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Multiplication Modules and Tensor Product

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Abstract. All rings are commutative with identity and all modules are unital. The tensor product of projective (resp. flat, multiplication) modules is a projective (resp. flat, multiplication) module but not conversely. In this paper we give some conditions under which the converse is true. We also give necessary and sufficient conditions for the tensor product of faithful multiplication Dedekind (resp. Prüfer, finitely cogenerated, uniform) modules to be a faithful multiplication Dedekind (resp. Prüfer, finitely cogenerated, uniform) module. Necessary and sufficient conditions for the tensor product of pure (resp. invertible, large, small, join principal) submodules of multiplication modules to be a pure (resp. invertible, large, small, join principal) submodule are also considered.

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0. Introduction

Let R be a commutative ring with identity and M an R-module. M is a *multiplication* module if every submodule N of M has the form IM for some ideal I of R. Equivalently, $N = [N : M]M$, [9]. A submodule K of M is multiplication if and only if $N \cap K = [N : K]K$ for all submodules N of M, [22, Lemma 1.3].

Anderson [8], defined $\theta(M) = \sum_{m\in M} [Rm : M]$ and showed the usefulness of this ideal in studying multiplication modules. He proved for example that if M is multiplication then $M = \theta(M)M$, and a finitely generated module M is multiplication if and only if $\theta(M) = R$, [8, Proposition 1 and Theorem 1]. M is multiplication if and only if $R = \theta(M) + \mathrm{ann}(m)$ for each $m \in M$, equivalently, $Rm = \theta(M)m$ for each $m \in M$, [4, Corollary 1.2] and [24, Theorem 2].

Let P be a maximal ideal of R. An R-module M is called P-*torsion* provided for each $m \in M$ there exists $p \in P$ such that $(1-p)\,m = 0$. On the other hand M is called P-*cyclic* provided there exist $x \in M$ and $q \in P$ such that $(1-q)\,M \subseteq Rx$. El-Bast and Smith, [11, Theorem 1.2], showed that M is multiplication if and only if M is P-torsion or P-cyclic for each maximal ideal P of R. If M is a faithful multiplication R-module then it is easily verified that $\mathrm{ann}N = \mathrm{ann}\,[N : M]$ for every submodule N of M.

Let R be a ring and M an R-module. Then M is *faithfully flat* if M is flat and for all R-modules N, $N \otimes M = 0$ implies that $N = 0$, equivalently, M is flat and $PM \neq M$ for each maximal ideal P of R, [18, Theorem 7.2]. Faithfully flat modules are faithful flat but not conversely. The $\mathbb{Z}$-module Q is faithful flat but not faithfully flat. Finitely generated faithful multiplication modules are faithfully flat, [4, Corollary 2.7], [19, Theorem 4.1] and [24, Corollary 2 to Theorem 9].

A submodule N of an R-module M is called *pure in* M if the sequence $0 \to N \otimes E \to M \otimes E$ is exact for every R-module E, [18]. Let N be a submodule of a flat R-module M. Then N is pure in M if and only if $AN = AM \cap N$ for all ideals A of R, [12, Corollary 11.21]. In particular, an ideal I of R is pure if and only if $AI = A \cap I$ for all ideals A of R. Consequently, an ideal I of R is pure if and only if $A = AI$ for all ideals $A \subseteq I$. Pure ideals are locally either zero or R. Pure submodules of flat modules are flat, from which it follows that pure ideals are flat ideals. It is shown, [4, Corollary 2.7] and [19, Theorem 4.1], that if M is a multiplication module with pure annihilator then M is flat. Anderson and Al Shaniafi, [7, Theorem 2.3], showed that if M is a faithful multiplication module then $\theta\,(M)$ is a pure ideal of R, equivalently, $\theta\,(M)$ is multiplication and idempotent, [3, Theorem 1.1].

