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The Affine Bundle Theorem in Synthetic Differential Geometry of Jet Bundles

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Abstract. In this paper we will establish the affine bundle theorem in our synthetic approach to jet bundles in terms of infinitesimal spaces D_n's. We will then compare these affine bundles with the corresponding ones constructed in our previous synthetic approach to jet bundles in terms of infinitesimal spaces D^n's both in the general microlinear setting and in the finite-dimensional setting.

1. Introduction

In our [11] we have established the affine bundle theorem in our synthetic approach to jet bundles in terms of infinitesimal spaces D^n's. In our succeeding [12] we have introduced another synthetic approach to jet bundles in terms of infinitesimal spaces D_n's, and have compared it with the former approach both in the general microlinear setting and in the finite-dimensional setting. However our comparison in the finite-dimensional setting was incomplete, for our use of dimension counting techniques has tacitly assumed that jet bundles in terms of D^n's and D_n's are vector bundles, which is not the case unless the given bundle is already a vector bundle. The principal objective in this paper is to establish the affine bundle theorem in our second synthetic approach to jet bundles and then to compare not merely the jet bundles based on both approaches but the affine bundles constructed as a whole both in the general microlinear setting and in the finite-dimensional setting. This completes our comparison between the two approaches in the finite-dimensional setting. Last but not least, we should say that the notion of a simple polynomial and correspondingly the notion of a D^n-tangential are modified essentially.

2. Preliminaries

2.1. Convention

Throughout the rest of the paper, unless stated to the contrary, E and M denote microlinear spaces, and $\pi : E \to M$ denotes a bundle, i.e., a mapping from E to M. The fiber of π over $x \in M$, namely, the set $\{y \in E \mid \pi(y) = x\}$ is denoted by E_x, as is usual. We denote by $\mathbb{R}$ the extended set of real numbers awash in nilpotent infinitesimals, which is expected to acquiesce in the so-called general Kock axiom (cf. [4]). The bundle $\pi : E \to M$ is called a *vector bundle* if the fiber E_x of π over x is an $\mathbb{R}$-module for every $x \in M$. Various canonical injections among infinitesimal spaces are simply denoted by the same symbol i. Mappings denoted by symbols with subscripts are sometimes denoted without subscripts, provided that the intended subscripts are clear from the context.

2.2. Infinitesimal spaces

As is depicted in [4], an *infinitesimal space* corresponds to a Weil algebra $\mathbb{R}[X_1, \ldots, X_n]/I$ in finite presentation. Given two infinitesimal spaces $\mathbb{E}_1$ and $\mathbb{E}_2$ corresponding to Weil algebras $\mathbb{R}[X_1, \ldots, X_n]/I$ and $\mathbb{R}[Y_1, \ldots, Y_m]/J$ respectively, the infinitesimal space corresponding to the Weil algebra $\mathbb{R}[X_1, \ldots, X_n, Y_1, \ldots, Y_m]/K$, where K is the ideal generated by I, J and $\{X_iY_j \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}$, is denoted by $\mathbb{E}_1 \oplus \mathbb{E}_2$, from which there are canonical projections onto $\mathbb{E}_1$ and $\mathbb{E}_2$ denoted by $p_{\mathbb{E}_1}$ and $p_{\mathbb{E}_2}$. Given two mappings $f : \mathbb{E}_1 \to \mathbb{F}_1$ and $g : \mathbb{E}_2 \to \mathbb{F}_2$ of infinitesimal spaces, there exists a unique mapping $f \oplus g : \mathbb{E}_1 \oplus \mathbb{E}_2 \to \mathbb{F}_1 \oplus \mathbb{F}_2$ making the following diagram commutative:

$$\begin{tikzcd}
\mathbb{E}_1 \arrow[r, "f"] & \mathbb{F}_1 \\
\mathbb{E}_1 \oplus \mathbb{E}_2 \arrow[u, "p_{\mathbb{E}_1}"] \arrow[r, "f\oplus g"] \arrow[d, "p_{\mathbb{E}_2}"'] & \mathbb{F}_1 \oplus \mathbb{F}_2 \arrow[u, "p_{\mathbb{F}_1}"'] \arrow[d, "p_{\mathbb{F}_2}"] \\
\mathbb{E}_2 \arrow[r, "g"'] & \mathbb{F}_2
\end{tikzcd}$$

We denote by D_1 or D the totality of elements of $\mathbb{R}$ whose squares vanish. More generally, given a natural number n, we denote by D_n the set

$$\{d \in \mathbb{R} | d^{n+1} = 0\}.$$

Given natural numbers m, n, we denote by $D(m)_n$ the set

$$\{(d_1, \ldots, d_m) \in D^m | d_{i_1} \cdots d_{i_{n+1}} = 0\},$$

where $i_1, \ldots, i_{n+1}$ shall range over natural numbers between 1 and m including both ends. We will often write $D(m)$ for $D(m)_1$. By convention $D^0 = D_0 = \{0\}$. A polynomial ρ of $d \in D_n$ is called a *simple* polynomial of $d \in D_n$ if the constant term is 0. A simple polynomial ρ of $d \in D_n$ is said to be *of dimension* m, in notation $\dim_n \rho = m$, provided that m is the least integer with $\rho^{m+1} = 0$. By way of example, letting $d \in D_3$, we have $\dim_3 d = \dim_3(d + d^2) = \dim_3(d + d^3) = 3$ and $\dim_3 d^2 = \dim_3 d^3 = \dim_3(d^2 + d^3) = 1$. Letting $e \in D_1$ or $e \in D_2$, we have $\dim_3 ed = 1$ or $\dim_3 ed = 2$ respectively. The reader should note that our present definition of a simple polynomial is different from that in [12].

Simplicial infinitesimal spaces are spaces of the form

$$D(m; \mathcal{S}) = \{(d_1, \dots, d_m) \in D^m | d_{i_1} \cdots d_{i_k} = 0 \text{ for any } (i_1, \dots, i_k) \in \mathcal{S}\},$$

where $\mathcal{S}$ is a finite set of sequences $(i_1, \dots, i_k)$ of natural numbers with $1 \leq i_1 < \dots < i_k \leq m$. To give examples of simplicial infinitesimal spaces, we have $D(2) = D(2; (1, 2))$ and $D(3) = D(3; (1, 2), (1, 3), (2, 3))$, which are all symmetric. The number m is called the *degree* of $D(m; \mathcal{S})$, in notation: $m = \mathrm{deg} D(m; \mathcal{S})$, while the maximum number n such that there exists a sequence $(i_1, \dots, i_n)$ of natural numbers of length n with $1 \leq i_1 < \dots < i_n \leq m$ containing no subsequence in $\mathcal{S}$ is called the *dimension* of $D(m; \mathcal{S})$, in notation: $n = \dim D(m; \mathcal{S})$. By way of example, $\mathrm{deg} D(3) = \mathrm{deg} D(3; (1, 2)) = \mathrm{deg} D(3; (1, 2), (1, 3)) = \mathrm{deg} D^3 = 3$, while $\dim D(3) = 1$, $\dim D(3; (1, 2)) = \dim D(3; (1, 2), (1, 3)) = 2$ and $\dim D^3 = 3$. It is easy to see that if $n = \dim D(m; \mathcal{S})$, then $d_1 + \dots + d_m \in D_n$ for any $(d_1, \dots, d_m) \in D(m; \mathcal{S})$. Given two simplicial infinitesimal spaces $D(m; \mathcal{S})$ and $D(m'; \mathcal{S}')$, a mapping $\varphi = (\varphi_1, \dots, \varphi_{m'}) : D(m; \mathcal{S}) \to D(m'; \mathcal{S}')$ is called a *monomial mapping* if every φ_j is a monomial in $d_1, \dots, d_m$ with coefficient 1.

