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A Staircase Illumination Theorem for Orthogonal Polygons

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Abstract. Let S be a simply connected orthogonal polygon in the plane, and let T be a horizontal (or vertical) segment such that $T' \cap S$ is connected for every translate T' of T. If every two points of S see via staircase paths a common translate of T, then there is a translate of T seen via staircase paths by every point of S. That is, some translate of T is a staircase illuminator for S. Clearly the number two is best possible. The result fails without the requirement that each set $T' \cap S$ be connected.

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1. Introduction

We begin with some definitions from [7]. For points x and y in the plane, $[x, y]$ will denote the corresponding line segment. Let S be a nonempty subset of the plane. Set S is called an *orthogonal polygon* (rectilinear polygon) if and only if S is a connected union of finitely many convex polygons (possibly degenerate) whose edges are parallel to the coordinate axes. Let λ denote a simple polygonal path in the plane. Path λ is an *orthogonal* path if and only if its edges $[v_{i-1}, v_i]$, $1 \le i \le n$, are parallel to the coordinate axes. The orthogonal path λ is called an *$x - y$ path* (or a $y - x$ path) if and only if λ lies in S and contains points x and y; λ is an *$x - y$ geodesic* if and only if λ is an $x - y$ path of minimal length in S. (Clearly an $x - y$ geodesic need not be unique.) Subset S' of S is *geodesically convex* if

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and only if for each pair of points x, y in S', S' contains every $x - y$ geodesic in S. Set S' is *horizontally convex* if and only if for each pair x, y in S' with $[x, y]$ horizontal, it follows that $[x, y] \subseteq S'$. *Vertically convex* is defined analogously. Finally, S' is *orthogonally convex* if and only if S' is both horizontally convex and vertically convex.

The path λ is a *staircase path* if and only if the associated vectors $[v_{i-1}, v_i]$ alternate in direction. That is, for an appropriate labeling, for i odd the vectors $\overrightarrow{v_{i-1}v_i}$ have the same horizontal direction, and for i even the vectors $\overrightarrow{v_{i-1}v_i}$ have the same vertical direction. We say that point v_i is *north, south, east,* or *west* of v_{i-1} according to the direction of vector $\overrightarrow{v_{i-1}v_i}$. Similarly, we use the terms *north, south, east, west, northeast, northwest, southeast, southwest* to describe the relative position of points. For points x and y in set S, we say *x sees y* (x is *visible* from y) *via staircase paths* if and only if there is a staircase path in S which contains both x and y. For set T in the plane with $T \cap S \neq \phi$ and point x in S, x *sees T via staircase paths* (*T illumines x via staircase paths*) if and only if x sees via staircase paths in S at least one point of T. Set T is a *staircase illuminator* for set S if and only if T illumines via staircase paths every point of S. Finally, set S is *starshaped via staircase paths* if and only if for some point p in S, p sees via staircase paths each point of S.

Many results in convexity that involve the usual idea of visibility via straight line segments have interesting analogues that use the notion of visibility via staircase paths. (See [15], [2], [3], [6], [8].) For instance, the familiar Krasnosel'skij theorem [12] in the plane states that for S nonempty and compact in $\mathbb{R}^2$, S is starshaped via segments if and only if every three points of S are visible (via segments in S) from a common point. In the staircase analogue [3], for S a simply connected orthogonal polygon in the plane, S is starshaped via staircase paths if and only if every two points of S are visible (via staircase paths in S) from a common point. Notice that in the staircase version, the Helly number three is reduced to two.

In this paper, we consider a variation of the starshaped set problem. However, instead of showing that a set S is starshaped, the idea is to show that S has a convex illuminator. Some related results using segment visibility appear in a paper by Bezdek, Bezdek, and Bisztriczky [1]. Among their results is the following theorem: For S a smooth domain in $\mathbb{R}^2$, if every three points of S are illumined by some translate in S of segment T, then S contains an illuminator which is a translate of T. Analogues are established in [4] for S compact in $\mathbb{R}^2$, when every three points of S are illumined by a translate of compact convex set T, and in [5] for S a finite union of boxes in $\mathbb{R}^d$, when every two boundary points of S are illumined by a translate of box T. Here we ask if a corresponding result holds for orthogonal polygons, using visibility via staircase paths rather than visibility via segments.

Concerning notation, throughout the paper, *int S*, *cl S*, and *bdry S* will denote the interior, the closure, and the boundary, respectively, for set S. For points x and y, $dist(x, y)$ will be the distance from x to y. If λ is an ordered path containing x and y, $\lambda(x, y)$ will represent a subpath of λ from x to y. When x and y are distinct,

$L(x, y)$ will denote their corresponding line. The reader may refer to Valentine [16], to Lay [13], to Danzer, Grünbaum, Klee [9], and to Eckhoff [10] for discussions concerning Helly-type theorems, visibility via segments, and starshaped sets.

2. The results

We will establish the following theorem.

