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Equivariant Higher Algebraic K-Theory for Waldhausen Categories

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1. Introduction

The aim of this paper is to generalize the equivariant higher algebraic K-theory constructions in [3] from exact categories to Waldhausen categories. So, let W be a Waldhausen category, G a finite group, X a G-set and $\underline{X}$ translation category of X (see 4.1). Then the covariant functors from $\underline{X}$ to W also form a Waldhausen category under cofibrations and weak equivalences induced from W (see 4.2). We denote this category by $[\underline{X}, W]$ and we write $\mathbb{K}(X, W)$ for the Waldhausen K-theory space/spectrum for $[\underline{X}, W]$ and write $\mathbb{K}_n^G(X, W) := \pi_n(\mathbb{K}(X, W))$ for the n-th Waldhausen K-theory group for all $n \geq 0$. To construct a relative theory, let X, Y be G-sets, and ${}^Y[\underline{X}, W]$ a Waldhausen category defined such that $\mathrm{ob}({}^{\mathrm{Y}}[\underline{\mathrm{X}}, \mathrm{W}]) = \mathrm{ob}[\underline{\mathrm{X}}, \mathrm{W}]$, cofibrations are Y-cofibrations defined in 4.5 and weak equivalences are those defined for $[\underline{X}, W]$. This new Waldhausen category yields a K-theory space/spectrum $\mathbb{K}({}^Y[X, W])$ and new K-theory groups $\mathbb{K}_n^G(X, W, Y) := \pi_n(\mathbb{K}({}^Y[X, W])$ (see 5.1.1). Next, we define, for G-sets X, Y, a new Waldhausen category $[\underline{X}, W]_Y$ consisting of "Y-projective" objects in $[\underline{X}, W]$ with appropriate cofibrations and weak equivalences (see 4.6), leading to a new Waldhausen K-theory space/spectrum $\mathbb{K}([X, W]_Y)$ and new K-theory groups $\mathbb{P}_n^G(X, W, Y) := \pi_n(\mathbb{K}([\underline{X}, W]_Y))$ for all $n \geq 0$ (see 5.1.1). Next, we prove that the functors $\mathbb{K}_n^G(-, W)$, $\mathbb{K}_n^G(-, W, Y)$ and $\mathbb{P}_n^G(-, W, Y)\colon G\mathrm{Sets} \to Ab$ are Mackey functors (see 5.1.2). Under suitable hypothesis on W, we show that $\mathbb{K}_0^G(-W)$, $\mathbb{K}_0^G(-, W, Y)$ are Green functors and that $\mathbb{K}_n^G(-, W)$ are $\mathbb{K}_0^G(-, W)$ modules and that $\mathbb{K}_n^G(-, W, Y)$ and $\mathbb{P}_n^G(-, W, Y)$ are $K_0^G(-, W, Y)$-modules for all $n \geq 0$. We highlight in 5.1.5 some consequences of these results. While still on general Waldhausen categories we present equivariant consequences of Waldhausen K-theory,

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Additivity theorem (5.1.8) and fibration theorem (5.1.9). In Section 6, we focus on applications of the foregoing to Thomason's "complicial bi-Waldhausen categories" of the form $Ch_b(\mathcal{C})$, where $\mathcal{C}$ is any exact category. First we obtain connections between the foregoing theory and those in [3] (see 6.1) and then interprete the theories in terms of group-rings (6.2). In the process we prove a striking result that if R is the ring of integers in a number field, G a finite group, then the Waldhausen's K-groups of the category $(Ch_b(\underline{M}(RG), w)$ of bounded complexes of finitely generated RG-modules with stable quasi-isomorphisms as weak equivalences are finite abelian groups (see 6.4). Finally we present in 6.5 an equivariant approximation theorem for complicial bi-Waldhausen categories (see 6.6).

Even though we have focussed in this paper on finite group actions, we observe that it should be possible to construct equivariant K-theory for Waldhausen categories for the actions of profinite and compact Lie groups as was done for exact categories in [8] and [13]. We also feel that it should be possible to interprete the foregoing theory for $Ch_b(\mathcal{C})$ for exact categories $\mathcal{C}$ like $P(X)$ the category of locally free sheaves of O_X-modules (X a scheme) as well as $\underline{\underline{M}}(X)$, the category of coherent sheaves of O_X-modules where X is a Noetherian scheme.

2. Notes on notation

For a Waldhausen category W, we shall write $\mathbb{K}(W)$ for the Waldhausen K-theory space/spectrum of W. So, if $\mathbb{K}(W)$ is the space $\Omega|\omega S_* W|$ or spectrum $\{\Omega|\omega S_*^n W|\}$ we shall write $\mathbb{K}_n(W) := \pi_n \mathbb{K}(W)$.

For an exact category $\mathcal{C}$, we shall write $K(\mathcal{C})$ for the Quillen K-theory space/ spectrum of $\mathcal{C}$. Hence if $K(\mathcal{C})$ is the space $\Omega BQ\mathcal{C}$ or spectrum $\{\Omega BQ^n\mathcal{C}\}$, we shall write $\pi_n(K(\mathcal{C})) := K_n(\mathcal{C})$.

For any ring A with identity, we shall write $\underline{P}(A)$ for the category of finitely generated projective A-modules, $\underline{\underline{M}}'(A)$ for the category of finitely presented A-modules, $\underline{\underline{M}}(A)$ the category of finitely generated A-modules and write $K_n(A)$ for $K_n(\underline{P}(A))$, $G'_n(A)$ for $K_n(\underline{\underline{M}}'(A))$ and $G_n(A)$ for $K_n(\underline{\underline{M}}(A))$. The inclusions $\underline{P}(A) \subseteq \underline{\underline{M}}'(A)$, $\underline{P}(A) \subseteq \underline{\underline{M}}(A)$ induce Cartan maps $K_n(A) \to G'_n(A)$, $K_n(A) \to G_n(A)$. Note that if A is Noetherian, $G'_n(A) = G_n(A)$ since $\underline{\underline{M}}'(A) = \underline{\underline{M}}(A)$. If A is an R-algebra finitely generated as an R-module (R a commutative ring with identity), we shall write $G_n(R, A)$ for $K_n(\underline{P}_R(A))$ where $\underline{P}_R(A)$ is the category of finitely generated A-modules that are projective over R. Similarly, we shall write $G'_n(R, A)$ for $K_n(\underline{P}'_R(A)$ where $\underline{P}'_R(A)$ is the category of finitely presented A-modules that are projective over R. Note that if R is Noetherian, then $G'_n(R, A) = G_n(R, A)$. If G is a finite group and $A = RG$, we shall write $G'_n(R, G)$ for $G'_n(R, RG)$, $G_n(R, G)$ for $G_n(R, RG)$.

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3. Some preliminaries on Waldhausen categories; Mackey functors

3.1. Generalities on Waldhausen categories

3.1.1. Definition. *A category with cofibrations is a category $\mathcal{C}$ with zero object together with a sub category $co(\mathcal{C})$ whose morphisms are called cofibrations written $A \rightarrowtail B$ and satisfying axioms:*

(C1) *Every isomorphism in $\mathcal{C}$ is a cofibration.*

(C2) *If $A \rightarrowtail B$ is a cofibration, and $A \to C$ any $\mathcal{C}$-map, then the pushout $B \cup_A C$ exists in $\mathcal{C}$ and the horizontal arrow in the diagram* (I) *is a cofibration.*

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ C & \rightarrowtail & B \cup_A C \end{array} \qquad \text{(I)}$$

- *Hence coproducts exist in $\mathcal{C}$ and each cofibration $A \rightarrowtail B$ has a cokernel $C = B/A$.*
- *Call $A \rightarrowtail B \twoheadrightarrow B/A$ a cofibration sequence.*

(C3) *The unique map $0 \to B$ is a cofibration for all $\mathcal{C}$-objects B.*

3.1.2. Definition. *A Waldhausen category W is a category with cofibrations together with a subcategory $w(W)$ of weak equivalences (w.e for short) containing all isomorphisms and satisfying:*

(W1) **Gluing axiom for weak equivalences:** *For any commutative diagram*

$$\begin{array}{ccccc} C & \leftarrow & A & \rightarrowtail & B \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ C' & \leftarrow & A' & \rightarrowtail & B' \end{array}$$

in which the vertical maps are weak equivalences and the two right horizontal maps are cofibrations, the induced map $B \cup_A C \to B' \cup_{A'} C'$ is also a weak equivalence. We shall sometimes denote W by (W, w).