Given an infinitesimal space $\mathbb{E}$, a mapping γ from $\mathbb{E}$ to M is called an $\mathbb{E}$-*microcube* on M. We denote by $\mathbf{T}^{\mathbb{E}}(M)$ the totality of $\mathbb{E}$-microcubes on M. Given $x \in M$, we denote by $\mathbf{T}^{\mathbb{E}}_x(M)$ the totality of $\mathbb{E}$-microcubes γ on M with $\gamma(0, \dots, 0) = x$. A mapping $f : M \to M'$ of microlinear spaces naturally gives rise to a mapping $\gamma \in \mathbf{T}^{\mathbb{E}}(M) \mapsto f \circ \gamma \in \mathbf{T}^{\mathbb{E}}(M')$, which is denoted by $f^{\mathbb{E}}_*$ or f_*. It is well known that the canonical projection $\tau_E : \mathbf{T}^D(E) \to E$ is a vector bundle. Its subbundle $\nu_\pi : \mathbf{V}(\pi) \to E$ with $\mathbf{V}(\pi) = \{t \in \mathbf{T}^D(E) : \pi_*(t) = 0\}$, called the *vertical bundle* of π, is also a vector bundle.

Given $\gamma \in \mathbf{T}^{D_n}(M)$ and a simplicial infinitesimal space $D(m; \mathcal{S})$ with $\dim D(m; \mathcal{S}) \leq n$, the mapping $(d_1, \dots, d_m) \in D(m; \mathcal{S}) \mapsto \gamma(d_1 + \dots + d_m) \in M$ is denoted by $\gamma_{D(m;\mathcal{S})}$. The following lemma should be obvious.

Lemma 1. *Let* $\gamma \in \mathbf{T}^{D_n}(M)$. *Let* $D(m; \mathcal{S}_1) \subseteq D(m; \mathcal{S}_2)$ *(this is the case if* $\mathcal{S}_1 \supseteq \mathcal{S}_2$*) with* $n_1 = \dim D(m; \mathcal{S}_1) \leq n_2 = \dim D(m; \mathcal{S}_2) \leq n$. *Let* $n_1 \leq n_3 \leq n$. *Then we have*

$$\gamma_{D(m;\mathcal{S}_2)} |_{D(m;\mathcal{S}_1)} = (\gamma |_{D_{n_3}})_{D(m;\mathcal{S}_1)}.$$

We denote by $\mathfrak{S}_n$ the symmetric group of the set $\{1, \dots, n\}$, which is well known to be generated by $n-1$ transpositions $< i, i+1 >$ exchanging i and $i+1$ $(1 \leq i \leq n-1)$ while keeping the other elements fixed. Given $\sigma \in \mathfrak{S}_n$ and $\gamma \in \mathbf{T}^{D^n}_x(M)$, we define $\Sigma_\sigma(\gamma) \in \mathbf{T}^{D^n}_x(M)$ to be

$$\Sigma_\sigma(\gamma)(d_1, \dots, d_n) = \gamma(d_{\sigma(1)}, \dots, d_{\sigma(n)})$$

for any $(d_1, \dots, d_n) \in D^n$. Given $\alpha \in \mathbb{R}$ and $\gamma \in \mathbf{T}^{D^n}(M)$, we define $\alpha \underset{i}{\cdot} \gamma \in \mathbf{T}_x^{D^n}(M)$ $(1 \leq i \leq n)$ to be

$$(\alpha \underset{i}{\cdot} \gamma)(d_1, \dots, d_n) = \gamma(d_1, \dots, d_{i-1}, \alpha d_i, d_{i+1}, \dots, d_n)$$

for any $(d_1, \dots, d_n) \in D^n$. Given $\alpha \in \mathbb{R}$ and $\gamma \in \mathbf{T}^{D_n}(M)$, we define $\alpha\gamma \in \mathbf{T}_x^{D_n}(M)$ $(1 \leq i \leq n)$ to be

$$(\alpha\gamma)(d) = \gamma(\alpha d)$$

for any $d \in D_n$. For any $\gamma \in \mathbf{T}_x^{D_n}(M)$ and any $d \in D_n$, we define $\mathbf{i}_d(\gamma) \in \mathbf{T}_x^{D_{n+1}}(M)$ to be

$$\mathbf{i}_d(\gamma)(d') = \gamma(dd')$$

for any $d' \in D_{n+1}$.

2.3. Affine bundles

A bundle $\pi : E \to M$ is called an *affine bundle* over a vector bundle $\xi : P \to M$ if E_x is an affine space over the vector space P_x for any $x \in M$. Following the lines of our previous paper [11], we will establish a variant of Theorem 0.3.6 of that paper.

As in Lemma 0.3.2 of [11], we have

Lemma 2. *The diagram*

$$\begin{array}{ccc} D_n & \xrightarrow{i} & D_{n+1} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \psi_{D_{n+1}}} \\ D_{n+1} & \xrightarrow[\varphi_{D_{n+1}}]{} & D_{n+1} \oplus D \end{array}$$

is a quasi-colimit diagram, where $\varphi_{D_{n+1}}(d) = (d, 0)$ and $\psi_{D_{n+1}}(d) = (d, d^{n+1})$ for any $d \in D_{n+1}$.

Remark 3. We will write $\varphi_{D^{n+1}} : D^{n+1} \to D^{n+1} \oplus D$ and $\psi_{D^{n+1}} : D^{n+1} \to D^{n+1} \oplus D$ for the mappings referred to in Lemma 0.3.2 of [11].