Theorem 1. *Let S be a simply connected orthogonal polygon in the plane, and let T be a horizontal (or vertical) segment such that $T' \cap S$ is connected for every translate T' of T. If every two points of S see via staircase paths a common translate of T, then there is a translate of T seen via staircase paths by every point of S. That is, some translate of T is a staircase illuminator for S. Clearly the number two is best possible.*

Proof. If T is a singleton set, then the result is an immediate consequence of [3, Corollary 1]. Hence we assume that T is nondegenerate. For convenience of notation and without loss of generality, throughout the proof we assume that T is a closed segment with one endpoint at the origin and the other endpoint on the positive x axis. To each point x in S, associate sets

$$V_x = \{y : x \text{ sees } y \text{ via staircase paths}\}, \text{ and}$$
$$A_x = \{y : x \text{ sees via staircase paths some point of } y + T\}.$$

We will show that each set A_x is simply connected and compact, every two of these sets have a path connected intersection, and every three of these sets have a nonempty intersection. Then the result will follow from this version of Molnár's theorem [14] by Karimov, Repovš, and Željko [11, Theorem 2]: Let $\mathcal{F}$ be a family of simply connected compact sets in the plane. If every two members of $\mathcal{F}$ have a path connected intersection and every three members have a nonempty intersection, then $\cap\{F : F \text{ in } \mathcal{F}\} \neq \phi$.

We will also make use of the following result from [7, Theorem 1]: If S is a simply connected orthogonal polygon in the plane, then for each point t of S, the corresponding set V_t is geodesically convex.

To make the argument easier to follow, we separate it into three parts.

Part 1. We show that each set A_x is compact and simply connected. We begin with the observation that each set A_x is an orthogonal polygon. It is easy to show that $A_x = V_x - T$. By an argument like the one in [2, Lemma 2], set V_x is a finite (and connected) union of rectangular regions, hence an orthogonal polygon. Since T is a closed segment, $A_x = V_x - T$ is an orthogonal polygon as well, hence compact and connected.

Next we prove that each set A_x is simply connected. Let λ denote a simple closed curve in A_x, and let p be a point in the (open) bounded region determined by λ. We will show that x sees via staircase paths a point of $p + T$ and hence $p \in A_x$. We consider cases according to the relative positions of x and p.

Case 1. In case x is west of (possibly on) the vertical line at p, without loss of generality assume that x is southwest of p. Select on λ points n and e, north and east of p, respectively, so that each point of $\lambda(n, e)$ is northeast of p and so that no point of $\lambda(n, e) \backslash \{n, e\}$ is north or east of p. Assume $\lambda(n, e)$ is ordered from n to e.

Point x sees via staircase paths some point of $n + T$. If such a staircase path contains a point q east of p, then it is easy to show that $q \in p + T$. Hence x sees via staircase paths a point of $p + T$, the desired result. Otherwise, every staircase path in S from x to $n + T$ must contain a point strictly west of p and a point strictly north of p. □

The following proposition will be helpful:

Proposition 1. *Let $\{z'_m\}$ be a sequence in $\lambda(n, e)$, and assume that, for every m, x sees a point of $z'_m + T$ via a staircase path which contains a point strictly north (south) of p. If $\{z'_m\}$ converges to z_o, then x sees a point of $z_o + T$ via a staircase path which contains a point north (south) of p, possibly p itself.*

Proof. Using an argument from [6, Lemma 1], the lines determined by edges of S give rise to a collection of nondegenerate closed rectangular regions which share no interior points. This allows us to establish an upper bound k for the number of segments in our staircase paths. That is, if x sees y via such a path, then x sees y via such a path consisting of at most k segments. Then a standard convergence argument finishes the proof. □

Using Proposition 1, relative to our order on $\lambda(n, e)$, we may choose the last point z_o on $\lambda(n, e)$ such that x sees a point of $z_o + T$ via a path passing north of p. (See Figure 1.) If $z_o = e$, then such a path cannot contain a point strictly north of p, so p itself lies on the path. Hence x sees via staircase paths a point of $p+T$, the desired result. If $z_o \neq e$, then examine $\lambda(z_o, e) \subseteq \lambda(n, e)$. For each w on $\lambda(z_o, e) \backslash \{z_o\}$, x sees via staircase paths a point of $w+T$, and no such path contains a point north of p (or p itself). Hence each of these staircase paths contains a point strictly south of p. Again by Proposition 1, x sees a point of $z_o + T$ via a staircase path passing south of p (possibly through p).

We have the existence of some staircase path μ_1 in S from x to a point z_1 of $z_o + T$, with μ_1 passing north of p. Similarly, we have some staircase path μ_2 in S from x to a point z_2 of $z_o + T$, with μ_2 passing south of p. Clearly p belongs to the region R bounded by $\mu_1 \cup \mu_2 \cup [z_1, z_2]$. Since $(z_o + T) \cap S$ is connected, $[z_1, z_2] \subseteq S$. Hence $\mu_1 \cup \mu_2 \cup [z_1, z_2] \subseteq S$, and, since S is simply connected, $R \subseteq S$. Moreover, curves μ_1, μ_2 and $[z_1, z_2]$ are staircase paths, so by [6, Lemma 2], R is orthogonally convex. We have x and p in the orthogonally convex subset R of S, so x sees p via a staircase path in S. Again x sees via staircase paths a point of $p + T$, finishing the argument in Case 1.