3.1.3. Definition. *A Waldhausen subcategory W' of a Waldhausen category W is a subcategory which is also Waldhausen-category such that*

(a) *the inclusion $W' \subseteq W$ is an exact functor,*

(b) *the cofibrations in W' are the maps in W' which are cofibrations in W and whose cokernels lie in W' and*

(c) *the weak equivalences in W' are the weak equivalences of W which lie in W'.*

3.1.4. Definition. *A Waldhausen category W is said to be saturated if whenever f, g are composable maps and fg is a w.e. then f is a w.e. iff g is*

- *The cofibrations sequences in a Waldhausen category W form a category $\mathcal{E}$. Note that $ob(\mathcal{E})$ consists of cofibrations sequences $E\colon A \rightarrowtail B \twoheadrightarrow C$ in W. A morphism $E \to E'\colon A' \rightarrowtail B' \twoheadrightarrow C'$ in $\mathcal{E}$ is a commutative diagram*

$$\text{(I)} \qquad \begin{array}{ccccccc} E & & A & \rightarrowtail & B & \twoheadrightarrow & C \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E' & & A' & \rightarrowtail & B' & \twoheadrightarrow & C' \end{array}$$

To make $\mathcal{E}$ a Waldhausen category, we define a morphism $E \to E'$ in $\mathcal{E}$ to be a cofibration if $A \to A'$, $C \to C'$ and $A' \cup_A B \to B'$ are cofibrations in W while $E \to E'$ is a w.e. if its component maps $A \to A'$, $B \to B'$, $C \to C'$ are w.e. in W. We shall sometimes write $\mathcal{E}(W)$ for $\mathcal{E}$.

3.1.5. Extension axiom *A Waldhausen category W is said to satisfy extension axiom if for any morphism $f\colon E \to E'$ as in* 3.1.4, *maps $A \to A'$, $C \to C'$ being w.e. implies that $B \to B'$ is also a w.e.*

3.1.6. Examples.

(i) Any exact category $\mathcal{C}$ is a Waldhausen-category where cofibrations are the admissible monomorphisms and w.e. are isomorphisms.

(ii) If $\mathcal{C}$ is any exact category, then the category $Ch_b(\mathcal{C})$ of bounded chain complexes in $\mathcal{C}$ is a Waldhausen category where w.e. are quasi-isomorphisms (i.e. isomorphisms on homology) and a chain map $\underline{A}. \to \underline{B}.$ is a cofibration if each $A_i \to B_i$ is a cofibration (admissible monomorphisms) in $\mathcal{C}$.

(iii) $Ch_b(\mathcal{C})$ as in (ii) above is an example of Thomason's "complicial bi-Waldhausen category" i.e., a full subcategory of $Ch_b(\mathcal{A})$ where $\mathcal{A}$ is an Abelian category (see [22]). This is because there exists a faithful embedding of $\mathcal{C}$ in an abelian category $\mathcal{A}$ such that $\mathcal{C} \subset \mathcal{A}$ is closed under extensions and the exact functor $C \to \mathcal{A}$ reflects exact sequences. Thus a morphism in $Ch_b(\mathcal{C})$ is a quasi-isomorphism iff its image in $Ch_b(\mathcal{A})$ is a quasi-isomorphism. We shall be particularly interested in the complicial bi-Waldhausen categories $Ch_b(\mathcal{P}(R))$, $Ch_b(\mathcal{M}'(R))$ and $Ch_b(\mathcal{M}(R))$.

    Note: Neeman and Ranicki [19] have used the terminology "permissible Waldhausen categories" for Thomason's complicial bi-Waldhausen category.

(iv) **Stable derived categories and Waldhausen categories** Let $\mathcal{C}$ be an exact category and $H^b(\mathcal{C})$ the (bounded) homotopy category of $\mathcal{C}$. So, $ob(H^b(\mathcal{C})) = Ch_b(\mathcal{C})$ and morphisms are homotopy classes of bounded complexes. Let $A(\mathcal{C})$ be the full subcategory of $H^b(\mathcal{C})$ consisting of acyclic complexes (see [4]). The derived category $D^b(\mathcal{C})$ of $\mathcal{E}$ is defined by $D^b(\mathcal{C}) = H^b(\mathcal{C})/A(\mathcal{C})$. A morphism of complexes in $Ch_b(\mathcal{C})$ is called a quasi-isomorphism if its image in $D^b(\mathcal{C})$ is an isomorphism. We could also define unbounded derived category $D(\mathcal{C})$ from unbounded complexes $Ch(\mathcal{C})$. Note that there exists a faithful embedding of $\mathcal{C}$ in an Abelian category $\mathcal{A}$ such that $\mathcal{C} \subset \mathcal{A}$ is closed under extensions and the exact functor $\mathcal{C} \to \mathcal{A}$ reflects exact sequences. So, a complex in $Ch(\mathcal{C})$ is a cyclic iff its image in $Ch(\mathcal{A})$ is acyclic. In particular, a morphism in $Ch(\mathcal{C})$ is a quasi-isomorphism iff its image in $Ch(\mathcal{A})$ is a quasi-isomorphism. Hence, the derived category $D(\mathcal{C})$ is the category obtained from $Ch(\mathcal{C})$ by formally inverting quasi-isomorphisms. Now let $\mathcal{C} = \mathcal{M}'(R)$. A complex M in $\mathcal{M}'(R)$ is said to be compact if the functor $\mathrm{Hom}(M, -)$ commutes with arbitrary set-valued coproducts. Let $\underline{\mathrm{Comp}(R)}$ denote the full subcategory of $D(\mathcal{M}'(R))$ consisting of compact objects. Then we have $\underline{\mathrm{Comp}(R)} \subset D^b(\mathcal{M}'(R)) \subset D(\mathcal{M}'(R))$. Define the stable derived category of bounded complexes $\underline{D}^b(\mathcal{M}'(R))$ as the quotient category of $D^b(\mathcal{M}'(R))$ with respect to $\underline{\mathrm{Comp}(R)}$. A morphism of complexes in $Ch_b(\mathcal{M}'(R))$ is called a stable quasi-isomorphism of its image in $\underline{D}^b(\mathcal{M}'(R))$ is an isomorphism. The family of stable quasi-isomorphism in $\mathcal{A} = Ch_b(\mathcal{M}'(R))$ is denoted $\omega\mathcal{A}$.

(v) Theorem [4]. $w(Ch_b(\mathcal{M}'(R))$ forms a set of weak equivalence and satisfies the saturation and extension axioms.

3.2. Higher *K*-theory of Waldhausen categories

In order to define the K-theory space $\mathbb{K}(W)$ such that

$$\pi_n(\mathbb{K}(W)) = \mathbb{K}_n(W)$$

for a W-category W, we construct a simplicial Waldhausen category S_*W, where S_nW is the category whose objects A are sequences of n cofibrations in W i.e.,

$$A : 0 = A_0 \rightarrowtail A_1 \rightarrowtail A_2 \to \cdots \rightarrowtail A_n$$

together with a choice of every subquotient $A_{ij} = A_j/A_i$ in such a way that we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & & & A_{n-1,n} \\
 & & & & & & & & \uparrow \\
 & & & & & & & & \uparrow \\
 & & & & A_{23} & \rightarrowtail & \cdots & \rightarrowtail & A_{2n} \\
 & & & & \uparrow & & & & \uparrow \\
 & & A_{12} & \rightarrowtail & A_{13} & \rightarrowtail & \cdots & \rightarrowtail & A_{1n} \\
 & & \uparrow & & \uparrow & & & & \uparrow \\
A_1 & \rightarrowtail & A_2 & \rightarrowtail & A_3 & \rightarrowtail & \cdots & \rightarrowtail & A_n
\end{array}$$

By convention put $A_{jj} = 0$ and $A_{0j} = A_j$. A morphism $A \to B$ is a natural transformation of sequences. A weak equivalence in S_nW is a map $A \to B$ such that each $A_i \to B_i$ (and hence each $A_{ij} \to B_{ij}$) is a w.e. in W. A map $A \to B$ is a cofibration if for every $0 \leq i < j < k \leq n$ the map of cofibration sequences is a cofibration in $\mathcal{E}(W)$

$$\begin{array}{ccccc}
A_{ij} & \rightarrowtail & A_{ik} & \twoheadrightarrow & A_{jk} \\
\downarrow & & \downarrow & & \downarrow \\
B_{ij} & \rightarrowtail & B_{ik} & \twoheadrightarrow & B_{jk}\,.
\end{array}$$

For $0 < i \leq n$, define exact functors $\delta_i\colon S_nW \to S_{n-1}W$ by omitting A_i from the notation and re-indexing the A_{jk} as needed. Define $\delta_0\colon S_nW \to S_{n-1}W$ where δ_0 omits the bottom arrow. We also define $s_i\colon S_nW \to S_{n+1}W$ by duplicating A_i and re-indexing (see [23]). We now have a simplicial category $n \to wS_nW$ with degree-wise realisation $n \to B(wS_n)$ and denote the total space by $|wS_*W|$ (see [24]).