Given two D_{n+1}-microcubes γ_+ and γ_- on M with $\gamma_+|_{D_n} = \gamma_-|_{D_n}$, there exists a unique function $f : D_{n+1} \oplus D \to M$ with $f \circ \psi_{D_{n+1}} = \gamma_+$ and $f \circ \varphi_{D_{n+1}} = \gamma_-$. We define $(\gamma_+ \dot{-} \gamma_-) \in \mathbf{T}^D(M)$ to be

$$(\gamma_+ \dot{-} \gamma_-)(d) = f(0, d)$$

for any $d \in D$. It is easy to see that

$$\alpha\gamma_+ \dot{-} \alpha\gamma_- = \alpha^{n+1}(\gamma_+ \dot{-} \gamma_-) \tag{1}$$

for any $\alpha \in \mathbb{R}$ and any $\gamma_\pm \in \mathbf{T}^{D_{n+1}}(M)$ with $\gamma_+|_{D_n} = \gamma_-|_{D_n}$, while we know well that

$$\alpha \underset{i}{\cdot} \gamma_+ \dot{-} \alpha \underset{i}{\cdot} \gamma_- = \alpha(\gamma_+ \dot{-} \gamma_-) \quad (1 \leq i \leq n+1) \tag{2}$$

for any $\alpha \in \mathbb{R}$ and any $\gamma_\pm \in \mathbf{T}^{D^{n+1}}(M)$ with $\gamma_+|_{D(n+1)_n} = \gamma_-|_{D(n+1)_n}$. By the very definition of $\dot{-}$, we have

Proposition 4. *Let $f : M \to M'$. Given $\gamma_+, \gamma_- \in \mathbf{T}^{D_{n+1}}(M)$ with $\gamma_+|_{D_n} = \gamma_-|_{D_n}$, we have $f_*(\gamma_+)|_{D_n} = f_*(\gamma_-)|_{D_n}$ and*

$$f_*(\gamma_+ \dot{-} \gamma_-) = f_*(\gamma_+) \dot{-} f_*(\gamma_-).$$

As in Lemma 0.3.4 of [11], we have

Lemma 5. *The diagram*

$$\begin{array}{ccc} 1 & \xrightarrow{i} & D \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \epsilon_{D_{n+1}}} \\ D_{n+1} & \xrightarrow[\varphi_{D_{n+1}}]{} & D_{n+1} \oplus D \end{array}$$

is a quasi-colimit diagram, where $\varepsilon_{D_{n+1}}(d) = (0, d)$ for any $d \in D$.

Remark 6. We will write $\varepsilon_{D^{n+1}} : D \to D^{n+1} \oplus D$ for the mapping referred to in Lemma 0.3.4 of [11].

Given $t \in \mathbf{T}^D(M)$ and $\gamma \in \mathbf{T}^{D_{n+1}}(M)$ with $t(0) = \gamma(0)$, there exists a unique function $f : D_{n+1} \oplus D \to M$ with $f \circ \varphi_{D_{n+1}} = \gamma$ and $f \circ \varepsilon_{D_{n+1}} = t$. We define $(t \dot{+} \gamma) \in \mathbf{T}^{D_{n+1}}(M)$ to be

$$(t \dot{+} \gamma)(d) = f(d, d^{n+1})$$

for any $d \in D_{n+1}$. It is easy to see that

$$\alpha^{n+1} t \dot{+} \alpha\gamma = \alpha(t \dot{+} \gamma) \tag{3}$$

for any $\alpha \in \mathbb{R}$. By the very definition of $\dot{+}$ we have

Proposition 7. *Let $f : M \to M'$. Given $t \in \mathbf{T}^D(M)$ and $\gamma \in \mathbf{T}^{D_{n+1}}(\gamma)$ with $t(0) = \gamma(0, \dots, 0)$, we have $f_*(t)(0) = f_*(\gamma)(0, \dots, 0)$ and*

$$f_*(t \dot{+} \gamma) = f_*(t) \dot{+} f_*(\gamma).$$

As in Theorem 0.3.6 of [11], we have the following affine bundle theorem.

Theorem 8. *The canonical projection $\mathbf{T}^{D_{n+1}}(M) \to \mathbf{T}^{D_n}(M)$ is an affine bundle over the vector bundle $\mathbf{T}^D(M) \underset{M}{\times} \mathbf{T}^{D_n}(M) \to \mathbf{T}^{D_n}(M)$.*

2.4. The first approach to jet bundles

In [11] we have discussed a synthetic approach to jet bundles in terms of infinitesimal spaces D^n's. We have no intention to reproduce the paper, but our present notation and terminology are slightly different from those of that paper. First of all, what were called n-pseudoconnections and n-preconnections in that paper are now called D^n*-pseudotangentials* and D^n*-tangentials* respectively. We will write $\mathbb{J}_x^{D^n}(\pi)$ and $\mathbb{J}^{D^n}(\pi)$ for the space of D^n-tangentials for $\pi : E \to M$ at $x \in E$

and that of D^n-tangentials for $\pi : E \to M$ in place of $\mathbb{J}^n_x(\pi)$ and $\mathbb{J}^n(\pi)$ in that paper respectively. We will denote the canonical projection $\mathbb{J}^{D^{n+1}}(\pi) \to \mathbb{J}^{D^n}(\pi)$ by $\pi_{n+1,n}$ in place of $\underline{\pi}_{n+1,n}$ in that paper. What is more than a matter of notation or terminology is that, in our present definition of a D^{n+1}-tangential f for $\pi : E \to M$ at $x \in E$, we require the following condition besides conditions (1.11) and (1.12) of our previous paper [11]:

$$\begin{aligned} &\mathsf{f}((d_1, \dots, d_{n+1}) \in D^{n+1} \mapsto \gamma(ed_1, \dots, ed_{n+1}) \in M) \\ &\qquad = (d_1, \dots, d_{n+1}) \in D^{n+1} \mapsto \mathsf{f}_{D(n+1)_n}(\gamma)(ed_1, \dots, ed_{n+1}) \in E \end{aligned}$$

for any $e \in D_n$ and any $\gamma \in \mathbf{T}^{D(n+1)_n}_{\pi(x)}(M)$, where $\mathsf{f}_{D(n+1)_n}$ stands for the induced $D(n+1)_n$-pseudotangential of f depicted in Theorem 7 of our previous paper [12].

2.5. Symmetric forms

Given a vector bundle $\xi : P \to E$, a *symmetric D_n-form at $x \in E$ with values in ξ* is a mapping $\omega : \mathbf{T}^{D_n}_{\pi(x)}(M) \to P_x$ subject to the following conditions:

1. For any $\gamma \in \mathbf{T}^{D_n}_{\pi(x)}(M)$ and any $\alpha \in \mathbb{R}$, we have
$$\omega(\alpha\gamma) = \alpha^n \omega(\gamma.) \tag{4}$$
2. For any simple polynomial ρ of $d \in D_n$ and any $\gamma \in \mathbf{T}^{D_l}_{\pi(x)}(M)$ with $\dim_n \rho = l < n$, we have
$$\omega(\gamma \circ \rho) = 0. \tag{5}$$

We denote by $\mathbb{S}^{D_n}_x(\pi; \xi)$ the totality of symmetric D_n-forms at x with values in ξ. We denote by $\mathbb{S}^{D_n}(\pi; \xi)$ the set-theoretic union of $\mathbb{S}^{D_n}_x(\pi; \xi)$'s for all $x \in E$. The canonical projection $\mathbb{S}^{D_n}(\pi; \xi) \to E$ is obviously a vector bundle.