The *trace* ideal of an R-module M is $\mathrm{Tr}(M) = \sum_{f\in \mathrm{Hom}(M,R)} f(M)$, [13]. If M is projective then $M = \mathrm{Tr}(M)M$, $\mathrm{ann}M = \mathrm{ann}\mathrm{Tr}(M)$, and $\mathrm{Tr}(M)$ is a pure ideal of R, [12, Proposition 3.30]. It is shown, [7, Theorem 2.6], that if M is a faithful multiplication module then $\theta(M) = \mathrm{Tr}(M)$. If M is a finitely generated multiplication R-module such that $\mathrm{ann}M = Re$ for some idempotent e, then M is projective, [24, Theorem 11]. In particular, finitely generated faithful multiplication modules are projective.

Let R be a ring and M an R-module. M is called *finitely cogenerated* if for every non-empty collection of submodules $N_\lambda\,(\lambda \in \Lambda)$ of M with $\bigcap_{\lambda\in\Lambda} N_\lambda = 0$, there exists a finite subset Λ' of Λ such that $\bigcap_{\lambda\in\Lambda'} N_\lambda = 0$. A submodule N of M is called *large* (or *essential*) *in* M if for all submodules K of M, $K \cap N = 0$ implies $K = 0$. Dually, N is *small* (or *superfluous*) *in* M if for all submodules K of M,

$K + N = M$ implies $K = M$. For properties of finitely cogenerated, large and small modules, see [15].

Let R be a commutative ring with identity. Generalizing the case for ideals, an R-module M is defined to be a *cancellation* module if $IM = JM$ for ideals I and J of R implies $I = J$, [6] and [20]. Examples of cancellation modules include invertible ideals, free modules and finitely generated faithful multiplication modules. It is easy to check that if M is a finitely generated faithful multiplication (hence cancellation) module, then $I\,[N : M] = [IN : M]$ for each submodule N of M and each ideal I of R. It is also defined that M is a *weak cancellation* module if $IM = JM$ implies $I + \mathrm{ann}M = J + \mathrm{ann}M$ and M is a *restricted cancellation* module if $IM = JM \neq 0$ implies $I = J$. An R-module M is cancellation if and only if it is a faithful weak cancellation module. A submodule N of M is said to be *join principal* if for all ideals A of R and all submodules K of $M, [(AN + K) : N] = A + [K : N]$. Setting $K = 0, N$ becomes weak cancellation.

In Section 1 we show that the tensor product of faithful multiplication (resp. faithfully flat) modules is a faithful multiplication (resp. faithfully flat) module, Theorem 2. We also give sufficient conditions on the tensor product of modules for them to be finitely generated (resp. multiplication, flat, finitely generated projective, faithfully flat), Proposition 3. Theorem 11 gives necessary and sufficient conditions for the tensor product of faithful multiplication Dedekind (resp. Prüfer) modules to be a faithful multiplication Dedekind (resp. Prüfer) module.

In Section 2 we investigate large and small submodules of multiplication modules. Several properties of these modules are given in Proposition 12. We also give necessary and sufficient conditions for the tensor product of large (resp. small, finitely cogenerated, uniform) modules to be a large (resp. small, finitely cogenerated, uniform) module, Corollary 13 and Proposition 17.

Section 3 is concerned with join principal submodules. Propositions 18, 19, 21 and 22 give necessary and sufficient conditions for the product, intersection, sum and tensor product of join principal submodules (ideals) of multiplication modules to be join principal modules.

All rings considered in this paper are commutative with 1 and all modules are unital. For the basic concepts used, we refer the reader to [12]–[16] and [18].

1. Projective, flat and multiplication modules

It is well known that the tensor product of projective (flat) R-modules is projective (flat), see for example [12, pp. 431, 435]. In [4, Theorem 2.1] it is proved that the tensor product of multiplication R-modules is multiplication. The converses of these statements are not necessarily true. Let M be the maximal ideal of a non-discrete rank one valuation ring R, [8, p. 466]. Then M is an idempotent but not a multiplication ideal of R. Note that $R/M \otimes M \cong M/M^2 = 0$ is a multiplication module. Let $R = \mathbb{Z}_4$ and $I = R2$. Then I is not a projective ideal of R, and the R-module $M = I \oplus I$ is not projective (in fact it is not flat), but $M \otimes M$ is a projective and flat R-module. In this note we give some conditions under which the converse is true.