3.2.1. Definition. *The K-theory space of a W-category W is $\mathbb{K}(W) = \Omega|wS_*W|$. For each $n \geq 0$, the K-groups are defined as $\mathbb{K}_n(W) = \pi_n(\mathbb{K}(W))$.*

Note. By iterating the S construction, one can show (see [23]) that the sequence

$$\{\Omega|wS_*W|,\, \Omega|wS, S_*W|,\, \dots,\, \Omega|wS_*W|\}$$

forms a connective spectrum $\mathbb{K}(W)$ called the K-theory spectrum of W. Hence $\mathbb{K}(W)$ is an infinite loop space, see [23].

3.2.2. Examples.

(i) Let $\mathcal{C}$ be an exact category, $Ch_b(\mathcal{C})$ the category of bounded chain complexes over $\mathcal{C}$. It is a theorem of Gillet-Waldhausen that $K(\mathcal{C}) \cong K(Ch_b(\mathcal{C}))$ and so, $K_n(\mathcal{C}) \simeq K_n(Ch_b(\mathcal{C})$ for every $n \geq 0$ (see [22]).

(ii) **Perfect Complexes** Let R be any ring with identity and $\underline{\underline{M}}'(R)$ the exact category of finitely presented R-modules. (Note that $\underline{\underline{M}}'(R) = \underline{\underline{M}}(R)$ if R is Noetherian). An object M of $Ch_b(\underline{\underline{M}}'(R))$ is called a perfect complex if M is quasi isomorphic to a complex in $Ch_b(\underline{\underline{P}}(R))$. The perfect complexes form a Waldhausen subcategory Perf(R) of $Ch_b(\underline{\underline{M}}'(R))$. So, we have

$$K(R) \simeq K(Ch_b(\underline{\underline{P}}(R)) \cong K(\mathrm{Perf}(R))$$

(iii) For a Waldhausen category W, $\mathbb{K}_0(W)$ is the Abelian group generated by objects of W with relations (i) $A \simeq B \Rightarrow [A] = [B]$ and (ii) $A \rightarrowtail B \twoheadrightarrow C \Rightarrow [B] = [A] + [C]$. Note that this description agrees with the $K_0(\mathcal{C})$ for an exact category $\mathcal{C}$.

3.3. Mackey functors

In this subsection, we briefly introduce Mackey functors in a way relevant to our context. For more general definition and presentation, see [1], [9] or [14].

3.3.1. Definition. *Let G be a finite group, $GSet$ the category of (finite) $GSets$. A pair (M_*, M^*) of functors $GSet \to R - mod$ is a Mackey functor if*

(i) *$M_* : GSet \to R - mod$ is covariant and $M^* : GSet \to R - mod$ is contravariant and $M_*(X) = M^*(X) := M(X)$ for any $GSet$ X*

(ii) *M^* transforms finite disjoint unions in $G\mathcal{S}et$ into finite products in $R-mod$, i.e., the embeddings $X_i \hookrightarrow \dot{U} X_i$ induce isomorphism $M(X_1 \dot{U} X_2 \dot{U} \ldots \dot{U} X_n) \simeq M(X_1) \times M(X_2) \times \cdots \times M(X_n)$*

(iii) *For any pull-back diagram*

$$\begin{array}{ccc} X_1 \underset{Y}{\times} X_2 & \xrightarrow{p_2} & X_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f_2} \\ X_1 & \xrightarrow[f_1]{} & Y \end{array} \quad \textit{in Gset,}$$

the diagram

$$\begin{array}{ccc} M(X_1 \underset{Y}{\times} X_2) & \xrightarrow{M_*(p_2)} & M(X_2) \\ {\scriptstyle M^*(p_1)}\uparrow & & \uparrow{\scriptstyle M^*(f_2)} \\ M(X_1) & \xrightarrow[M_*(f_1)]{} & M(Y) \end{array}$$

commutes (Mackey subgroup property).

A morphism (or natural transformation) of Mackey functors $\tau\colon M \to N$ consists of a family of homomorphisms $\tau(X)\colon M(X) \to N(X)$, indexed by the objects X in $GSet$, such that τ is a natural transformation of M_* as well as of M^*, i.e. such that for any G-map $f\colon X \to Y$ the diagrams

$$\begin{array}{ccc} M(X) & \xrightarrow{M^*(f)} & M(Y) \\ \downarrow{\scriptstyle \tau(X)} & & \downarrow{\scriptstyle \tau(Y)} \\ N(X) & \xrightarrow{N^*(f)} & N(Y) \end{array} \quad \textit{and} \quad \begin{array}{ccc} M(Y) & \xrightarrow{M^*(f)} & M(X) \\ \downarrow{\scriptstyle \tau(Y)} & & \downarrow{\scriptstyle \tau(X)} \\ N(Y) & \xrightarrow{N^*(f)} & N(X) \end{array}$$

are commutative.

A pairing $M \times N \to L$ of two Mackey functors M and N into a third one, called L is a family of R-bilinear maps

$$M(X) \times N(X) \to L(X)\colon (m, n) \mapsto m \cdot n$$

such that for any G-map $f\colon X \to Y$ the following diagrams commute

$$\begin{array}{ccc} M(Y) \times N(Y) & \longrightarrow & L(Y) \\ {\scriptstyle M^*(f)\times N^*(f)}\downarrow & & \downarrow{\scriptstyle L^*(f)} \\ M(X) \times N(X) & \longrightarrow & L(X) \end{array}$$

$$\begin{array}{ccccc} M(X) \times N(Y) & \xrightarrow{Id \times M^*(f)} & M(X) \times N(X) & \longrightarrow & L(X) \\ \downarrow{\scriptstyle M_*(f)\times Id} & & & & \downarrow{\scriptstyle L_*(f)} \\ M(Y) \times N(Y) & & \longrightarrow & & L(Y) \end{array}$$

$$\begin{array}{ccccc} M(Y)\times N(X) & \xrightarrow{M^*(f)\times Id} & M(X)\times N(X) & \longrightarrow & L(X) \\ {\scriptstyle Id\times M_*(f)}\big\downarrow & & & & \big\downarrow{\scriptstyle L_*(f)} \\ M(Y)\times N(Y) & & \longrightarrow & & L(Y) \end{array}$$

(the last two being related to Frobenius reciprocity).

A Green functor is a Mackey functor $\mathfrak{G}\colon Gset \to R-mod$ together with a pairing $\mathfrak{G}\times\mathfrak{G}\to\mathfrak{G}$ such that an R-bilinear map $\mathfrak{G}(X)\times\mathfrak{G}(X)\to\mathfrak{G}(X)$ turns $\mathfrak{G}(X)$ into an R-algebra with unit $1_{\mathfrak{G}(X)}$ and such that for each G-map $f\colon X\to Y$, the equation $f^*(\mathfrak{G})(1_{\mathfrak{G}(Y)}) = 1_{\mathfrak{G}(X)}$ holds.

If $\mathfrak{G}$ is a Green functor, M a Mackey functor and $\mathfrak{G}\times M\to M$ a pairing such that $1_{\mathfrak{G}(X)}$ acts as identity on $M(X)$, we shall call M with respect to this pairing a $\mathfrak{G}$-module.

4. Equivariant Waldhausen categories

4.1. Definiton. *Let G be a finite group, X a G-set. The translation category of X is a category $\underline{X}$ whose objects are elements of X and whose morphisms $Hom_{\underline{X}}(x,x')$ are triples (g,x,x') where $g\in G$ and $gx=x'$.*

4.2. Theorem. *Let W be a Waldhausen category, G a finite group, $\underline{X}$ the translation category of a G-set X, $[\underline{X},W]$ the category of covariant functors from $\underline{X}$ to W. Then $[\underline{X},W]$ is a Waldhausen category.*

Proof. Say that a morphism $\zeta\to\eta$ in $[\underline{X},W]$ is a cofibration if $\zeta(x)\rightarrowtail\eta(x)$ is a cofibration in W. So, isomorphisms are cofibrations in $[\underline{X},W]$. Also if $\zeta\rightarrowtail\eta$ is a cofibration and $\eta\to\delta$ is a morphism in $[\underline{X},W]$, then the push-out $\zeta\cup\delta$ defined by $(\zeta\underset{\eta}{\cup}\delta)(x)=\zeta(x)\underset{\eta(x)}{\cup}\delta(x)$ exists since $\zeta(x)\underset{\eta(x)}{\cup}\delta(x)$ is a push-out in W for all $x\in X$. Hence coproducts also exist in $[\underline{X},W]$. Also, define a morphism $\zeta\to\eta$ in $[\underline{X},W]$ as a weak equivalence if $\zeta(x)\to\eta(x)$ is a weak equivalence in W for all $x\in X$. It can be easily checked that the weak equivalences contain all isomorphisms and also satisfy the gluing axiom i.e. if $\begin{array}{ccccc}\delta & \leftarrow & \zeta & \rightarrowtail & \eta\\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim\\ \delta' & \leftarrow & \zeta' & \rightarrowtail & \eta'\end{array}$ is a commutative diagram where the vertical maps are weak equivalences and the two right horizontal maps are cofibrations, then the induced maps $\eta\underset{\zeta}{\cup}\delta\to\eta'\underset{\zeta'}{\cup}\delta'$ is also a weak equivalence.