What were simply called symmetric n-forms along π with values in ξ in our previous paper [11] are now called symmetric D^n*-forms* with values in ξ in distinction to symmetric D_n-forms with values in ξ. Corresponding to our modification of the notion of D^{n+1}-tangential in the preceding subsection, a D^n-form ω is required to satisfy

$$\omega((d_1, \dots, d_n) \in D^n \mapsto \gamma(ed_1, \dots, ed_n) \in M) = 0$$

for any $e \in D_{n-1}$ and any $\gamma \in \mathbf{T}^{D(n)_{n-1}}_{\pi(x)}(M)$ besides conditions (0.4.1)–(0.4.3) of our previous paper [11]. The spaces $\mathbb{S}^{D^n}_x(\pi; \xi)$ and $\mathbb{S}^{D^n}(\pi; \xi)$ shall be such as are expected.

3. The second approach to jet bundles

We have already discussed the second approach to jet bundles in terms of infinitesimal spaces D_n's in [12]. What has remained there to be discussed is the affine bundle theorem as a D_n-variant of Theorem 1.9 in [11]. To begin with, let us recall the fundamental concepts of our second approach.

Definition 9. *A D_n-pseudotangential for π at $x \in E$ is a mapping $\mathfrak{f} : \mathbf{T}^{D_n}_{\pi(x)}(M) \to \mathbf{T}^{D_n}_x(E)$ subject to the following two conditions:*

1. *We have*
$$\pi \circ (\mathfrak{f}(\gamma)) = \gamma \tag{6}$$
for $\gamma \in \mathbf{T}^{D_n}_{\pi(x)}(M)$.
2. *We have*
$$\mathfrak{f}(\alpha\gamma) = \alpha\mathfrak{f}(\gamma) \tag{7}$$
for any $\gamma \in \mathbf{T}^{D_n}_{\pi(x)}(M)$ and any $\alpha \in \mathbb{R}$.

We denote by $\hat{\mathbb{J}}^{D_n}_x(\pi)$ the totality of D_n-pseudotangentials for π at $x \in E$. We denote by $\hat{\mathbb{J}}^{D_n}(\pi)$ the totality of D_n-pseudotangentials for π, i.e., the set-theoretic union of $\hat{\mathbb{J}}^{D_n}_x(\pi)$'s for all $x \in E$. We have the canonical projection $\hat{\pi}_{n+1,n} : \hat{\mathbb{J}}^{D_{n+1}}(\pi) \to \hat{\mathbb{J}}^{D_n}(\pi)$ such that

$$\mathfrak{f}(\mathbf{i}_d(\gamma)) = \mathbf{i}_d(\hat{\pi}_{n+1,n}(\mathfrak{f})(\gamma))$$

for any $\mathfrak{f} \in \hat{\mathbb{J}}^{D_{n+1}}(\pi)$, any $d \in D_n$ and any $\gamma \in \mathbf{T}^{D_n}(M)$, for which the reader is referred to Proposition 15 of our previous paper [12]. By assigning $\pi(x) \in M$ to each D^n-pseudotangential for $\pi : E \to M$ at $x \in E$ we have the canonical projection $\hat{\pi}_n : \hat{\mathbb{J}}^{D_n}(\pi) \to M$, which is easily seen to be a vector bundle providing that π is already a vector bundle. Note that $\hat{\pi}_n \circ \hat{\pi}_{n+1,n} = \hat{\pi}_{n+1}$. For any natural numbers n, m with $m \leq n$, we define $\hat{\pi}_{n,m} : \hat{\mathbb{J}}^{D_n}(\pi) \to \hat{\mathbb{J}}^{D_m}(\pi)$ to be the composition $\hat{\pi}_{m+1,m} \circ \cdots \circ \hat{\pi}_{n,n-1}$.

Definition 10. *The notion of a D_n-tangential for π at $x \in E$ is defined inductively on n. The notion of a D_0-tangential for π at $x \in E$ and that of a D_1-tangential for π at $x \in E$ shall be identical with that of a D_0-pseudotangential for π at $x \in E$ and that of a D_1-pseudotangential for π at $x \in E$ respectively. Now we proceed by induction on n. A D_{n+1}-pseudotangential $\mathfrak{f} : \mathbf{T}^{D_{n+1}}_{\pi(x)}(M) \to \mathbf{T}^{D_{n+1}}_x(E)$ for π at $x \in E$ is called a D_{n+1}-tangential for π at x if it acquiesces in the following two conditions:*

1. *$\hat{\pi}_{n+1,n}(\mathfrak{f})$ is a D_n-tangential for π at x.*
2. *For any simple polynomial ρ of $d \in D_{n+1}$ with* $\dim_n\rho = l \leq n$ *and any $\gamma \in \mathbf{T}^{D_l}_{\pi(x)}(M)$, we have $\mathfrak{f}(\gamma \circ \rho) = (\hat{\pi}_{n+1,l}(\mathfrak{f})(\gamma)) \circ \rho$.*

We denote by $\mathbb{J}^{D_n}_x(\pi)$ the totality of D_n-tangentials for π at x. We denote by $\mathbb{J}^{D_n}_x(\pi)$ the totality of D_n-tangentials for π, namely, the set-theoretic union of $\mathbb{J}^{D_n}_x(\pi)$'s for all $x \in E$. By the very definition of a D_n-tangential for π, the projection $\hat{\pi}_{n+1,n} : \hat{\mathbb{J}}^{D_{n+1}}(\pi) \to \hat{\mathbb{J}}^{D_n}(\pi)$ is naturally restricted to a mapping $\pi_{n+1,n} : \mathbb{J}^{D_{n+1}}(\pi) \to \mathbb{J}^{D_n}(\pi)$. Similarly for $\pi_n : \mathbb{J}^{D_n}(\pi) \to M$ and $\pi_{n,m} : \mathbb{J}^{D_n}(\pi) \to \mathbb{J}^{D_m}(\pi)$ with $m \leq n$. We can see easily that $\pi_n : \mathbb{J}^{D_n}(\pi) \to M$ is naturally a vector bundle providing that π is already a vector bundle.

Now we will establish a D_n-variant of Theorem 1.9 of [11]. Let us begin with a D_n-variant of Proposition 1.7 in [11].