Case 2. In case x is east of the vertical line at p, without loss of generality assume that x is southeast of p. Select on λ points n and w, north and west of p, respectively, so that all points of $\lambda(n, w)$ are northwest of p and so that no point of $\lambda(n, w) \backslash \{n, w\}$ is north or west of p. Assume $\lambda(n, w)$ is ordered from n to w. Clearly x sees a point of $n + T$, and each associated staircase path passes east of

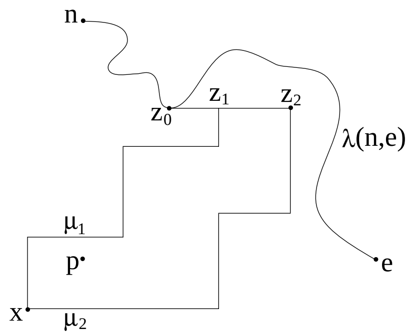

Figure 1.

p. If for each z on $\lambda(n, w)$, no staircase from x to $z + T$ contains a point west of p, then x sees $w + T$ via a path passing east of p. Hence x sees via staircase paths some w' on $w + T$ where w' is east of p. (Possibly $w' = p$.) Clearly $w' \in p + T$, so x sees via staircase paths a point of $p + T$, the desired result. Otherwise, using an analogue of Proposition 1 and the argument in Case 1, for some z_o on $\lambda(n, w)$, x sees a point z_1 of $z_o + T$ via a staircase path passing east of p, and x sees a point z_2 of $z_o + T$ via a staircase path passing west of p. As in Case 1, the associated paths together with $[z_1, z_2]$ determine an orthogonally convex subset of S, and x sees p via staircase paths. Again x sees a point of $p + T$, completing the argument in Case 2. We conclude that A_x is simply connected, finishing Part 1.

Part 2. We show that for each pair x, y in S, the associated intersection $A_x \cap A_y$ is path connected. We select points s_1 and s_2 in $A_x \cap A_y$ to find a path from s_1 to s_2 in $A_x \cap A_y$. Assume that x sees via staircase paths points a_1 on $s_1 + T$ and a_2 on $s_2 + T$. Similarly, assume y sees points b_1 on $s_1 + T$ and b_2 on $s_2 + T$. Let λ be an $a_1 - a_2$ geodesic in S, μ a $b_1 - b_2$ geodesic in S. By [7, Theorem 1], x (respectively y) sees via staircase paths each point of λ (respectively μ).

For the moment, assume that the paths λ, μ meet, if at all, only at one or both endpoints. Let G denote the region bounded by $\lambda \cup \mu \cup [a_1, b_1] \cup [a_2, b_2]$. The following proposition will be helpful.

Proposition 2. *The region G is horizontally convex.*

Proof. Let L be a horizontal line meeting G, to show that $L \cap G$ is connected. Suppose, on the contrary, that $L \cap G$ has two or more components, to obtain a contradiction. Let $[p_1, p_2], [q_1, q_2]$ be consecutive components of $L \cap G \subseteq S$. Since λ and μ are disjoint except possibly for endpoints, it is easy to see that $p_1 \neq p_2$ and $q_1 \neq q_2$. Assume that the points are ordered on L with $p_1 < p_2 < q_1 < q_2$. Clearly $(p_2, q_1) \cap G = \phi$ and p_i, q_i belong to $\lambda \cup \mu$, $i = 1, 2$. There are two cases to consider.

Case 1. Consider the case in which one of the pairs p_1, p_2 or q_1, q_2 belongs to the same curve λ or μ. Say p_1, p_2 belong to λ. Since $[p_1, p_2] \subseteq G \subseteq S$ and λ is a geodesic in $S, [p_1, p_2] \subseteq \lambda$.

If L is not the line of a_1, a_2 (nor the line of b_1, b_2), then λ contains a previous edge to $[p_1, p_2]$ and a successive edge to $[p_1, p_2]$. For convenience, label these edges $[p_o, p_1]$ and $[p_2, p_3]$. If these edges were to lie in opposite closed halfplanes determined by L, then p_1 and p_2 could not be endpoints of a component of $L \cap G$, impossible. If these edges were to lie in the same closed halfplane determined by L, observe that near $[p_1, p_2]$, the associated horizontal segments from (p_o, p_1) to (p_2, p_3) also would lie in G: Otherwise, such segments would belong to the unbounded region $cl(\mathbb{R}^2 \backslash G)$ and again p_1 and p_2 could not be endpoints of a component of $L \cap G$, impossible. But the existence in G of a horizontal segment from (p_o, p_1) to (p_2, p_3) would allow us to replace λ by a shorter orthogonal path in S, contradicting the fact that λ is a geodesic.