4.3. Remarks/Definitions

If W is saturated, then so is $[\underline{X},W]$. For if $f\colon\zeta\to\zeta'$, $g\colon\zeta'\to\eta$ are composable arrows in $[\underline{X},W]$ and gf is a weak equivalence, then for any $x\in X$, $(gf)(x)=g(x)f(x)$ is a weak equivalence in W. But then, $f(x)$ is a w.e. iff $g(x)$ is for all $x\in X$. Hence f is a w.e. iff g is

4.4. Example. (i) Let $W = Ch_b(\mathcal{C})$ ($\mathcal{C}$ an exact category) be a complicial bi-Waldhausen category. Then for any small category ℓ, $[\ell, W]$ is also a complicial bi-Waldhausen category (see [5]). Hence for any GSet X, $[\underline{X}, Ch_b(\mathcal{C})]$ is a complicial bi-Waldhausen category. We shall be interested in the cases $[\underline{X}, Ch_b(\underline{P}(R))]$, $[\underline{X}, Ch_b(\underline{\underline{M}}'(R)]$ and $[\underline{X}, Ch_b(\underline{\underline{M}}(R))]$, R a ring with identity.

(ii) Here is another way to see that $[\underline{X}, Ch_b(\mathcal{C})]$ is a complicial bi-Waldhausen category. One can show that there is an equivalence of categories $[\underline{X}, Ch_b(\mathcal{C})] \xrightarrow{F} Ch_b([\underline{X}, \mathcal{C}])$ where F is defined as follows: For $\zeta_* \in [\underline{X}, Ch_b(\mathcal{C})]$, $\zeta_*(x) = \{\zeta_r(x)\}$, $\zeta_r(x) \in \mathcal{C}$ where $a \le r \le b$ for some $a, b \in \mathbb{Z}$, and where each $\zeta_r \in [\underline{X}, \mathcal{C}]$. Put $F(\zeta_*) = \zeta'_* \in Ch_b[\underline{X}, \mathcal{C}]$ where $\zeta'_* = \{\zeta'_r\}$, $\zeta'_r(x) = \zeta_r(x)$.

4.5. Definition. *Let X, Y be G-sets, and $\underline{X} \times \underline{Y} \xrightarrow{\varphi} \underline{X}$ the functor induced by the projection $X \times Y \xrightarrow{\hat{\varphi}} X$. Let W be a Waldhausen category. If $\zeta \in \mathrm{ob[X,W]}$, we shall write ζ' for $\zeta \circ \varphi \colon X \times Y \to X \to W$. Call a cofibration $\zeta \rightarrowtail \eta$ in $[X, W]$ a Y-cofibration if $\zeta' \to \eta'$ is a split cofibration in $[\underline{X \times Y}, W]$. Call a cofibration sequence $\zeta \rightarrowtail \eta \twoheadrightarrow \delta$ in $[X, W]$ a Y-cofibration sequence if $\zeta' \to \eta' \to \delta'$ is a split cofibration sequence in $[X \times Y, W]$.*
We now define a new Waldhausen category ${}^Y[\underline{X}, W]$ as follows:
$\mathrm{ob}({}^{\mathrm{Y}}[\underline{\mathrm{X}}, \mathrm{W}]) = \mathrm{ob}[\underline{\mathrm{X}}, \mathrm{W}]$. *Cofibrations are Y-cofibrations and weak equivalences are the weak equivalence in $[\underline{X}, W]$.*

4.6. Definition. *With the notations as in 2.5, an object $\zeta \in [\underline{X}, W]$ is said to be Y-projective if every Y-cofibration sequence $\zeta \rightarrowtail \eta \twoheadrightarrow \delta$ in $[\underline{X}, W]$ is a split cofibration sequence. Let $[\underline{X}, W]_Y$ be the full subcategory of $[\underline{X}, W]$ consisting of Y-projective functors. Then $[\underline{X}, W]_Y$ becomes a Waldhausen category with respect to split cofibrations and weak equivalences in $[\underline{X}, W]$.*

5. Equivariant higher *K*-theory constructions for Waldhausen categories

5.1. Absolute and relative equivariant theory

5.1.1. Definitions. *Let G be a finite group X a G-set, W a Waldhausen category, $[\underline{X}, W]$ the Waldhausen category defined in Section 4. We shall write $\mathbb{K}^G(X, W)$ for the Waldhausen K-theory space (or spectrum) $\mathbb{K}([\underline{X}, W])$ and $\mathbb{K}_n^G$ (X, W) for the Waldhausen K-theory group $\pi_n(\mathbb{K}([\underline{X}, W])$. For the Waldhausen category ${}^Y[\underline{X}, W]$, we shall write $\mathbb{K}^G(X, W, Y)$ for the Waldhausen K-theory space (or spectrum) $\mathbb{K}({}^Y[\underline{X}, W])$ with corresponding nth K-theory groups $\mathbb{K}_n^G(X, W, Y)$ $:= \pi_n(\mathbb{K}^Y[\underline{X}, W])$.*

Finally, we denote by $\mathbb{P}^G(X, W, Y)$ the Waldhausen K-theory space (or spectrum) $\mathbb{K}([X, W]_Y)$ with corresponding n-th K-theory group $\pi_n(\mathbb{K}([X, W]_Y)))$ which we denote by $\mathbb{P}_n^G(X, W, Y)$.

5.1.2. Theorem. *Let W be a Waldhausen category, G a finite group, X any G-set. Then, in the notation of* 5.1.1, *we have:* $\mathbb{K}_n^G(-,W)$, $\mathbb{K}_n^G(-,W,Y)$ *and* $\mathbb{P}_n^G(-,W,Y)$ *are Mackey functors:* $GSet \to Ab$.

Proof. Any G-map $f\colon X_1 \to X_2$ defines a covariant functor $\underline{f}\colon \underline{X}_1 \to \underline{X}_2$ given by $x \to f(x)$, $(g,x,x') \mapsto (g,f(x),f(x'))$, and an exact restriction functor $f^*\colon [\underline{X}_2, W] \to [\underline{X}_1, W]$ given by $\zeta \to \zeta \circ f$. Also, f^* maps cofibrations to cofibrations and weak equivalence to weak equivalences. So, we have an induced map $\mathbb{K}_n^G(f,W)^*\colon \mathbb{K}_n^G(X_2,W) \to K_n^G(X_1,W)$ making $\mathbb{K}_n^G(-,W)$ contravariant functor: GSet $\to$ *Ab*. The restriction functor $[\underline{X}_2, W] \to [X_1, W]$ caries Y-cofibrations over $\underline{X}_2$ to Y-cofibrations over X_1 and also Y-projective functors in $[\underline{X}_2, W]$ to Y-projective functors in $[\underline{X}_1, W]$. Moreover, it takes w.e. to w.e. in both cases. Hence we have induced maps

$$\mathbb{K}_n^G(f,W,Y)^*\colon \mathbb{K}_n^G(X_2,W,Y) \to \mathbb{K}_n^G(X_1,W,Y)$$
$$\mathbb{P}_n^G(f,W,Y)^*\colon \mathbb{P}_n^G(X_2,W,Y) \to \mathbb{P}_n^G(X_1,W,Y)$$