We begin with the following lemma that collects results from [2, Propositions 2.2 and 3.7], [17, Lemma 1.4], and [24, Theorem 10].

Lemma 1. *Let R be a ring and N a submodule of a finitely generated faithful multiplication R-module M.*

(1) *N is finitely generated if and only if $[N : M]$ is a finitely generated ideal of R.*
(2) *N is multiplication if and only if $[N : M]$ is a multiplication ideal of R.*
(3) *N is flat if and only if $[N : M]$ is a flat ideal of R.*
(4) *If N is finitely generated then N is projective if and only if $[N : M]$ is a projective ideal of R.*

Let M be a flat R-module. Then $IM \cong I \otimes M$ for each ideal I of R, [12, Theorem 11.20]. The condition that M is flat can not be discarded. For let $R = \mathbb{Z}$, $M = \mathbb{Z}_2$ and $I = 2\mathbb{Z}$. Then $IM = 0$ but $I \otimes M = 2\mathbb{Z} \otimes \mathbb{Z}_2 \cong \mathbb{Z} \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2$. Suppose that R is a ring and M_1, M_2 flat (in particular, faithful multiplication) R-modules. Then

$$(I \otimes J)(M_1 \otimes M_2) \cong IM_1 \otimes JM_2 \cong JM_1 \otimes IM_2$$

for all ideals I and J of R. The next theorem shows that the tensor product of faithful multiplication (resp. faithfully flat) modules is a faithful multiplication (resp. faithfully flat) module.

Theorem 2. *Let R be a ring and M_1, M_2 R-modules.*

(1) *If M_1 and M_2 are faithful multiplication then so too is $M_1 \otimes M_2$.*
(2) *If M_1 and M_2 are faithfully flat then so too is $M_1 \otimes M_2$.*
(3) *If M_1 and M_2 are finitely generated faithful multiplication then so too is $M_1 \otimes M_2$.*

Proof. (1) The fact that $M_1 \otimes M_2$ is a multiplication R-module can be found in [1, Theorem 2.3], see also [4, Theorem 2.1]. We show that $M_1 \otimes M_2$ is faithful. We obtain from [8, Proposition 1] and [7, Theorem 2.6], that $M_i = \mathrm{Tr}\,(M_i)\, M_i$ for $i = 1, 2$. Since $\mathrm{ann}\mathrm{Tr}\,(M_i) = \mathrm{ann}M_i$ for $i = 1, 2$, we infer that $\mathrm{Tr}\,(M_1)\,\mathrm{Tr}\,(M_2)$ is a faithful ideal of R. Next,

$$M_1 \otimes M_2 = \mathrm{Tr}\,(M_1)\, M_1 \otimes \mathrm{Tr}\,(M_2)\, M_2 \cong (\mathrm{Tr}\,(M_1) \otimes \mathrm{Tr}\,(M_2))\,(M_1 \otimes M_2)\,.$$

And $\mathrm{Tr}\,(M_1 \otimes M_2) \cong \mathrm{Tr}\,(M_1) \otimes \mathrm{Tr}\,(M_2)$. For, if $f \in \mathrm{Hom}(M_1, R), g \in \mathrm{Hom}\,(M_2, R)$, $m_1 \in M_1, m_2 \in M_2, (f \otimes g)(m_1 \otimes m_2) = f(m_1) \otimes g(m_2)$, [12, Proposition 11.3] and [15, p. 19]. Since $\mathrm{Tr}\,(M_2)$ is idempotent and multiplication, [7, Theorem 2.3], and hence pure, [3, Theorem 1.1], it follows that $\mathrm{Tr}\,(M_2)$ is a flat ideal of R and hence

$$\mathrm{Tr}\,(M_1)\,\mathrm{Tr}\,(M_2) \cong \mathrm{Tr}\,(M_1) \otimes \mathrm{Tr}\,(M_2) \cong \mathrm{Tr}\,(M_1 \otimes M_2)\,.$$