The only remaining possibility is that L be the line of a_1, a_2 or of b_1, b_2, say the former. The preceding argument yields $L \cap \lambda = [p_1, p_2]$. However, this implies that $q_1, q_2 \in \mu$, and $L \cap \mu = [q_1, q_2]$. But then $a_1 \in [p_1, p_2]$, $b_1 \in [q_1, q_2]$, and since $[a_1, b_1] \subseteq G$, $[p_2, q_1] \subseteq G$, a contradiction. We conclude that the situation in Case 1 cannot occur.

Case 2. Assume that for each pair p_1, p_2 and q_1, q_2, one of the points belongs to λ, the other to μ. There are two possibilities to consider.

If p_1, q_2 lie on the same curve, say λ, and p_2, q_1 are on μ, then we can replace $\lambda(p_1, q_2)$ by $[p_1, p_2] \cup \mu(p_2, q_1) \cup [q_1, q_2]$. (See Figure 2a.) Since λ and μ are disjoint (except possibly for endpoints), the new curve would be shorter than $\lambda(p_1, q_2)$, impossible since λ is a geodesic. Thus this situation cannot occur.

The only other possibility is that p_1, q_1 belong to the same curve, say λ, while p_2, q_2 are in μ. (See Figure 2b.) For an appropriate labeling of λ and μ, $\lambda(p_1, q_1)$ can be replaced by the shorter curve $[p_1, p_2] \cup \mu(p_2, q_2) \cup [q_2, q_1]$, again impossible. We have a contradiction, our original supposition is false, and $L \cap G$ is connected for every horizontal line L. This finishes the proof of Proposition 2. □

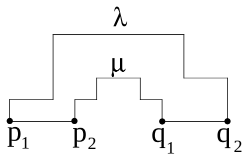

Figure 2a.

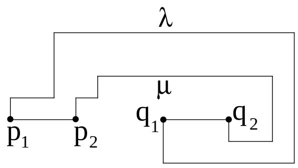

Figure 2b.

Using Proposition 2, for L any horizontal line which meets $\lambda, L \cap \lambda \subseteq G \subseteq S$. Then since λ is a geodesic in $S, L \cap \lambda$ must be either a point or a segment in λ. A parallel statement holds for μ. Notice that for L any horizontal line which meets G, L meets $\lambda \cup \mu$. In fact, it is not hard to see that L meets both λ and μ: If, on the contrary, L met (say) λ but not μ, then μ would be a positive distance from L. For (infinitely many) horizontal lines L' near L and meeting $G, L' \cap \mu = \phi$, and $L' \cap G$ would be a (nondegenerate) segment with both endpoints in λ. Hence $L' \cap \lambda$, would be a segment in λ, clearly impossible for orthogonal path λ.

Since we are assuming that λ meets μ, if at all, only at one or both endpoints, using the comments above, without loss of generality we may assume that λ is west of μ. That is, for L any horizontal line which meets G, points of $L \cap \lambda$

are west of points of $L \cap \mu$. Furthermore, if R is the smallest rectangular region containing $[a_1, b_1] \cup [a_2, b_2], G \subseteq R$, for otherwise, since λ is west of μ, we could replace one of λ, μ by a shorter path in G, impossible.

We will show that $\lambda \subseteq A_x \cap A_y$. That is, for every point a on λ, both x and y see via staircase paths some point of $a + T$. Since $a \in V_x \cap (a + T)$, the result is trivial for x, so we need only establish it for y. Select point c on $(a+T) \cap S$ as far as possible from a. Since $(a+T) \cap S$ is connected, $[a, c] \subseteq S$. We will show that y sees via staircase paths a point of $[a, c]$. For L a horizontal line at a, let b belong to $L \cap \mu \neq \phi$, where b is as close as possible to a. If $a \leq b \leq c$, then certainly y sees point b of $[a, c] \subseteq a + T$, the desired result. Hence we restrict our attention to the case in which $a \leq c < b$.

We assert that there is in G a $b_1 - b_2$ geodesic which contains c (and hence y sees c via staircase paths). Note that since G is horizontally convex, $[a, b] \subseteq G \subseteq S$. Since T is not a singleton set and $a < b, a < c$ also. Moreover, $dist(a, c)$ is the full length of T, for otherwise we could have chosen c further from a.

Without loss of generality, assume that a_1 is northwest of a_2. By previous comments, λ and hence point a lie in the rectangular region R, so a is southeast of a_1. Likewise, point b is in $\mu \subseteq R$, and c is in $[a, b] \subseteq G \subseteq R$. Thus at least one of b_1 or b_2 is east of the vertical line M_c at c. If b_1 were strictly east of line M_c, then (since a_1 is northeast of a) $dist(a, c) < dist(a_1, b_1) \leq$ length of T, impossible since $dist(a, c)$ is the full length of T. Thus b_1 is west of (possibly on) line M_c. Also, for $i = 1, 2, dist(a_i, b_i) \leq$ length of $T = dist(a, c)$, and since $[a, b] \subseteq G \subseteq R, a_2$ must lie strictly east of the vertical line at a and b_2 must lie east of the vertical line at b, hence strictly east of M_c. (See Figure 3.)