making $K_n^G(-,W,Y)^*$, $\mathbb{P}_n^G(-,W,Y)^*$ contravariant functors GSet $\to$ *Ab*. Now, any G-map $f\colon X_1 \to X_2$ also induces an "induction functor" $f_*\colon [\underline{X}_1, W] \to [\underline{X}_2, W]$ as follows. For any functor $\zeta \in \mathrm{ob}[\underline{\mathrm{X}}_1, \mathrm{W}]$, define $f_*(\zeta) \in [\underline{X}_2, W]$ by $f_*(\zeta)(x_2) = \bigoplus_{x_1 \in f^{-1}(x_2)} \zeta(x_1)$; $f_*(\zeta)(g,x_2,x_2') = \bigoplus_{x_1 \in f^{-1}(x_2)} \zeta(g,x_1,gx_1)$. Also for any morphism $\zeta \to \zeta'$ in $[\underline{X}_1, W]$ define $(f_*)(\alpha)(x_2) = \bigoplus_{x_1 \in f^{-1}(x_2)} \alpha(x_1)$; $f_*(\zeta)(x_2) = \bigoplus_{x_1 \in f^{-1}(x_2)} \zeta(x_1) \to f_*(\zeta')(x_2) = \bigoplus_{x_1 \in f^{-1}(x_2)} \zeta'(x_1)$. Also, f_* preserves cofibrations and weak equivalences. Hence we have induced homomorphisms $\mathbb{K}_n^G(f,W)\colon K_n^G(X_1, W) \to K_n^G(X_2,W)$ and $K_n^G(-,W)$ is a covariant functor GSet $\to$ *Ab*. Also the induction functor preserves Y-cofibrations and Y-projective functors as well as weak equivalences. Hence we also have induced homomorphisms

$$\mathbb{K}_n^G(f,W,Y)_*\colon K_n^G(X_1,W,Y) \to \mathbb{K}_n^G(X_2,W,Y)$$
$$\text{and } \mathbb{P}_n^G(f,W,Y)_*\colon \mathbb{P}_n^G(X_1,W,Y) \to \mathbb{P}_n^G(X_2,W,Y)$$

making $\mathbb{K}_n^G(-,W,Y)$, and $\mathbb{P}_n^G(-,W,Y)$ covariant functions GSet $\to$ *Ab*. Also for morphisms $f_1\colon X_1 \to X$, $f_2\colon X_2 \to X$ in GSet and any pullback diagram

$$\begin{array}{ccc} X_1 \underset{X}{\times} X_2 & \xrightarrow{f_2} & X_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow {\scriptstyle f_2} \\ X_1 & \xrightarrow[f_1]{} & X \end{array} \qquad \text{(I)}$$

we have a commutative diagram

$$\begin{array}{ccc} [X_1 \underset{X}{\times} X_2, W] & \longrightarrow & [\underline{X}_2, W] \\ \downarrow & & \downarrow \\ [X_1, W] & \longrightarrow & [X, W] \end{array} \qquad \text{(II)}$$

and hence the commutative diagram obtained by applying $\mathbb{K}_n^G(-,W)$, $\mathbb{K}_n^G(-,W,Y)$ to diagram II above and applying $\mathbb{P}_n^G(-,W,Y)$ to diagram III below:

$$\begin{array}{ccc} [X_1 \underset{X}{\times} X_2, W]_Y & \longrightarrow & [\underline{X}_2, W]_Y \\ \big\downarrow & & \big\downarrow \\ [\underline{X}_2, W]_Y & \longrightarrow & [X, W]_Y \end{array} \qquad \text{(III)}$$

shows that Mackey properties are satisfied. Hence $\mathbb{K}_n^G(-,W)$, $\mathbb{K}_n^G(-,W,Y)$ and $\mathbb{P}_n^G(-,W,Y)$ are Mackey functors.

5.1.3. Theorem. *Let W_1, W_2, W_3 be Waldhausen categories and $W_1 \times W_2 \to W_3$, $(A_1, A_2) \to A_1 \circ A_2$ an exact pairing of Waldhausen categories. Then the pairing induces, for any GSet X, a pairing $[\underline{X}, W_1] \times [\underline{X}, W_2] \to [\underline{X}, W_3]$ and hence a pairing*

$$\mathbb{K}_0^G(X, W_1) \times K_n^G(X, W_2) \to K_n^G(X, W_3).$$

Suppose that W is a Waldhausen category such that the pairing is naturally associative and commutative and there exists $E \in W$ such that $E \circ X \backsimeq X \circ E \backsimeq X$, then $K_0^G(-,W)$ is a Green functor and $K_n^G(-,W)$ is a unitary $K_0^G(-,W)$-module.

Proof. The pairing $W_1 \times W_2 \to W_3$ $(X_1, X_2) \to X_1 \circ X$ induces a pairing $[\underline{X}, W_1) \times [\underline{X}, W_2] \to [\underline{X}, W_3]$ given by $(\zeta_1, \zeta_2) \to \zeta_1 \circ \zeta_2$ where $(\zeta_1 \circ \zeta_2)(x) = \zeta_1(x) \circ \zeta_2(x)$. Now, any $\zeta_1 \in [\underline{X}, W_1]$ induces a functor $\zeta_1^*\colon [\underline{X}, W_2] \to [\underline{X}, W_3]$ given by $\zeta_2 \to \zeta_1 \circ \zeta_2$ which preserves cofibrations and weak equivalences and hence a map

$$\mathbb{K}_n^G(\zeta_1^*)\colon \mathbb{K}_n^G(X, W_2) \to \mathbb{K}_n^G(X, W_3).$$

Now, define a map:

$$\mathbb{K}_0^G(X, W_1) \xrightarrow{\delta} \mathrm{Hom}(\mathbb{K}_n^G(X, W_2), \mathbb{K}_n^G(X, W_3)) \ \ (I)$$

by $[\zeta_1] \to \mathbb{K}_n^G(\zeta_1^*)$. We now show that this map is a homomorphism. Let $\zeta_1' \rightarrowtail \zeta_1 \twoheadrightarrow \zeta_1''$ be a cofibration sequence in $[\underline{X}, W_1]$. Then, we obtain a cofibration sequence of functors $\zeta_1'^* \rightarrowtail \zeta_1^* \twoheadrightarrow \zeta_1''^*\colon [\underline{X}, W_2] \to [\underline{X}, W_3]$ such that for each $\zeta_2 \in [\underline{X}, W_2]$, the sequence $\zeta_1'^*(\zeta_2) \to \zeta_1^*(\zeta_2) \to \zeta_1''^*(\zeta_2)$ is a cofibration sequence in $[\underline{X}, W_3]$. Then by applying the additivity theorem for Waldhausen categories (see [22] or [23]) we have $\mathbb{K}_n^G(\zeta_1'^*) + \mathbb{K}_n^G(\zeta_1''^* = \mathbb{K}_n^G(\zeta_1^*)$. So, δ is a homomorphism and hence we have a pairing $\mathbb{K}_0^G(X, W_1) \times \mathbb{K}_n^G(X, W_2) \to K_n^G(X, W_3)$. One can check easily that far for any G-map $\varphi\colon X' \to X$ the Frobenius reciprocity law holds, i.e.,. For $\xi_i \in [\underline{X}, W_i]$, $\eta_i \in [\underline{X}', W_i]$, $i = 1, 2$, we have canonical isomorphisms

$$\begin{aligned} &f_*(f^*(\zeta_1) \circ \zeta_2) \cong \zeta_1 \circ f_*(\zeta_2) \\ &f_*(\zeta_1 \circ f^*(\zeta_2)) \cong f_*(\zeta_1) \circ \zeta_2 \quad \text{and} \\ &f^*(\zeta_1 \circ \zeta_2) \cong f^*(\zeta_1) \circ f^*(\zeta_2) \end{aligned}$$

Now, the pairing $W \times W \to W$ induces $\mathbb{K}_0^G(X, W) \times \mathbb{K}_0^G(X, W) \to K_0^G(X, W)$ which turns $K_0^G(X, W)$ into a ring with unit such that for any G-map $f\colon X \to Y$, we have $\mathbb{K}_0^G(f, W)_*({}^1\mathbb{K}_0^G(X, W)) \equiv {}^1\mathbb{K}_0^G(Y, W)$. Then ${}^1\mathbb{K}_0^G(X, W)$ acts as the identity on $K_0^G(X, W)$. So, $\mathbb{K}_0^G(X, W)$ is a $K_0^G(X, W)$-module.

5.1.4. Theorem. *Let Y be a G-set, W a Waldhausen category. If the pairing $W \times W \to W$ is naturally associative, commutative and exact and W contains a natural unit, then $\mathbb{K}_0^G(-,W,Y)\colon Gset \to Ab$ is a Green functor and $\mathbb{K}_n^G(-,W,Y)$ and $\mathbb{P}_n^G(-,W,Y)$ are $\mathbb{K}_0^G(-,W,Y)$-modules.*

Proof. Note that for any G-set Y, the pairing $[\underline{X},W] \times [\underline{X},W] \to [\underline{X},W]$ takes Y-cofibration sequence to Y-cofibration sequences and Y-projective functors to Y-projective functors and so, we have induced pairing ${}^Y[\underline{X},W] \times {}^Y[\underline{X},W] \to {}^Y[\underline{X},W]$ inducing a pairing $\mathbb{K}_0^G(X,W,Y) \times \mathbb{K}_n^G(X,W,Y) \to \mathbb{K}_n^G(X,W,Y)$ as well as induced pairing ${}^Y[\underline{X},W] \times [\underline{X},W]_Y \to [\underline{X},W]_Y$ yielding K-theoretic pairing $\mathbb{K}_0^G(X,W,Y) \times \mathbb{P}_n^G(X,W,Y) \to \mathbb{P}_n^G(X,W,Y)$. If $W \times W$ is naturally associative and commutative and W has a natural unit, then $K_0^G(-,W,Y)$ is a Green functor and $P_n^G(-,W,Y)$ and $K_n^G(-,W,Y)$ are $K_0^G(-,W,Y)$-modules.