Proposition 11. *Let* $\mathfrak{f}^+$, $\mathfrak{f}^- \in \mathbb{J}_x^{D_{n+1}}(\pi)$ *with* $\pi_{n+1,n}(\mathfrak{f}^+) = \pi_{n+1,n}(\mathfrak{f}^-)$. *Then the assignment* $\gamma \in \mathbf{T}_{\pi(x)}^{D_{n+1}}(M) \longmapsto \mathfrak{f}^+(\gamma)\dot{-}\mathfrak{f}^-(\gamma)$ *belongs to* $\mathbb{S}_{\pi(x)}^{D_{n+1}}(\pi; v_\pi)$.

Proof. Since we have

$$\begin{aligned}
&\pi_*(\mathfrak{f}^+(\gamma)\dot{-}\mathfrak{f}^-(\gamma)) \\
&= \pi_*(\mathfrak{f}^+(\gamma))\dot{-}\pi_*(\mathfrak{f}^-(\gamma)) \\
&\text{[by Proposition 4]} \\
&= 0
\end{aligned}$$

$\mathfrak{f}^+(\gamma)\dot{-}\mathfrak{f}^-(\gamma)$ belongs in $\mathbf{V}_x^1(\pi)$. For any $\alpha \in \mathbb{R}$ we have

$$\begin{aligned}
&\mathfrak{f}^+(\alpha\gamma)\dot{-}\mathfrak{f}^-(\alpha\gamma) \\
&= \alpha\mathfrak{f}^+(\gamma)\dot{-}\alpha\mathfrak{f}^-(\gamma) \\
&= \alpha^{n+1}(\mathfrak{f}^+(\gamma)\dot{-}\mathfrak{f}^-(\gamma)) \\
&\text{[by (1)]}
\end{aligned}$$

which implies that the assignment $\gamma \in \mathbf{T}_{\pi(x)}^{D_{n+1}}(M) \longmapsto \mathfrak{f}^+(\gamma)\dot{-}\mathfrak{f}^-(\gamma) \in \mathbf{V}_x^1(\pi)$ abides by (4). It remains to show that the assignment $\gamma \in \mathbf{T}_{\pi(x)}^{n+1}(M) \longmapsto \mathfrak{f}^+(\gamma)\dot{-}\mathfrak{f}^-(\gamma) \in \mathbf{V}_x^1(\pi)$ abides by (5), which follows directly from the second condition in Definition 10 and the assumption that $\pi_{n+1,n}(\mathfrak{f}^+) = \pi_{n+1,n}(\mathfrak{f}^-)$. □

Now we will establish a D_n-variant of Proposition 1.8 in [11].

Proposition 12. *Let* $\mathfrak{f} \in \mathbb{J}_x^{D_{n+1}}(\pi)$ *and* $\omega \in \mathbb{S}_{\pi(x)}^{D_{n+1}}(\pi; v_\pi)$. *Then the assignment* $\gamma \in \mathbf{T}_{\pi(x)}^{D_{n+1}}(M) \longmapsto \omega(\gamma)\dot{+}\mathfrak{f}(\gamma)$ *belongs to* $\mathbb{J}_x^{D_{n+1}}(\pi)$.

Proof. Since we have

$$\begin{aligned}
&\pi_*(\omega(\gamma)\dot{+}\mathfrak{f}(\gamma)) \\
&= \pi_*(\omega(\gamma))\dot{+}\pi_*(\mathfrak{f}(\gamma)) \\
&\text{[by Proposition 7]} \\
&= \gamma
\end{aligned}$$

the assignment $\gamma \in \mathbf{T}_{\pi(x)}^{D_{n+1}}(M) \longmapsto \omega(\gamma)\dot{+}\mathfrak{f}(\gamma)$ stands to (6). For any $\alpha \in \mathbb{R}$ we have

$$\begin{aligned}
&\omega(\alpha\gamma)\dot{+}\mathfrak{f}(\alpha\gamma) \\
&= \alpha^{n+1}\omega(\gamma)\dot{+}\alpha\mathfrak{f}(\gamma) \\
&\text{[by (3)]} \\
&= \alpha(\omega(\gamma)\dot{+}\mathfrak{f}(\gamma))
\end{aligned}$$

so that the assignment $\gamma \in \mathbf{T}_{\pi(x)}^{D_{n+1}}(M) \longmapsto \omega(\gamma)\dot{+}\mathfrak{f}(\gamma)$ stands to (7). That the assignment $\gamma \in \mathbf{T}_{\pi(x)}^{D_{n+1}}(M) \longmapsto \omega(\gamma)\dot{+}\mathfrak{f}(\gamma)$ stands to the first condition of Definition 10 follows from the simple fact that the image of the assignment under $\hat{\pi}_{n+1,n}$

coincides with $\hat{\pi}_{n+1,n}(\mathfrak{f})$, which is consequent upon (5). It remains to show that the assignment $\gamma \in \mathbf{T}^{D_{n+1}}_{\pi(x)}(M) \longmapsto \omega(\gamma)\dot{+}\mathfrak{f}(\gamma)$ abides by the second condition of Definition 10, which follows directly from (5) and the assumption that $\mathfrak{f}$ satisfies the second condition of Definition 10. □

For any $\mathfrak{f}^+, \mathfrak{f}^- \in \mathbb{J}^{D_{n+1}}(\pi)$ with $\pi_{n+1,n}(\mathfrak{f}^+) = \pi_{n+1,n}(\mathfrak{f}^-)$, we define $\mathfrak{f}^+\dot{-}\mathfrak{f}^- \in \mathbb{S}^{D_{n+1}}(M; v_\pi)$ to be

$$(\mathfrak{f}^+\dot{-}\mathfrak{f}^-)(\gamma) = \mathfrak{f}^+(\gamma)\dot{-}\mathfrak{f}^-(\gamma)$$

for any $\gamma \in \mathbf{T}^{D_{n+1}}_{\pi(x)}(M)$. For any $\mathfrak{f} \in \mathbb{J}^{D_{n+1}}_x(\pi)$ and any $\omega \in \mathbb{S}^{D_{n+1}}_{\pi(x)}(\pi; v_\pi)$ we define $\omega\dot{+}\mathfrak{f} \in \mathbb{J}^{D_{n+1}}_x(\pi)$ to be

$$(\omega\dot{+}\mathfrak{f})(\gamma) = \omega(\gamma)\dot{+}\mathfrak{f}(\gamma)$$

for any $\gamma \in \mathbf{T}^{D_{n+1}}_{\pi(x)}(M)$.

With these two operations we have the following microlinear generalization of the classical affine bundle theorem (cf. Theorem 6.2.9 of Saunders [13]), which has been concerned merely with the finite-dimensional setting.