Select points p_1 and p_2 in $M_c \cap bdry\ G$ such that $c \in [p_1, p_2] \subseteq G$. Since all points of $[a_1, b_1]$ are west of line M_c, at least one of p_1 or p_2 is in $\lambda \cup \mu$. There are two cases to consider, determined by the positions of p_1 and p_2.

Case 1. In case p_1 or p_2 is in μ, without lost of generality assume that $p_1 \in \mu(b_1, b)$. (See Figure 3.) We may replace $\mu(p_1, b)$ by the geodesic $[p_1, c] \cup [c, b]$. Clearly the lengths of these two paths are equal. Then $\mu(b_1, p_1) \cup [p_1, c] \cup [c, b] \cup \mu(b, b_2)$ will be a $b_1 - b_2$ geodesic in $G \subseteq S$ and containing point c. By [7, Theorem 1], V_c is geodesically convex. Thus y sees via staircase paths each point of this geodesic, and hence y sees c via staircase paths, the desired result.

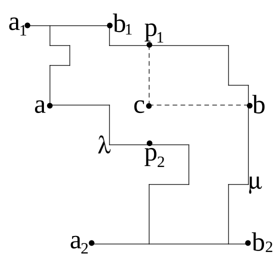

Figure 3.

Case 2. In case neither p_1 nor p_2 is in μ, by previous comments, we may assume that p_1 is in λ. If p_2 were also in λ, then we could replace $\lambda(p_1, p_2)$ by the strictly shorter path $[p_1, p_2]$ in S, impossible since λ is a geodesic. (See Figure 4a.) Hence $p_2 \notin \lambda$, and for an appropriate labeling $p_2 \in (a_2, b_2)$. (See Figure 4b.) However, since a_2 is strictly east of the vertical line at a, path $[p_1, p_2] \cup [p_2, a_2]$ would be strictly shorter than the geodesic $\lambda(p_1, a_2)$. Again we have a contradiction. We conclude that the situation in Case 2 cannot occur, and Case 1 must occur, finishing this part of the argument.

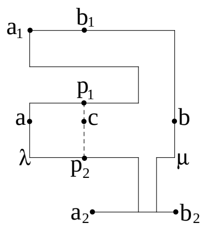

Figure 4a.

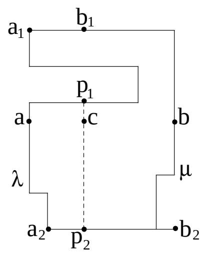

Figure 4b.

We have proved that both x and y see points of $a + T$ for each a on λ, and hence $\lambda \equiv \lambda(a_1, a_2) \subseteq A_x \cap A_y$. Since x and y see via staircase paths points a_i and b_i, respectively, on $s_i + T$, and $s_i \leq a_i \leq b_i$ on $s_i + T$, it is clear that $[s_i, a_i] \subseteq A_x \cap A_y, i = 1, 2$. Thus $A_x \cap A_y$ contains the polygonal path $[s_1, a_1] \cup \lambda(a_1, a_2) \cup [a_2, s_2]$.

In case λ and μ meet at other points, we may write λ and μ as unions of consecutive subpaths $\lambda_1, \ldots, \lambda_k$ and $\mu_1, \ldots, \mu_k$, respectively, such that for each i, either $\lambda_i = \mu_i$ or λ_i and μ_i meet only in endpoints. By applying the argument above to each appropriate pair λ_i, μ_i, then fitting together the corresponding paths, we obtain a polygonal (orthogonal) path in $A_x \cap A_y$ from s_1 to s_2. Thus $A_x \cap A_y$ is path connected, finishing Part 2.

For future reference, using the notation above, observe that with the possible exception of points on $[s_1, a_1] \cup [a_2, s_2]$, all points of the selected path lie in $V_x \cup V_y \subseteq S$. Moreover, a much easier version of the argument shows that set $A_x \cap S$ is path connected as well.