Let $r \in \mathrm{ann}\,(M_1 \otimes M_2)$. Then $r\,(M_1 \otimes M_2) = 0$, and hence $r\mathrm{Tr}\,(M_1 \otimes M_2) = \mathrm{Tr}\,(r\,(M_1 \otimes M_2)) = 0$. This implies that $r \in \mathrm{ann}\mathrm{Tr}\,(M_1 \otimes M_2) = \mathrm{ann}\,(\mathrm{Tr}\,(M_1)\,\mathrm{Tr}\,(M_2)) = 0$. Hence $M_1 \otimes M_2$ is faithful.

(2) $M_1 \otimes M_2$ is a flat R-module. Let N be any R-module such that $(M_1 \otimes M_2) \otimes N = 0$. Then $M_1 \otimes (M_2 \otimes N) = 0$, and hence $M_2 \otimes N = 0$ which implies that $N = 0$. Hence $M_1 \otimes M_2$ is faithfully flat.

(3) If M_1 and M_2 are finitely generated R-modules then so too is $M_1 \otimes M_2$. By (1), $M_1 \otimes M_2$ is faithful multiplication. Alternatively, since $M_1$and M_2 are finitely generated multiplication modules, we infer from [8, Theorem1] that $\theta(M_1) = R = \theta(M_2)$. It is easily verified that $\theta(M_1) \otimes \theta(M_2) \subseteq \theta(M_1 \otimes M_2)$. Hence $R \cong R \otimes R = \theta(M_1 \otimes M_2)$, and again by [8, Theorem 1], $M_1 \otimes M_2$ is a finitely generated multiplication R-module. Now, M_1 and M_2 are faithfully flat and by (2) $M_1 \otimes M_2$ is faithfully flat. Hence $M_1 \otimes M_2$ is faithful. □

Theorem 2 has two corollaries which we wish to mention. The first one gives some basic properties of the tensor product of finitely generated faithful multiplication modules. It will be useful for our results in this paper.

Corollary 3. *Let R be a ring and M_1, M_2 finitely generated faithful multiplication R-modules. Let K be a submodule of M_1 and N a submodule of M_2.*

(1) $[K \otimes N : M_1 \otimes M_2] \cong [K : M_1] \otimes [N : M_2]$. *If K or N is flat then* $[K \otimes N : M_1 \otimes M_2] \cong [K : M_1]\,[N : M_2]$.

(2) $[K \otimes M_2 : M_1 \otimes M_2] \cong [K : M_1]$.

(3) $K \otimes M_2 \cong N \otimes M_1$ *if and only if* $K \cong N$.

Proof. By Theorem 2, $M_1 \otimes M_2$ is a finitely generated faithful multiplication R-module, hence it is cancellation.

(1) $[K \otimes N : M_1 \otimes M_2] = [[K : M_1]\, M_1 \otimes [N : M_2]\, M_2 : M_1 \otimes M_2] \cong ([K : M_1] \otimes [N : M_2])\,(M_1 \otimes M_2) : M_1 \otimes M_2 = [K : M_1] \otimes [N : M_2]$. For the second assertion, suppose N is flat. By Lemma 1, $[N : M_2]$ is flat and the result follows.

(2) Follows by (1).

(3) By (2), $K \otimes M_2 \cong N \otimes M_1$ if and only if $[K \otimes M_2 : M_1 \otimes M_2] \cong [N \otimes M_1 : M_1 \otimes M_2]$ if and only if $[K : M_1] \cong [N : M_2]$ if and only if $K = [K : M_1]\, M_1 \cong [N : M_2]\, M_2 = N$. □

Corollary 4. *Let R be a ring and M an R-module.*

(1) *If M is faithfully flat then for all flat R-modules K and N, $K \otimes M \cong N \otimes M$ implies* $\text{ann}K = \text{ann}N$.