Theorem 13. *The bundle $\pi_{n+1,n} : \mathbb{J}^{D_{n+1}}(\pi) \to \mathbb{J}^{D_n}(\pi)$ is an affine bundle over the vector bundle $\mathbb{S}^{D_{n+1}}(\pi; v_\pi)\underset{E}{\times}\mathbb{J}^{D_n}(\pi) \to \mathbb{J}^{D_n}(\pi)$.*

Proof. This follows simply from Theorem 8. □

4. The comparison without coordinates

The relationship between $\mathbb{J}^{D^n}(\pi)$ and $\mathbb{J}^{D_n}(\pi)$ without any reference to the affine bundle structures stated in Theorem 1.9 of our [11] and Theorem 13 of this paper has already been discussed in our [12]. We remind the reader that all the results of our previous paper [12] are valid with due modifications, though our definition of a simple polynomial and that of a D^n-tangential have been modified.

Let f be a D^n-tangential for $\pi : E \to M$ at $x \in E$. We have a function $\Phi_n(\mathsf{f}) : \mathbf{T}^{D_n}_{\pi(x)}(M) \to \mathbf{T}^{D_n}_x(E)$, which is characterized by

$$\mathsf{f}(\gamma_{D^n}) = \Phi_n(\mathsf{f})(\gamma)_{D^n}$$

for any $\gamma \in \mathbf{T}^{D_n}_{\pi(x)}(M)$. For the existence and uniqueness of such a function, the reader is referred to Lemma 18 of our previous [12], from which we quote two crucial results.

Theorem 14. *For any $\mathsf{f} \in \mathbb{J}^{D^n}_x(\pi)$, we have $\Phi_n(\mathsf{f}) \in \mathbb{J}^{D_n}_x(\pi)$, so that we have a canonical mapping $\Phi_n : \mathbb{J}^{D^n}(\pi) \to \mathbb{J}^{D_n}(\pi)$.*

Proposition 15. *The diagram*

$$\begin{array}{ccc}
\mathbb{J}^{D^{n+1}}_x(\pi) & \xrightarrow{\Phi_{n+1}} & \mathbb{J}^{D_{n+1}}_x(\pi) \\
{\scriptstyle \pi_{n+1,n}}\downarrow & & \downarrow{\scriptstyle \pi_{n+1,n}} \\
\mathbb{J}^{D^n}_x(\pi) & \xrightarrow[\Phi_n]{} & \mathbb{J}^{D_n}_x(\pi)
\end{array}$$

is commutative.

Now we are in a position to investigate the relationship between the affine bundles discussed in Theorem 1.9 of our [11] and Theorem 13 of this paper. Let us begin with

Lemma 16. *Let $\gamma^{\pm} \in \mathbf{T}_x^{D_{n+1}}(E)$ with $\gamma^+ \mid_{D_n} = \gamma^- \mid_{D_n}$. Then*

$$\gamma^+_{D^{n+1}} \mid_{D(n+1)_n} = \gamma^-_{D^{n+1}} \mid_{D(n+1)_n}, \tag{8}$$

and we have

$$\gamma^+ \dot{-} \gamma^- = \gamma^+_{D^{n+1}} \dot{-} \gamma^-_{D^{n+1}}. \tag{9}$$

Proof. Since $D(n+1)_n \subseteq D^{n+1}$, the first statement follows from the following simple calculation:

$$\begin{aligned} &\gamma^+_{D^{n+1}} \mid_{D(n+1)_n} \\ &\quad = (\gamma^+ \mid_{D_n})_{D(n+1)_n} \quad \text{[by Lemma 1]} \\ &\quad = (\gamma^- \mid_{D_n})_{D(n+1)_n} \\ &\quad = \gamma^-_{D^{n+1}} \mid_{D(n+1)_n} \quad \text{[by Lemma 1]}. \end{aligned} \tag{10}$$

The second statement follows simply from the following commutative diagram where + stands for addition of components.

$$\begin{tikzcd}
& D^{n+1} \arrow[rr, "+"] \arrow[dd, "\psi"] & & D_{n+1} \arrow[dd, "\psi"] \\
D(n+1)_n \arrow[ur, "i"] \arrow[rr, "+"] \arrow[dd, "i"'] & & D_n \arrow[ur, "i"'] \arrow[dd, "i"] & \\
& D^{n+1} \oplus D \arrow[rr, "+ \oplus \mathrm{id}"] & & D_{n+1} \oplus D \\
D^{n+1} \arrow[ur, "\varphi"] \arrow[rr, "+"'] & & D_{n+1} \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

□

Lemma 17. *Let $t \in \mathbf{T}_x^1(E)$ and $\gamma \in \mathbf{T}_x^{D_n}(E)$. Then we have*

$$(t \dot{+} \gamma)_{D^n} = t \dot{+} \gamma_{D^n}. \tag{11}$$

Proof. This follows simply from the following commutative diagram.

$$\begin{tikzcd}
& D \arrow[rr, "\mathrm{id}"] \arrow[dd, "\varepsilon"] & & D \arrow[dd, "\varepsilon"] \\
1 \arrow[ur, "i"] \arrow[rr] \arrow[dd, "i"'] & & 1 \arrow[ur, "i"'] \arrow[dd, "i"] & \\
& D^{n+1} \oplus D \arrow[rr, "+ \oplus \mathrm{id}"] & & D_{n+1} \oplus D \\
D^{n+1} \arrow[ur, "\varphi"] \arrow[rr, "+"'] & & D_{n+1} \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

□

Now we are ready to state the main result of this section.

Theorem 18. *We have the following.*

1. *For any* $\mathsf{f}^+, \mathsf{f}^- \in \mathbf{J}_x^n(\pi)$ *and any* $\gamma \in \mathbf{T}_{\pi(x)}^{D_n}(M)$, *we have*

$$\Phi_n(\mathsf{f}^+)(\gamma)\dot{-}\Phi_n(\mathsf{f}^-)(\gamma) = \mathsf{f}^+(\gamma_{D^n})\dot{-}\mathsf{f}^-(\gamma_{D^n}). \tag{12}$$

2. *For any* $\mathsf{f} \in \mathbf{J}_x^n(\pi)$, *any* $t \in \mathbf{T}_{\pi(x)}^{D}(M)$ *and any* $\gamma \in \mathbf{T}_{\pi(x)}^{D_n}(M)$, *we have*

$$(t\dot{+}\Phi_n(\mathsf{f})(\gamma))_{D^n} = t\dot{+}\mathsf{f}(\gamma_{D^n}). \tag{13}$$

Proof.