5.1.5. Remarks. (1) It is well known that the Burnside functor $\Omega\colon G\mathrm{Set} \to Ab$ is a Green functor and that any Mackey functor $M\colon G\mathrm{Set} \to Ab$ is an Ω-module and that any Green functor is an Ω-algebra (see [1], [9], [14]). Hence the above K-functors $\mathbb{K}_n^G(-,W,Y)$, $\mathbb{P}_n^G(-,W,Y)$ and $\mathbb{K}_n^G(-W)$ are Ω-modules, and $\mathbb{K}_o^G(-,W,Y)$ and $\mathbb{K}_o^G(-,W)$ are Ω-algebra.

(2) Let M be any Mackey functor: $G\mathcal{S}et \to Ab$, X a GSet. Define $K_M(X)$ as the kernel of $M(G/G) \to M(X)$ and $I_M(X)$ as the image of $M(X) \to M(G/G)$. An important induction result is that $|G|M(G/G) \subseteq K_M(X) + I_M(X)$ for any Mackey functor M and GSet X. This result also applies to all the K-theory functors defined above.

(3) If M is any Mackey functor $G\mathcal{S}et \to Ab$ X a GSet, define a Mackey functor $M_X\colon G\mathcal{S}et \to Ab$ by $M_X(Y) = M(X \times Y)$. The projection map $\mathrm{pr}\colon X \times Y \to Y$ defines a natural transformation $\Theta_X\colon M_X \to M$ where $\Theta_X(Y) = \mathrm{pr}\colon M(X \times Y) \to M(Y)$. M is said to be X-projective if Θ_X is split surjective (see [1], [14]). Now define the defect base D_M of M by $D_M = \{H \le G \mid X^H \ne \phi\}$ where X is a GSet (called the defect set of M) such that M is Y-projective iff there exists a G-map $f\colon X \to Y$ (see [14]). If M is a module over a Green functor $\mathcal{G}$, then M is X-projective iff $\mathcal{G}$ is X-projective iff the induction map $\mathcal{G}(X) \to \mathcal{G}(G/G)$ is surjective. In general proving induction results reduce to determining G-sets X for which $\mathcal{G}(X) \to \mathcal{G}(G/G)$ is surjective and this in turn reduces to computing $D_{\mathcal{G}}$. Thus one could apply induction techniques to obtain results on higher K-groups which are modules over the Green functors $\mathbb{K}_0^G(-,W)$ and $K_0^G(-,W,Y)$ for suitable W (e.g. $W = Ch_b(\mathcal{C})$, $\mathcal{C}$ a suitable exact category (see §5, as well as [3]).

(4) One can show via general induction theory principles that for suitably chosen W all the higher K-functors $\mathbb{K}_n^G(-,W)$, $\mathbb{K}_n^G(-,W,Y)$ and $\mathbb{P}_n^G(-,W,Y)$ are "hyper-elementary computable" – see [2], [6], [9], [13].

5.2. Equivariant additivity theorem

In this subsection, we present an equivariant version of additivity theorem below (5.2.3) for Waldhausen categories. First we review the non-equivariant situation.

5.2.1. Definition. *Let W, W' be Waldhausen categories. Say that a sequence $F' \rightarrowtail F \twoheadrightarrow F''$ of exact functors $F', F, F''\colon W \to W'$ is a cofibration sequence of exact functors if each $F'(A) \rightarrowtail F(A) \twoheadrightarrow F''(A)$ is a cofibration in W' and if for every cofibration $A \rightarrowtail B$ in W $F(A) \bigcup\limits_{F'(A)} F'(B) \to F(B)$ is a cofibration in W'.*

5.2.2. Theorem. (Additivity theorem) ([17], [24]). *Let W, W' be Waldhausen categories, and $F' \rightarrowtail F \twoheadrightarrow F''$ a cofibration sequence of exact functors from W to W'. Then $F_* \simeq F'_* + F''_*\colon \mathbb{K}_n(W) \to \mathbb{K}_n(W')$.*

5.2.3. Equivariant additivity theorem. *Let W, W' be Waldhausen categories, X, Y, $GSets$, and $F' \rightarrowtail F \twoheadrightarrow F''$ cofibration sequence of exact functors from W to W'. Then $F' \rightarrowtail F \twoheadrightarrow F''$ induces a cofibration sequence $\widehat{F}' \rightarrowtail \widehat{F} \twoheadrightarrow \widehat{F}''$ of exact functors from $[\underline{X}, W]$ to $[\underline{X}, W']$; from ${}^Y[\underline{X}, W]$ to ${}^Y[\underline{X}, W']$; and from $[\underline{X}, W]_Y$ to $[\underline{X}, W']_Y$ and hence so we have induced homomorphisms*

$$\begin{aligned} \widehat{F}_* \cong \widehat{F}'_* + \widehat{F}''_*\colon \mathbb{K}_n^G(\underline{X}, W) &\to \mathbb{K}_n^G(\underline{X}, W') \\ \mathbb{K}_n^G(\underline{X}, W, Y) &\to \mathbb{K}_n^G(\underline{X}, W', Y) \\ \text{and} \quad \mathbb{P}_n^G(\underline{X}, W, Y) &\to \mathbb{P}_n^G(\underline{X}, W', Y) \end{aligned}$$

Proof. First note that $[\underline{X}, W]$, $[\underline{X}, W']$; ${}^Y[\underline{X}, W]$, ${}^Y[\underline{X}, W']$ and $[\underline{X}, W]_Y$, $[\underline{X}, W']_Y$ are all Waldhausen categories. Now define $\widehat{F}'$, $\widehat{F}$ and $\widehat{F}''\colon [\underline{X}, W] \to [\underline{X}, W']$ by $\widehat{F}'(\zeta)(x) = F'(\zeta(x))$, $\widehat{F}(\zeta)(x) = F(\zeta(x))$ and $\widehat{F}''(\zeta)(x) = F''(\zeta(x))$. Then one can check that $\widehat{F}' \to \widehat{F} \to \widehat{F}''$ is a cofibration sequence of exact functors $[X, W] \to [\underline{X}, W']$. ${}^Y[\underline{X}, W] \to {}^Y[X, W']$. and $[\underline{X}, W]_Y \to [\underline{X}, W']_Y$. Result then follows by applying 5.2.2.

5.3. Equivariant Waldhausen fibration sequence

In this subsection, we present an equivariant version of Waldhausen fibration sequence. First we define the necessary notion and state the non-equivariant version.

5.3.1. Definition. **Cylinder functors** *A Waldhausen category has a cylinder functor if there exists a functor $T\colon ArW \to W$ together with three natural transformations p, j_1, j_2 such that to each morphism $f\colon A \to B$, T assigns an object Tf of W and $j_1\colon A \to Tf$, $j_2\colon B \to Tf$, $p\colon Tf \to B$ satisfying certain properties (see* [4], [24]*).*

Cylinder Axiom. *For all f, $p\colon Tf \to B$ is in $w(W)$.*

5.3.2. Let W be a Waldhausen category. Suppose that W has two classes of weak equivalences $\nu(W)$, $w(W)$ such that $\nu(W) \subset w(W)$. Assume that $w(W)$ satisfies the saturation and extension axioms and has a cylinder functor T which satisfies the cylinder axiom. Let W^w be the full subcategory of W whose objects are those $A \in W$ such that $0 \to A$ is in $w(W)$. Then W^w becomes a Waldhausen category with $co(W^w) = co(W) \cap W^w$ and $\nu(W^w) = \nu(W) \cap (W^w)$.