Part 3. It remains to show that every three of the A_x sets intersect. For convenience of notation, for $1 \leq i \leq 3$, let $A_{x_i} = A_i$ and $V_{x_i} = V_i$ denote any three of the A_x and associated V_x sets, to show that $A_1 \cap A_2 \cap A_3 \neq \phi$. Parts of the proof follow arguments in [3, Theorem 1]. Choose a_{ij} in $A_i \cap A_j \neq \phi, 1 \leq i < j \leq 3$. Along $a_{12} + T$, select points c_1, c_2 in V_1, V_2, respectively, such that $dist(c_1, c_2)$ is as small as possible. For future reference, notice that if c_2 is west of c_1, then $c_1 \in c_2 + T$ so $c_2 \in A_1$ and in fact $[c_2, c_1] \subseteq A_1$. Since c_2, c_1 are in $S \cap (a_{12} + T), [c_2, c_1] \subseteq S$ as well, so $[c_2, c_1] \subseteq A_1 \cap S$. Similarly, if c_1 is west of c_2, then $[c_1, c_2] \subseteq A_2 \cap S$. Using a parallel argument, along $a_{13} + T$ select $c_1^{'}, c_3^{'}$ in V_1, V_3, respectively, with $dist(c_1^{'}, c_3^{'})$ minimal. If $c_3^{'}$ is west of $c_1^{'}$, then $[c_3^{'}, c_1^{'}] \subseteq A_1 \cap S$, and if $c_1^{'}$ is west of $c_3^{'}$, then $[c_1^{'}, c_3^{'}] \subseteq A_3 \cap S$. (Figure 5 may help the reader follow the argument.)

We may choose a geodesic $\lambda_2^{''}$ in S from c_2 to a point b_2 of $(a_{23} + T) \cap V_2$. By [7, Theorem 1], V_2 is geodesically convex and hence $\lambda_2^{''} \subseteq V_2$. Similarly, choose a geodesic $\lambda_3^{''}$ in S from $c_3^{'}$ to a point b_3 of $(a_{23} + T) \cap V_3$. Then $\lambda_3^{''} \subseteq V_3$. If

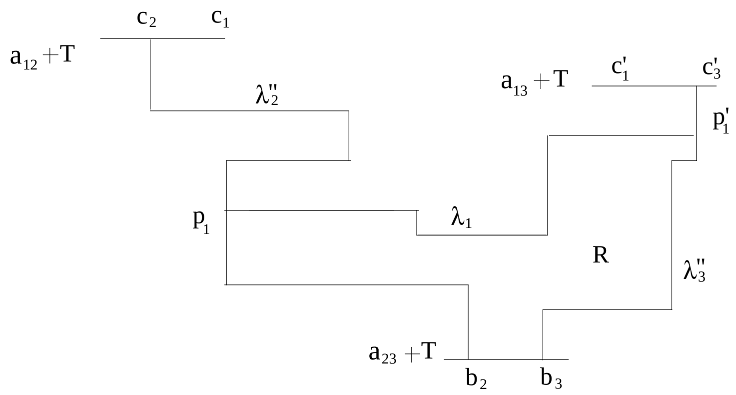

Figure 5.

$c_2 \in a_{13}+T$, then $a_{13} \in A_1 \cap A_2 \cap A_3$, finishing the argument. Similarly, if $c_3' \in a_{12}+T$, then $a_{12} \in A_1 \cap A_2 \cap A_3$. Hence we assume that neither situation occurs. Then certainly $c_2 \neq c_3'$. Without loss of generality, we assume that λ_2'' meets λ_3'', if at all, only in a common last endpoint. (Otherwise, we could delete appropriate parts of λ_2'', λ_3'' to obtain new paths ending at a common point of $V_2 \cap V_3 \subseteq A_2 \cap A_3$ and having the required property.) Also, without loss of generality we assume that λ_2'' meets $a_{23}+T$ only at b_2 and that λ_3'' meets $a_{23}+T$ only at b_3, with b_2 west of b_3 on $a_{23}+T$. Then it is easy to see that $b_3 \in b_2+T$ so $b_2 \in A_3$ and in fact $[b_2, b_3] \subseteq A_3$. Also, b_2 and b_3 are in $(a_{23}+T) \cap S$, so $[b_2, b_3] \subseteq S$. Thus $[b_2, b_3] \subseteq A_3 \cap S$.

Clearly λ_2'' meets $\lambda_3'' \cup [b_3, b_2]$ only in b_2. Notice that each of these paths is simple, with $\lambda_2'' \subseteq V_2 \subseteq A_2 \cap S, \lambda_3'' \cup [b_3, b_2] \subseteq A_3 \cap S$.

Define $\lambda_2' = [c_1, c_2] \cup \lambda_2''$ and $\lambda_3' = [c_1', c_3'] \cup \lambda_3'' \cup [b_3, b_2]$. Choose point p_1 on $\lambda_2' \cap A_1 \neq \phi$ closest to b_2 (relative to the order on λ_2'). By previous comments, if c_2 is west of c_1, then $[c_2, c_1] \subseteq A_1 \cap S$, and in this case $p_1 \in \lambda_2'' \subseteq A_2 \cap S$. If c_2 is east of c_1, then p_1 either is west of c_2 on $[c_1, c_2] \subseteq A_2 \cap S$ or is on λ_2''. Similarly, choose point p_1' on $\lambda_3' \cap A_1 \neq \phi$ closest to b_2 (relative to the order on λ_3'). Notice that p_1' either is west of c_3' on $[c_1', c_3'] \subseteq A_3 \cap S$ or is on $\lambda_3'' \cup [b_3, b_2] \subseteq A_3 \cap S$.