1. Since $\mathsf{f}^\pm(\gamma_{D^n}) = (\Phi_n(\mathsf{f}^\pm)(\gamma))_{D^n}$, we have

$$\begin{aligned} &\mathsf{f}^+(\gamma_{D^n})\dot{-}\mathsf{f}^-(\gamma_{D^n}) \\ &= (\Phi_n(\mathsf{f}^+)(\gamma))_{D^n}\dot{-}(\Phi_n(\mathsf{f}^-)(\gamma))_{D^n} \\ &= \Phi_n(\mathsf{f}^+)(\gamma)\dot{-}\Phi_n(\mathsf{f}^-)(\gamma) \quad \text{[by Lemma 16].} \end{aligned} \tag{14}$$

2. Since $\mathsf{f}(\gamma_{D^n}) = (\Phi_n(\mathsf{f})(\gamma))_{D^n}$, we have

$$\begin{aligned} &t\dot{+}\mathsf{f}(\gamma_{D^n}) \\ &= t\dot{+}(\Phi_n(\mathsf{f})(\gamma))_{D^n} \\ &= (t\dot{+}\Phi_n(\mathsf{f})(\gamma))_{D^n} \quad \text{[by Lemma 17].} \end{aligned} \tag{15}$$

□

Now we would like to discuss the relationship between $\mathbb{S}^{D^n}(\pi; v_\pi)$ and $\mathbb{S}^{D_n}(\pi; v_\pi)$.

Proposition 19. *For any* $\omega \in \mathbb{S}_x^{D^n}(\pi; v_\pi)$, *the mapping* $\gamma \in \mathbf{T}_x^{D_n}(M) \mapsto \omega(\gamma_{D^n})$, *denoted by* $\Psi_n(\omega)$, *belongs to* $\mathbb{S}_x^{D_n}(\pi; v_\pi)$, *thereby giving rise to a function* $\Psi_n : \mathbb{S}^{D^n}(\pi; v_\pi) \to \mathbb{S}^{D_n}(\pi; v_\pi)$.

Proof. For $n = 0, 1$, the statement is trivial. For any $\omega \in \mathbb{S}_x^{D^{n+1}}(M; v_\pi)$, there exist $\mathsf{f}^+, \mathsf{f}^- \in \mathbf{J}_x^{n+1}(\pi)$ such that $\pi_{n+1,n}(\mathsf{f}^+) = \pi_{n+1,n}(\mathsf{f}^-)$ and $\omega = \mathsf{f}^+\dot{-}\mathsf{f}^-$. Then we have the following:

1. Let $\alpha \in \mathbb{R}$ and $\gamma \in \mathbf{T}_x^{D_{n+1}}(M)$. Then we have

$$\begin{aligned} &\omega((\alpha\gamma)_{D^{n+1}}) \\ &= \mathsf{f}^+((\alpha\gamma)_{D^{n+1}})\dot{-}\mathsf{f}^-((\alpha\gamma)_{D^{n+1}}) \\ &= \Phi_{n+1}(\mathsf{f}^+)(\alpha\gamma)\dot{-}\Phi_{n+1}(\mathsf{f}^-)(\alpha\gamma) \quad \text{[by Theorem 18]} \\ &= \alpha(\Phi_{n+1}(\mathsf{f}^+)(\gamma))\dot{-}\alpha(\Phi_{n+1}(\mathsf{f}^-)(\gamma)) \\ &= \alpha^{n+1}(\Phi_{n+1}(\mathsf{f}^+)(\gamma)\dot{-}\Phi_{n+1}(\mathsf{f}^-)(\gamma)) \\ &= \alpha^{n+1}(\mathsf{f}^+(\gamma_{D^{n+1}})\dot{-}\mathsf{f}^-(\gamma_{D^{n+1}})) \quad \text{[by Theorem 18]} \\ &= \alpha^{n+1}\omega(\gamma_{D^{n+1}}) \end{aligned}$$

   so that $\Psi_{n+1}(\omega)$ abides by (4).

2. Let ρ be a simple polynomial of $d \in D_{n+1}$ and $\gamma \in \mathbf{T}^{D_l}_{\pi(x)}(M)$ with $\dim_{n+1}\rho = l < n+1$, we have

$$\begin{aligned}
&\omega((\gamma\circ\rho)_{D^{n+1}}) \\
&= \mathsf{f}^+((\gamma\circ\rho)_{D^{n+1}}) \dot{-} \mathsf{f}^-((\gamma\circ\rho)_{D^{n+1}}) \\
&= \Phi_{n+1}(\mathsf{f}^+)(\gamma\circ\rho) \dot{-} \Phi_{n+1}(\mathsf{f}^-)(\gamma\circ\rho) \quad \text{[by Theorem 18]} \\
&= (\pi_{n+1,l}(\Phi_{n+1}(\mathsf{f}^+))(\gamma))\circ\rho \dot{-} (\pi_{n+1,l}(\Phi_{n+1}(\mathsf{f}^-))(\gamma))\circ\rho \\
&= (\Phi_l(\pi_{n+1,l}(\mathsf{f}^+))(\gamma))\circ\rho \dot{-} (\Phi_l(\pi_{n+1,l}(\mathsf{f}^-))(\gamma))\circ\rho \quad \text{[by Proposition 15]} \\
&= 0,
\end{aligned}$$

so that $\Psi_{n+1}(\omega)$ abides by (5).

□

Let us fix our terminology. Given an affine bundle $\pi_1 : E_1 \to M_1$ over a vector bundle $\xi_1 : P_1 \to M_1$ and another affine bundle $\pi_2 : E_2 \to M_2$ over another vector bundle $\xi_2 : P_2 \to M_2$, a triple (f, g, h) of mappings $f : M_1 \to M_2$, $g : E_1 \to E_2$ and $h : P_1 \to P_2$ is called a *morphism of affine bundles* from the affine bundle $\pi_1 : E_1 \to M_1$ over the vector bundle $\xi_1 : P_1 \to M_1$ to the affine bundle $\pi_2 : E_2 \to E_2$ over the vector bundle $\xi_2 : P_2 \to M_2$ provided that they satisfy the following conditions:

1. (f, g) is a morphism of bundles from π_1 to π_2. In other words, the following diagram is commutative:
$$\begin{array}{ccc} E_1 & \xrightarrow{g} & E_2 \\ \pi_1\downarrow & & \downarrow\pi_2 \\ M_1 & \xrightarrow[f]{} & E_2 \end{array}$$
2. (f, h) is a morphism of bundles from ξ_1 to ξ_2. In other words, the following diagram is commutative:
$$\begin{array}{ccc} P_1 & \xrightarrow{h} & P_2 \\ \xi_1\downarrow & & \downarrow\xi_2 \\ M_1 & \xrightarrow[f]{} & E_2 \end{array}$$
3. For any $x \in M_1$, $(g|_{E_{1,x}}, h|_{P_{1,x}})$ is a morphism of affine spaces from $(E_{1,x}, P_{1,x})$ to $(E_{2,x}, P_{2,x})$.