(2) *Let M be finitely generated, faithful and multiplication. A submodule L of M is faithfully flat if and only if $[L : M]$ is a faithfully flat ideal of R.*

Proof. (1) Let $K \otimes M \cong N \otimes M$. Then

$$(\text{ann}N)\, K \otimes M = \text{ann}N\,(K \otimes M) \cong \text{ann}N(N \otimes M) \cong (\text{ann}N)\, N \otimes M = 0.$$

Since M is faithfully flat, $(\text{ann}N)\, K = 0$, and $\text{ann}N \subseteq \text{ann}K$. Similarly, $\text{ann}K \subseteq \text{ann}N$ and hence $\text{ann}N = \text{ann}K$.

(2) Suppose that L is faithfully flat. By Lemma 1, $[L : M]$ is a flat ideal of R. Suppose P is a maximal ideal of R and $P\,[L : M] = [L : M]$. Then $PL = L$, a contradiction. Hence $P\,[L : M] \neq [L : M]$, and hence $[L : M]$ is faithfully flat. Conversely, since M is finitely generated faithful multiplication, M is faithfully flat. The result follows by Theorem 2 since $L = [L : M]\,M \cong [L : M] \otimes M$. □

The condition that each of K and N are flat and M is faithfully flat is required. For example, for any positive integers $m \neq n, \mathbb{Z}_n \otimes Q \cong \mathbb{Z}_m \otimes Q = 0$, but $m\mathbb{Z} \neq n\mathbb{Z}$.

Proposition 5. *Let R be a ring and M_1 a finitely generated faithful multiplication R-module. Let N be a finitely generated faithful multiplication submodule of a multiplication R-module M_2.*

(1) *For all submodules K of M_1, if $K \otimes N$ is a finitely generated R-module then so too is K.*
(2) *For all submodules K of M_1, if $K \otimes N$ is a multiplication R-module then so too is K.*
(3) *For all submodules K of M_1, if $K \otimes N$ is a flat R-module then so too is K.*
(4) *For all finitely generated submodules K of M_1, if $K \otimes N$ is a projective R-module then so too is K.*
(5) *For all submodules K of M_1, if $K \otimes N$ is a faithfully flat R-module then so too is K.*

Proof. By [7, Lemma 1.1], $N = \theta\,(M_2)\,N$. Since N is a finitely generated faithful multiplication module, it is cancellation and hence $R = \theta\,(M_2)$. It follows by [8, Theorem 1], that M_2 is finitely generated. As $\mathrm{ann} M_2 \subseteq \mathrm{ann} N = 0, M_2$ is faithful. Alternatively, since $\mathrm{ann} N = 0 = \mathrm{ann} M_2$, we infer from, [17, Corollary 1 to Lemma 1.5] and [19, Theorem 3.1], that M_2 is finitely generated. By Theorem 2, $M_1 \otimes M_2$ is a finitely generated faithful multiplication R-module. Suppose $K \otimes N$ is a finitely generated (resp. multiplication, flat, finitely generated projective, faithfully flat) submodule of $M_1 \otimes M_2$. By Lemma 1 and Corollary 3, $[K : M_1]\,[N : M_2] \cong [K \otimes N : M_1 \otimes M_2]$ is a finitely generated (resp. multiplication, flat, finitely generated projective, faithfully flat) ideal of R. By Lemma 1, $[N : M_2]$ is a finitely generated faithful multiplication ideal of R, and hence $[K : M_1] = [[K : M_1]\,[N : M_2] : [N : M_2]]$ is a finitely generated (resp. multiplication, flat, finitely generated projective, faithfully flat) ideal of R. Again by Lemma 1, K is a finitely generated (resp. multiplication, flat, finitely generated projective, faithfully flat) submodule of M_1. □

A *presentation* of an R-module M is an exact sequence of R-modules

$$0 \to K \to F \xrightarrow{\pi} M \to 0$$

with F free. The proof of the next lemma can be found in [23, Theorem 2.1] and [12, Proposition 11.27].

