+)}IM \cong 0_{(+)}N/IM \end{aligned}$$

is a weak cancellation ideal of $R(M)/I_{(+)}IM \cong R(M/IM)$. It follows that N/IM is a weak cancellation submodule of M/IM, [1, Theorem 2] and [15, Theorem 3.1].

(2) Suppose $I_{(+)}N$ is join principal. The fact that I is join principal follows by [3, Theorem 9]. Note that if I is join principal then I is weak cancellation and that confirms part (1). For the converse, let H_1 and H_2 be ideals of $R(M)$ and let $H_1 + 0_{(+)}M = A_{(+)}M$ and $H_2 + 0_{(+)}M = B_{(+)}M$ for some ideals A and B of R. Then

$$\begin{aligned} &[(H_1(I_{(+)}N) + H_2) :_{R(M)} I_{(+)}N] \\ &\quad \subseteq [(H_1(I_{(+)}N) + 0_{(+)}IM) + H_2 + 0_{(+)}M) :_{R(M)} (I_{(+)}N + 0_{(+)}M)] \\ &\quad = [((H_1 + 0_{(+)}M)(I_{(+)}N) + H_2 + 0_{(+)}M) :_{R(M)} I_{(+)}M] \\ &\quad = [((A_{(+)}M)(I_{(+)}N) + B_{(+)}M) :_{R(M)} I_{(+)}M] \\ &\quad = [(AI + B)_{(+)}M :_{R(M)} I_{(+)}M] = [(AI + B) : I]_{(+)}M \\ &\quad = A + [B : I]_{(+)}M = A_{(+)}M + [B : I]_{(+)}M \\ &\quad = H_1 + 0_{(+)}M + [B_{(+)}M :_{R(M)} I_{(+)}M] \\ &\quad = H_1 + [H_2 + 0_{(+)}M :_{R(M)} I_{(+)}M] \\ &\quad \subseteq H_1 + [H_2(0_{(+)}M) :_{R(M)} 0_{(+)}IM] \\ &\quad \subseteq H_1 + [(H_2 \cap 0_{(+)}M) :_{R(M)} 0_{(+)}IM] \\ &\quad = H_1 + [H_2 :_{R(M)} 0_{(+)}IM] = H_1 + [H_2 :_{R(M)} (0_{(+)}M)(I_{(+)}N)] \\ &\quad \subseteq H_1 + [H_2 :_{R(M)} 0_{(+)}M] :_{R(M)} I_{(+)}N] \\ &\quad = H_1 + [H_2 :_{R(M)} I_{(+)}N]. \end{aligned}$$

The other inclusion is always true and hence

$$[H_1(I_{(+)}N) + H_2 :_{R(M)} I_{(+)}N] = H_1 + [H_2 :_{R(M)} I_{(+)}N].$$

This shows that $I_{(+)}N$ is join principal. Note that we used in the proof the fact that

$$0_{(+)}M \subseteq B_{(+)}M \subseteq [B_{(+)}M :_{R(M)} I_{(+)}M].$$

(3) Suppose M is cancellation and $I_{(+)}N$ a weak cancellation ideal of $R(M)$. Let A and B be ideals of R such that $AN = BN$. Since $AIM \subseteq AN = BN$, we infer that $0_{(+)}AIM \subseteq 0_{(+)}BN$, and hence

$$(0_{(+)}AM)(I_{(+)}N) \subseteq (B_{(+)}BM)(0_{(+)}N) \subseteq (B_{(+)}BM)(I_{(+)}N).$$

This implies that

$$0_{(+)}AM + (\mathrm{ann}I \cap \mathrm{ann}N)_{(+)}[0 :_M I] \subseteq B_{(+)}BM + (\mathrm{ann}I \cap \mathrm{ann}N)_{(+)}[0 :_M I],$$

from which one gets that $AM + [0 :_M I] \subseteq BM + [0 :_M I]$. Hence $AIM \subseteq BIM$, and this gives that $AI \subseteq BI$. It follows that $(A_{(+)}M)(I_{(+)}N) \subseteq (B_{(+)}M)(I_{(+)}N)$, and hence

$$A_{(+)}M + (\mathrm{ann}I \cap \mathrm{ann}N)_{(+)}[0 :_M I] \subseteq B_{(+)}M_{(+)}(\mathrm{ann}I \cap \mathrm{ann}N)_{(+)}[0 :_M I].$$

So $A + (\mathrm{ann}I \cap \mathrm{ann}N) \subseteq B + (\mathrm{ann}I \cap \mathrm{ann}N)$. Since M is cancellation (hence faithful) and $IM \subseteq N$, we have that $\mathrm{ann}N \subseteq \mathrm{ann}(IM) = \mathrm{ann}I$. This finally gives that $A + \mathrm{ann}N \subseteq B + \mathrm{ann}N$. Similarly, $B + \mathrm{ann}N \subseteq A + \mathrm{ann}N$. Hence N is weak cancellation.

(4) Suppose M is cancellation and $I_{(+)}N$ is cancellation. By (3), N is weak cancellation. By [1, Proposition 14] I is cancellation and hence IM is cancellation. It follows that $\mathrm{ann}N \subseteq \mathrm{ann}(IM) = 0$, and hence N is cancellation.

(5) Suppose $I_{(+)}N$ is a restricted cancellation ideal of $R(M)$. If $I^2 \neq 0$, then $0 \neq I^2{}_{(+)}IN = (I_{(+)}N)^2 = (I_{(+)}N)(I_{(+)}IM)$. So $I_{(+)}N = I_{(+)}IM$. The fact that I is restricted cancellation follows by Proposition 19. Suppose M is restricted cancellation. Since $N = IM$, it is easily verified that N is restricted cancellation. Next, let $I^2 = 0$. Then $I \subseteq \mathrm{ann}I$ and hence $IM \subseteq (\mathrm{ann}I)M \subseteq [0 :_M I]$. But $I_{(+)}N$ is weak cancellation, [15, Theorem 2.5]. Therefore $(I_{(+)}N)^2 = (I_{(+)}N)(I_{(+)}IM)$ implies that

$$I_{(+)}N + (\mathrm{ann}I \cap \mathrm{ann}N)_{(+)}[0 :_M I] = I_{(+)}IM + (\mathrm{ann}I \cap \mathrm{ann}N)_{(+)}[0 :_M I]).$$

So $N + [0 :_M I] = IM + [0 :_M I]$ and hence $IN = I^2M = 0$. This gives that $(I_{(+)}N)^2 = 0$, and hence $I_{(+)}N$ is nilpotent. If R is reduced then $I = 0$ is restricted cancellation. Moreover, $I_{(+)}N = 0_{(+)}N$ is restricted cancellation implies that N is restricted cancellation, see [1, Theorem 2] and [15, Theorem 3.1]. Finally, if N is faithful then $IN = 0$ implies $I = 0$ and again I is restricted cancellation and N is restricted cancellation (in fact, N is cancellation). □

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