5.3.3. Theorem. (Waldhausen fibration sequence [24]). *With the notations and hypothesis of* 5.3.2*, suppose that* W *has a cylinder functor* T *which is a cylinder functor for both* $\nu(W)$ *and* $\omega(W)$*. Then the exact inclusion functors* $(W^\omega, \nu) \to (W, \omega)$ *induce a homotopy fibre sequence of spectra*

$$\mathbb{K}(W^\omega, \nu) \to \mathbb{K}(W, \nu) \to \mathbb{K}(W, \omega)$$

and hence a long exact sequence

$$\mathbb{K}_{n+1}(W, \omega) \to \mathbb{K}_n(W^\omega) \to \mathbb{K}_n(W, \nu) \to \mathbb{K}_n(W, \omega) \to$$

5.3.4. Now let W be a Waldhausen category with two classes of weak equivalences $\nu(W)$ and $\omega(W)$ such that $\nu(W) \subset \omega(W)$. Then for any GSet X, $[\underline{X}, W]$ is a Waldhausen category with two choices of w.e. $\hat{\nu}[\underline{X}, W]$ and $\hat{\omega}[\underline{X}, W]$ and $\hat{\nu}[\underline{X}, W) \subseteq \hat{\omega}[\underline{X}, W]$ where a morphism $\zeta \xrightarrow{f} \zeta'$ in $\hat{\nu}[\underline{X}, W]$ (resp. $\hat{\omega}[\underline{X}, W]$ if $f(x)\colon \zeta(x) \to \zeta'(x)$ is in νW (resp. $\omega(W)$.) One can easily check that if $\omega(W)$ satisfies the saturation axiom so does $\hat{\omega}[\underline{X}, W]$ (see 2.3. iii). Suppose that $\omega(W)$ has a cylinder functor $T\colon Ar\,W \to W$ which also satisfies cylinder axiom. .... for all $f : A \to B$, in W, the map $p\colon Tf \to B$ is in $\omega(W)$, then T induces a functor $\widehat{T}\colon Ar([\underline{X}, W]) \to [\underline{X}, W]$ defined by $\widehat{T}(\zeta \to \zeta')(x) = T(\zeta(x) \to \zeta'(x))$ for any $x \in X$. Also, for an map $f\colon \zeta \to \zeta'$ in $[\underline{X}, W]$ the map $\hat{p}\colon \widehat{T}(f) \to \zeta' \in \hat{\omega}([\underline{X}, W])$. Let $[\underline{X}, W]^{\hat{\omega}}$ be the full subcategory of $[\underline{X}, W]$ such that $\zeta_0 \to \zeta \in \hat{\omega}[\underline{X}, W]$ where $\zeta_0(x) = 0 \in W$ for all $x \in X$. Then $[\underline{X}, W]^{\hat{\omega}}$ is a Waldhausen category with $co([\underline{X}, W]^{\hat{\omega}}) = co([\underline{X}, W) \cap [\underline{X}, W)^{\hat{\omega}})$ and $\nu([\underline{X}, W])^{\hat{\omega}} = \hat{\nu}[\underline{X}, W) \cap [\underline{X}, W]^{\hat{\omega}}$. We now have the following

5.3.5. Theorem. (Equivariant Waldhausen fibration sequence) *Let* W *be a Waldhausen category with a cylinder functor* T *and which also has a cylinder functor for* $\nu(W)$ *and* $\omega(W)$*. Then, in the notation of* 5.3.4*, we have exact inclusions* $([\underline{X}, W]^{\hat{\omega}}, \hat{\nu}) \to ([\underline{X}, W], \hat{\nu})$ *and* $([\underline{X}, W], \hat{\nu}) \to ([\underline{X}, W], \hat{\omega})$ *which induce a homotopy fibre sequence of spectra*

$$\mathbb{K}([\underline{X}, W]^{\hat{\omega}}, \hat{\nu}) \to \mathbb{K}([\underline{X}, W], \hat{\nu}) \to \mathbb{K}([\underline{X}, W], \hat{\omega})$$

and hence a long exact sequence

$$\ldots \mathbb{K}_{n+1}([\underline{X}, W], \hat{\omega}) \to \mathbb{K}_n([\underline{X}, W]^{\hat{\omega}}, \hat{\nu}) \to \mathbb{K}_n([\underline{X}, W], \hat{\nu}) \to \mathbb{K}_n([\underline{X}, W], \hat{\omega}) \ldots$$

Proof. Similar to that of 5.3.3.

6. Applications to complicial bi-Waldhausen categories

In this section, we shall focus attention on Waldhausen categories of the form $Ch_b(\mathcal{C})$ where $\mathcal{C}$ is an exact category. Recall from [3] that if $\mathcal{C}$ is an exact category and X, Y, Gsets, $K_n^G(X, \mathcal{C})$ is the nth (Quillen) algebraic K-group of the exact category $[\underline{X}, \mathcal{C}]$ with respect to fibre-wise exact sequences; $K_n^G(X, \mathcal{C}, Y)$ is the nth (Quillen) algebraic K-group of the exact category $[\underline{X}, \mathcal{C}]$ with respect to Y-exact sequences while $P_n^G(\underline{X}, \mathcal{C}, Y)$ is the nth (Quillen) algebraic K-group of the category $[\underline{X}, \mathcal{C}]$ of Y-projective functors in $[\underline{X}, \mathcal{C}]$ with respect to split exact sequences. We now have the following result

6.1. Theorem. *Let G be a finite group, X, Y GSets, $\mathcal{C}$ an exact category. Then*
(1) $K_n^G(X, \mathcal{C}) \cong \mathbb{K}_n^G(X, Ch_b(\mathcal{C}))$
(2) $K_n^G(X, \mathcal{C}, Y) \cong \mathbb{K}_n^G(X, Ch_b(\mathcal{C}), Y)$
(3) $P_n^G(X, \mathcal{C}, Y) \cong \mathbb{P}_n^G(X, Ch_b(\mathcal{C}), Y)$

Proof. (1) Note that $[\underline{X}, \mathcal{C}]$ is an exact category and $[\underline{X}, Ch_b(\mathcal{C})] \simeq Ch_b([\underline{X}, \mathcal{C}])$ is a complicial bi-Waldhausen category. Now identify $\zeta \in [\underline{X}, \mathcal{C}]$ with the object ζ_* in $Ch_b[\underline{X}, \mathcal{C}]$ defined by $\zeta_*(x)$ = chain complex consisting of a single object $\zeta(x)$ in degree zero and zero elsewhere. The result follows by applying the Gillet-Waldhausen theorem.

(2) Recall that $\mathbb{K}_n^G(X, Ch_b(\mathcal{C}), Y)$ is the Waldhausen K-theory of the Waldhausen category ${}^Y[\underline{X}, Ch_b(\mathcal{C})]$ where $\mathrm{ob}^{\mathrm{Y}}[\underline{\mathrm{X}}, \mathrm{Ch_b}(\mathcal{C})] = \mathrm{ob}[\underline{\mathrm{X}}, \mathrm{Ch_b}(\mathcal{C})]$, cofibrations are Y-cofibrations in $[\underline{X}, Ch_b(\mathcal{C})]$ and weak equivalences are the weak equivalences in $(\underline{X}, Ch_b(\mathcal{C})]$. Also, $K_*^G(X, \mathcal{C}, Y)$ is the Quillen K-theory of the exact category $[\underline{X}, \mathcal{C}]$ with respect to Y-exact sequences. Denote this exact category by ${}^Y[\underline{X}, \mathcal{C}]$. We can define an inclusion functor ${}^Y[\underline{X}, \mathcal{C}] \subseteq CH_b({}^Y[\underline{X}, \mathcal{C}]) \cong {}^Y[\underline{X}, Ch_b(\mathcal{C})]$ as in (1) and apply Gillet-Waldhausen theorem.

(3) Just as in the last two cases, we can define an inclusion functor from the exact category $[\underline{X}, \mathcal{C}]_Y$ to the Waldhausen category $Ch_b([\underline{X}, \mathcal{C}]_Y) \simeq [\underline{X}, Ch_b(\mathcal{C})]_Y$ and apply Gillet-Waldhausen theorem.

6.2. Remarks. Applications to higher K-theory of group-rings:

(1) Recall from [3] that if $X = G/H$ where H is a subgroup of G and R is a commutative ring with identity, we can identify $[G/H, \underline{\underline{M}}'(R)]$ with $\underline{\underline{M}}'(RH)$ and $[G/H, \underline{P}(R)]$ with $\underline{P}_R(RH)$. Hence we can identify $[G/H, Ch_b(\underline{\underline{M}}'(R)]$ with $Ch_b(\underline{\underline{M}}'(RH))$ and $[G/H, Ch_b(\underline{P}(R)]$ with $Ch_b(\underline{P}_R(RH))$. So, we can identify $K_n^G(G/H, \underline{\underline{M}}'(R))$ with $K_n(\underline{\underline{M}}'(RH)) = G_n(RH)$ when R is Noetherian. By 4.1, we can identify $\mathbb{K}_n^G(G/H, Ch_b(\underline{\underline{M}}'(R)))$ with $\mathbb{K}_n(Ch_b(\underline{\underline{M}}'(RH))) \simeq G_n(RH)$ by Gillet-Waldhausen theorem. Also $K_n^G(G/H, \underline{P}(R)) \simeq \mathbb{K}_n(Ch_b\underline{P}_R(RH)) \simeq K_n(\underline{P}_R(RH)) \simeq G_n(R, H)$ by Gillet-Waldhausen result.