Using this terminology, we can summarize Theorem 18 succinctly as follows:

Theorem 20. *The triple* $(\Phi_n, \Phi_{n+1}, \Psi_{n+1}\times\Phi_n)$ *of mappings is a morphism of affine bundles from the affine bundle* $\pi_{n+1,n} : \mathbb{J}^{D^{n+1}}(\pi) \to \mathbb{J}^{D^n}(\pi)$ *over the vector bundle* $\mathbb{S}^{D^{n+1}}(\pi; \upsilon_\pi)\underset{E}{\times}\mathbb{J}^{D^n}(\pi) \to \mathbb{J}^{D^n}(\pi)$ *in Theorem* 1.9 *of* [11] *to the affine bundle* $\pi_{n+1,n} : \mathbb{J}^{D_{n+1}}(\pi) \to \mathbb{J}^{D_n}(\pi)$ *over the vector bundle* $\mathbb{S}^{D_{n+1}}(\pi; \upsilon_\pi)\underset{E}{\times}\mathbb{J}^{D_n}(\pi) \to \mathbb{J}^{D_n}(\pi)$ *in Theorem* 13.

5. The comparison with coordinates

Throughout this section we assume that the bundle $\pi : E \to M$ is a formal bundle (cf. [10]), so that we can assume without any loss of generality that $E = \mathbb{R}^{p+q}$, $M = \mathbb{R}^p$ and π is the canonical projection of $\mathbb{R}^{p+q}$ to the first p coordinates, for our considerations to follow are always infinitesimal. We will let i with or without subscripts range over natural numbers between 1 and p (including endpoints). By the general Kock axiom any $\gamma \in \mathbf{T}^{D^n}(M)$ is of the form

$$(d_1, \dots, d_n) \in D^n \longmapsto (x^i) + \Sigma_{r=1}^n \Sigma_{1 \le k_1 < \cdots < k_r \le n} d_{k_1} \cdots d_{k_r} (a^i_{k_1 \cdots k_r}) \in \mathbb{R}^p,$$

while any $\gamma \in \mathbf{T}^{D_n}(M)$ is of the form

$$d \in D_n \longmapsto (x^i) + \Sigma_{r=1}^n d^r (a^i_{\mathbf{r}}) \in \mathbb{R}^p.$$

The principal objective in this section is to show that

Theorem 21. *For any natural number n, $\Phi_n : \mathbb{J}^{D^n}(\pi) \to \mathbb{J}^{D_n}(\pi)$ and $\Psi_n : \mathbb{S}^{D^n}(\pi; v_\pi) \to \mathbb{S}^{D_n}(\pi; v_\pi)$ are bijective.*

We proceed by induction on n. For $n = 0, 1$, the theorem holds trivially. We have shown in [12] that Φ_n is injective for any natural number n, so that Ψ_n is injective for any natural number n. With due regard to Theorem 20, it suffices to show that Ψ_n is bijective for any natural number n, for which we can and will use dimension counting techniques. Let us remark the following two plain propositions, which may belong to the folklore.

Proposition 22. *Let $x = (x^i) \in M$ and V a finite-dimensional $\mathbb{R}$-module. Let $\omega : \mathbf{T}_x^{D^n}(M) \to V$ be a function acquiescent in conditions* (0.4.1) *and* (0.4.2) *of our previous* [11]. *Then ω is of the following form:*

$$\begin{aligned} &\omega((d_1, \dots, d_n) \in D^n \longmapsto (x^i) + \Sigma_{r=1}^n \Sigma_{1 \le k_1 < \cdots < k_r \le n} d_{k_1} \cdots d_{k_r} (a^i_{k_1 \cdots k_r}) \in \mathbb{R}^p) \\ &\qquad = \Sigma \Omega^n_{\mathbf{J}_1, \dots, \mathbf{J}_s}((a^{i_{\mathbf{J}_1}}_{\mathbf{J}_1}), \dots, (a^{i_{\mathbf{J}_s}}_{\mathbf{J}_s})), \end{aligned}$$

where the last Σ is taken over all partitions of the set $\{1, \dots, n\}$ into nonempty subsets $\{\mathbf{J}_1, \dots, \mathbf{J}_s\}$, $\Omega^n_{\mathbf{J}_1, \dots, \mathbf{J}_s} : (\mathbb{R}^p)^s \to V$ is a symmetric s-linear mapping, and $a^{i_{\mathbf{J}}}_{\mathbf{J}}$ denotes $a^{i_{\mathbf{J}}}_{k_1 \cdots k_r}$ for $\mathbf{J} = \{k_1, \dots, k_r\}$ with $k_1 < \cdots < k_r$.

Proof. By the same token as in the proof of Proposition 11 (Section 1.2) of [4]. □

Proposition 23. *Let $x \in M$ and V a finite-dimensional $\mathbb{R}$-module. Let $\omega : \mathbf{T}_x^{D_n}(M) \to V$ be a function acquiescent in condition* (4). *Then ω is of the following form:*

$$\begin{aligned} &\omega(d \in D_n \longmapsto (x^i) + \Sigma_{r=1}^n d^r (a^i_{\mathbf{r}}) \in \mathbb{R}^p) \\ &\qquad = \Sigma(\Omega^n_{r_1, \dots, r_k}((a^{i_1}_{\mathbf{r}_1}), \dots, (a^{i_k}_{\mathbf{r}_k}))), \end{aligned}$$

where $\Omega^n_{r_1, \dots, r_k} : (\mathbb{R}^p)^k \to V$ is a symmetric k-linear mapping, and the last Σ is taken over all partitions of the positive integer n into positive integers $r_1, \dots, r_k$ (so that $n = r_1 + \cdots + r_k$) with $r_1 \le \cdots \le r_k$.

Proof. By the same token as in the proof of Proposition 11 (Section 1.2) of [4]. □

Proposition 24. *For any $x \in M$, $\mathbb{R}$-modules $\mathbb{S}_x^{D^n}(\pi; v_\pi)$ and $\mathbb{S}_x^{D_n}(\pi; v_\pi)$ are of the same dimension $q\binom{p+n-1}{n}$, so that $\Psi_n : \mathbb{S}^{D^n}(\pi; v_\pi) \to \mathbb{S}^{D_n}(\pi; v_\pi)$ is bijective.*

Proof. Taking into consideration the condition (0.4.3) of our previous [11] in Proposition 22, we can conclude that the $\mathbb{R}$-module $\mathbb{S}_x^{D^n}(\pi; v_\pi)$ is of dimension $q\binom{p+n-1}{n}$, for $\Omega^n_{\mathbf{J}_1,\dots,\mathbf{J}_s}$ is zero unless $\{\mathbf{J}_1, \dots, \mathbf{J}_s\} = \{\{1\}, \dots, \{n\}\}$. Similarly, taking into consideration the condition (5) in Proposition 23, we can conclude that the $\mathbb{R}$-module $\mathbb{S}^{D_n}(\pi; v_\pi)$ is of dimension $q\binom{p+n-1}{n}$, for $\Omega^n_{r_1,\dots,r_k}$ is zero except $\Omega^n_{1,\dots,1}$. □

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