Define $\lambda_2 = \lambda_2'(p_1, b_2), \lambda_3 = \lambda_3'(p_1', b_2)$. By earlier comments, $\lambda_2 \subseteq A_2 \cap S, \lambda_3 \subseteq A_3 \cap S$. Clearly λ_2 is a simple curve. Notice that if λ_3 fails to be simple, then p_1' must be west of c_3', with $[p_1', c_3'] \cap [b_2, b_3] \neq \phi$. If a_{13} is west of a_{23}, then for this intersection to be nonempty, b_2 must belong to $[a_{13}, c_3'] \subseteq a_{13}+T$, and $a_{13} \in A_1 \cap A_2 \cap A_3$, finishing the argument. Similarly, if a_{23} is west of a_{13}, then for this intersection to be nonempty, b_3 must be east of p_1', hence east of c_1'. But then $c_1' \in [a_{23}, b_3] \subseteq a_{23}+T$, so $a_{23} \in A_1 \cap A_2 \cap A_3$, again finishing the argument. Thus we may assume that curve λ_3 is simple as well.

Furthermore, we assert that λ_2 meets λ_3 only at b_2: Since λ_2' meets $\lambda_3' \cup [b_3, b_2]$ only at b_2, if the assertion fails, then p_1 must be west of c_2 and p_1' must be west of c_3', with $[p_1, c_2] \cap [p_1', c_3'] \neq \phi$. Since $c_2 \neq c_3'$, without loss of generality assume c_2 is west of c_3'. Since $[p_1, c_2] \cap [p_1', c_3'] \neq \phi$, then p_1' is west of c_2. However, then $c_2 \in [p_1', c_3'] \subseteq a_{13}+T$, contradicting an early assumption. Thus the assertion holds, and $\lambda_2 \cap \lambda_3 = \{b_2\}$.

Finally, we define λ_1 to be a geodesic in $A_1 \cap S$ from p_1 to $p_1^{'}$. By our choice of p_1 and $p_1^{'}, \lambda_1 \cap \lambda_2 = \{p_1\}$ and $\lambda_1 \cap \lambda_3 = \{p_1^{'}\}$. Moreover, λ_1 is simple. Let R denote the region bounded by $\lambda_1 \cup \lambda_2 \cup \lambda_3$. By previous comments, $bdry\, R = \lambda_1 \cup \lambda_2 \cup \lambda_3 \subseteq (A_1 \cup A_2 \cup A_3) \cap S$ and hence $R \subseteq S$.

We will show that $R \subseteq A_1 \cup A_2 \cup A_3$, and it suffices to show that $int\ R \subseteq A_1 \cup A_2 \cup A_3$. In $bdry\ R$, choose points n, s, e, w north, south, east, west, respectively, of p so that the union $(n, p] \cup (s, p] \cup (e, p] \cup (w, p]$ is interior to R. At least two of these points belong to the same set A_i. Since the argument does not depend on the particular selection of $\lambda_1, \lambda_2, \lambda_3$ above, for convenience of notation, we label this set A_1. We consider cases according to the points of $\{n, s, e, w\}$ involved.

Case 1. Assume that w belongs to A_1. Then x_1 sees via staircase paths a point q of $w + T$. If q is east of p, then $q \in p + T, x$ sees point q of $p + T$, and $p \in A_1$, the desired result. Hence suppose that q is west of p.

In case e belongs to A_1, then x_1 sees via staircase paths some point r on $e+T$, and certainly r is east of p. Then $[q, r] \subseteq [w, e] \cup [e, r] \subseteq S$. By [7, Theorem 1], x_1 sees via staircase paths each point of the $q - r$ geodesic $[q, r]$. Hence x_1 sees p via staircase paths, and $p \in V_1 \subseteq A_1$, again the desired result.

In case n or s belongs to A_1, without loss of generality assume $n \in A_1$. Then x_1 sees via staircase paths some point t of $n + T$, and t is east of n. The staircase path $[q, p] \cup [p, n] \cup [n, t]$ is a $q - t$ geodesic in S and hence lies in V_1. Again x_1 sees point p via staircase paths, finishing Case 1.

Case 2. If $w \notin A_1$, then without loss of generality assume that $n \in A_1$ and that one of s, e belongs to A_1.

If n, s are in A_1, assume that x_1 sees via staircase paths point t on $n + T$ and point u on $s + T$. Every $t - u$ geodesic lies in the rectangular region containing $[n, t] \cup [s, u]$, and so each $t - u$ geodesic meets $p + T$. Each of these geodesics lies in V_1. Hence x_1 sees via staircase paths a point of $p + T$, and $p \in A_1$ as desired.