(2) With the notations above, we can identify $K_n^G(G/H, \underline{\underline{M}}'(R), Y)$ (resp. K_n^G $(G/H, \underline{P}(R), Y)$ with Quillen K-theory of the exact category $\underline{\underline{M}}'(RH)$ (resp.

$\underline{P}_R(RH))$ with respect to exact sequences which split when restricted to the various subgroups H' of H with a non-empty fixed point set $Y^{H'}$ (see [3], [9]). In particular

$$K_n^G(G/H, \underline{\underline{M}}'(R), G/e) \simeq K_n^G(G/H, \underline{\underline{M}}'(R)) \simeq K_n(\underline{\underline{M}}'(RH) \simeq G'_n(RH)$$

and

$$K_n^G(G/H, \underline{P}(R), G/e) \simeq K_n^G(G/H, \underline{P}(R)) \simeq K_n(\underline{P}_R(RH) \cong G_n(R, H).$$

Hence we also have

$$\mathbb{K}_n^G(G/H, Ch_b(\underline{\underline{M}}'(R), G/e) \simeq \mathbb{K}_n^G(G/H, Ch_b(\underline{\underline{M}}'(R))$$
$$\mathbb{K}_n(Ch_b(\underline{\underline{M}}'(RG))) \simeq K_n(\underline{\underline{M}}'(RG)) \simeq G'_n(RG)$$

by Gillet-Waldhausen theorem.

(3) Recall from [3] $P_n^G(G/H, \underline{\underline{M}}'(R), Y)$ (resp. $P_n^G(G/H), \underline{P}(R), Y))$ are the Quillen K-groups of the exact category $\underline{\underline{M}}'(RH)$ (resp. $\mathbb{P}_R(RH))$ that are relatively projective with respect to $D(Y, H) = \{H' \leq H \mid Y^{H'} \neq \phi\}$. In particular $P_n^G(G/H, \underline{P}(R), G/e) \equiv K_n(\underline{P}(RH) \simeq K_n(RH)$. Hence we can identify $\mathbb{P}_n^G(G/H, Ch_b(\underline{P}(R)), G/e)$ with $\mathbb{K}_n(Ch_b(\underline{P}(RH)) \simeq K_n(RH)$ by Gillet-Waldhausen theorem.

(4) In view of 6.1, we recover the relevant results and computations in [3], [9].

6.3. We now record below (6.4) an application of Waldhausen fibration sequence 5.3.3, 5.3.5 and Garkusha's result [4] 3.1.

6.4. Theorem. (1) *In the notations of* 6.1, 6.2*, let R be a commutative ring with identity G a finite group, $\underline{\underline{M}}'(RG)$ the category of finitely presented RG-modules $Ch_b(\underline{\underline{M}}'(RG))$ the Waldhausen category of bounded complexes over $\underline{\underline{M}}'(RG)$ with weak equivalences being stable quasi-isomorphism (see* 3.1.6 (iv), (v)*). Then we have a long exact sequence for all $n \geq 0$*

$$\to \mathbb{K}_{n+1}(Ch_b(\underline{\underline{M}}'(RG), \omega) \to \mathbb{P}_n^G(G/G, Ch_b(\underline{P}(R)), G/e) \dots$$
$$\to \mathbb{K}_n^G(G/G, Ch_b(\underline{\underline{M}}'(R), G/e) \to \mathbb{K}_n(Ch_b(\underline{\underline{M}}'(RG), \omega) \to \dots$$

(2) *If in* (1)*, R is the ring of integers in a number field, then for all $n \geq 1$, $\mathbb{K}_{n+1}(Ch_b(\underline{\underline{M}}'(RG), \omega)$ is a finite Abelian group.*

Proof. From 6.1, 6.2 we have

$$\mathbb{P}_n^G(G/G, Ch_b(\underline{P}(R)), G/e) \cong P_n^G(G/G, \underline{P}(R), G/e) \simeq K_n(RG)$$

and

$$\mathbb{K}_n^G(G/G, Ch_b(\underset{=}{M}'(R), G/e) \simeq K_n^G(G/G, \underset{=}{M}'(R), G/e) \cong G'_n(RG).$$

Hence the long exact sequence follows from [4] 3.1. Now, if R is the ring of integers in a number field F, then RG is an R-order in a semi-simple F-algebra FG and so by [7], [10], $K_n(RG)$, $G_n(RG)$ are finitely generated Abelian groups for all $n \geq 1$. Hence for all $n \geq 1$, $K_{n+1}(Ch_b(\underset{=}{M}(RG), \omega)$ is finitely generated. So, to show that $K_{n+1}(Ch_b(\underset{=}{M}(RG), \omega)$ is finite, we only have to show that it is torsion. Now let $\alpha_n \colon K_n(RG) \to G_n(RG)$ be the Cartan map which is part of the exact sequence

$$\cdots \to K_{n+1}(Ch_b(\underset{=}{M}(RG), \omega) \to K_n(RG) \xrightarrow{\alpha_n} G_n(RG) \to K_n(Ch_b(\underset{=}{M}(RG), \omega) \to \cdots \quad \text{(I)}$$

From this sequence we have a short exact sequence

$$0 \to \text{Coker } \alpha_{n+1} \to K_{n+1}(Ch_b(\underset{=}{M}(RG), \omega) \to \text{Ker } \alpha_n \to 0 \quad \text{(II)}$$

for all $n \geq 1$. So, it suffices to prove that kerα_n is finite and Coker α_{n+1} is torsion. Now, from the commutative diagram

$$\begin{array}{ccc} K_n(RG) & \xrightarrow{\alpha_n} & G_n(RG) \\ \searrow \beta_n & & \swarrow \gamma_n \\ & K_n(FG) & \end{array}$$

we have an exact sequence $0 \to \text{Ker } \alpha_n \to SK_n(RG) \to SG_n(RG) \to \text{Coker } \alpha_n \to \text{Coker } \beta_n \to \text{Coker } \gamma_n \to 0$. Now for all $n \geq 1$, $SK_n(RG)$ is finite (see [10] or [11]). Hence Ker α_n is finite for all $n \geq 1$. Also, $SG_n(RG)$ is finite for all $n \geq 1$ (see [6] or [7]) and Coker β_n is torsion (see [12], 1.7). Hence Coker α_n is torsion. So, from (II), $K_{n+1}(Ch_b(\underset{=}{M}(RG), \omega)$ is torsion. Since it is also finitely generated, it is finite.

We close this section with a presentation of an equivariant approximation theorem for complicial bi-Waldhausen categories.

6.5. Theorem. (Equivariant approximation theorem) *Let $W = Ch_b(\mathcal{C})$ and $W' = Ch_b(\mathcal{C}')$ be two complicial bi-Waldhausen categories where $\mathcal{C}$, $\mathcal{C}'$ are exact categories. $F \colon W \to W'$ an exact functor. Suppose that the induced map of derived categories $D(W) \to D(W')$ is an equivalence of categories. Then for any GSet X, the induced map of spectra $\mathbb{K}(F) \colon \mathbb{K}([\underline{X}, W]) \to \mathbb{K}([\underline{X}, W'])$ is a homotopy equivalence.*

Proof. An exact functor $F : Ch_b(\mathcal{C}) \to Ch_b(\mathcal{C}')$ induces a functor

$$\widehat{F} \colon [\underline{X}, Ch_b(\mathcal{C})] \to [X, Ch_b(\mathcal{C}')], \quad \zeta \to \widehat{F}(\zeta),$$

where $\widehat{F}(\zeta(x) = F(\zeta(x))$. Now suppose that the induced map $D(Ch_b(\mathcal{C}) \to D(Ch_b(\mathcal{C}'))$ is an equivalence of categories. Note that $D(Ch_b(\mathcal{C}))$ (resp. $D(Ch_b(\mathcal{C}'))$

is obtained from $Ch_b(\mathcal{C})$ (resp. $Ch_b(\mathcal{C}')$) by formally inverting quasi-isomorphisms. Now a map $\zeta \to \eta$ in $[\underline{X}, Ch_b(\mathcal{C})]$ is a quasi-isomorphism iff $\zeta(x) \to \eta(x)$ is a quasi-isomorphism in $Ch_b(\mathcal{C})$. The proof is now similar to [5] 5.2.

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