If n, e are in A_1, assume that x_1 sees via staircase paths point t on $n + T$ and point r on $e + T$. Assume $r \notin p + T$ for otherwise the proof is immediate. We consider the position of x_1: If x_1 is east of the vertical line at t, then it is easy to show that x_1 sees via staircase paths a point of $p + T$. If x_1 is west of this line, by taking cases according to whether x_1 is north or south of line $L(p, t)$, again it is not hard to show that x_1 sees via staircase paths a point of $p + T$. Therefore $p \in A_1$, finishing Case 2.

An identical argument holds if two of n, s, e, w belong to A_2 or A_3, so we conclude that $R \subseteq A_1 \cup A_2 \cup A_3$.

We will show that $A_1 \cap R$ is a simply connected orthogonal polygon. Observe that both A_1 and R are simply connected orthogonal polygons, so it suffices to show that $A_1 \cap R$ is connected. For each point y in $A_1 \cap R$, there is a staircase path μ_y in S from x_1 to a point $y + t$ of $y + T$. Then $\mu_y \subseteq A_1 \cap S$. Also, by earlier arguments, $[y, y + t] \subseteq A_1 \cap S$, so $\mu_y \cup [y, y + t]$ is an (orthogonal) path in $A_1 \cap S$ from x_1 to y.

By earlier remarks, this path cannot meet $\lambda_2 \cup \lambda_3 \backslash \{p_1, p_1^{'}\}$. Either the path is in R or the path meets λ_1 whenever it leaves or enters R. If $x_1 \notin R$, then for each y in $A_1 \cap R, (\mu_y \cap R) \cup \lambda_1$ is connected, and so

$$\cup\{(\mu_y \cap R) \cup \lambda_1 : y \text{ in } A_1 \cap R\} \equiv A_1 \cap R$$

is connected, too. If $x_1 \in R$, then for each y in $A_1 \cap R, \mu_y \cap R$ is connected when μ_y is disjoint from λ_1 and $(\mu_y \cap R) \cup \lambda_1$ is connected when μ_y meets λ_1. Since $\lambda_1 \subseteq A_1 \cap R$, it is easy to see that

$$\cup\{(\mu_y \cap R) \cup \lambda_1 : y \text{ in } A_1 \cap R\} \equiv A_1 \cap R$$

again is connected. We conclude that $A_1 \cap R$ is a simply connected orthogonal polygon.

Next we will show that $bdry(R \cap A_1)$ contains an orthogonal path in $A_2 \cup A_3$ from p_1 to $p_1^{'}$: If $R \cap A_1 = \lambda_1$, then $int\ R \subseteq A_2 \cup A_3, \lambda_1 \subseteq cl(A_2 \cup A_3) = A_2 \cup A_3$, and λ_1 serves as the required path. Otherwise, select path δ from p_1 to $p_1^{'}, \delta \subseteq bdry(R \cap A_1) \subseteq A_1$, so that $R \cap A_1$ is bounded by $\delta \cup \lambda_1$. Choose δ so that $\delta \cap \lambda_1$ is minimal for all such paths. By our choice of p_1 and $p_1^{'}$, it is easy to see that δ is disjoint from $\lambda_2 \cup \lambda_3 \backslash \{p_1, p_1^{'}\}$. Clearly $\delta \subseteq A_2 \cup A_2 \cup A_3$ since there are points of $(int\, R) \cap (A_2 \cup A_3)$ near each point of δ. Hence δ serves as the required path.

Observe that δ is a connected subset of $A_1 \cap (A_2 \cup A_3)$. Since $p_1 \in \delta \cap A_2$ and $p_1^{'} \in \delta \cap A_3$, by properties of connected sets, $\delta \cap A_2 \cap A_3 \neq \phi$. Thus $\delta \cap A_2 \cap A_3 \subseteq A_1 \cap A_2 \cap A_3 \neq \phi$, which is what we wanted to establish. This finishes Part 3.

At last we may apply a version of Molnár's theorem by Karimov, Repovš, and Željko [11, Theorem 2] to conclude that $\cap\{A_x : x \text{ in } S\} \neq \phi$. For z_o in this intersection, every point of S sees via staircase paths in S a point of $z_o + T$. That is, $z_o + T$ satisfies the theorem.

Clearly the number two in the hypothesis is best possible.

In conclusion, it is interesting to observe that the result fails without the requirement that $T^{'} \cap S$ be connected for translates $T^{'}$ of T. Consider the following example.

Example 1. Let S and T be the sets in Figure 6. Every two and in fact every three points of S see via staircase paths a common translate of T. (Consider translates at a, x, y, z.) However, no such translate exists for points a, b, c, d of S.

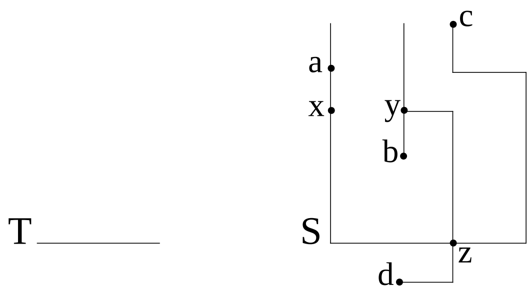

Figure 6.

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