Lemma 6. *Let R be a ring and M an R-module.*

(1) *M is projective if and only if for each such presentation of M, there exists an R-homomorphism $\theta : F \to F$ such that $\pi\theta = \pi$ and $\ker\theta = \ker\pi$.*

(2) *For any such presentation of M, the following conditions are equivalent.*

(i) *M is flat.*

(ii) *For any $u \in K$, there exists $\theta_u : F \to F$ such that $\pi\theta_u = \pi$, and $\theta_u(u) = 0$.*

(iii) *For any $u_1, \ldots, u_n \in K$, there exists $\theta : F \to F$ such that $\pi\theta = \pi$ and $\theta(u_i) = 0$ for all i.*

Proposition 7. *Let R be a ring and M_1, M_2 R-modules. Let L be a finitely generated faithful submodule of M_1. Then for all finitely generated flat submodules N of M_2, if $N \otimes L$ is a projective R-module, then so too is N.*

Proof. Let

$$0 \to K \to F \xrightarrow{\pi} N \to 0$$

be a presentation of N. Since N is flat, we infer that

$$0 \to K \otimes L \to F \otimes L \xrightarrow{\pi\otimes 1_L} N \otimes L \to 0$$

is exact. As $N \otimes L$ is finitely generated projective, this sequence splits and $K \otimes L$ is finitely generated, say $K \otimes L = \sum_{i=1}^{n} R(k_i \otimes l_i)$ where $k_i \in K$ and $l_i \in L$. By Lemma 6(2), there exists an R-homomorphism $\theta : F \to F$ such that $\pi\theta = \pi$ and $\theta(k_i) = 0$ for all i. Clearly $\ker\theta \subseteq \ker\pi$. On the other hand, for all $k \in K = \ker\pi$, if $l \in L$ then

$$k \otimes l = \sum_{i=1}^{n} r_i (k_i \otimes l_i) = \sum_{i=1}^{n} k_i \otimes r_i l_i$$

for some $r_i \in R$. Then $(\theta \otimes 1_l)(k \otimes l) = \sum_{i=1}^{n} \theta(k_i) \otimes 1_l (r_i l_i) = 0$, and hence $\theta(k) \otimes l = 0$. It remains to show that $\theta(k) = 0$. Assume that $\{x_1, \ldots, x_r\}$ is a basis of F, and that $\theta(k) = \sum_{i=1}^{r} a_i x_i$, with $a_i \in R$. Then

$$0 = \theta(k) \otimes l = \sum_{i=1}^{r} a_i x_i \otimes l = \sum_{i=1}^{r} x_i \otimes a_i l.$$

Since F is free, we infer from [15, p. 251] that $a_i l = 0$, and hence $a_i \in \mathrm{ann}(l)$. Since l is arbitrary, $a_i \in \bigcap_{l \in L} \mathrm{ann} l = \mathrm{ann} L = 0$ for all i. Thus $\theta(k) = 0$, and by Lemma 6(1), N is projective. This completes the proof. □

Suppose K is a finitely generated submodule of a finitely generated faithful multiplication R-module M_1. Let N be a finitely generated faithful multiplication submodule of a multiplication R-module M_2 such that $K \otimes N$ is projective. Then $K \otimes N$ is flat, and by Proposition 5(2), K is flat. By the above proposition, K is projective. This is an alternative proof of Proposition 5(3).

The following lemma extends [11, Corollary 1.7] to multiplication modules with pure annihilators.

Lemma 8. *Let R be a ring and M an R-module. Let $I_\lambda\,(\lambda \in \Lambda)$ be a non-empty collection of ideals of R. If M is multiplication with pure annihilator then* $\bigcap\limits_{\lambda\in\Lambda} I_\lambda M = \left(\bigcap\limits_{\lambda\in\Lambda} I_\lambda\right) M$.

Proof. By [11, Corollary 1.7], $\bigcap\limits_{\lambda\in\Lambda} I_\lambda M = \left(\bigcap\limits_{\lambda\in\Lambda} (I_\lambda + \mathrm{ann}M)\right) M$. Let P be a maximal ideal of R. Since $\mathrm{ann}M$ is pure, it follows by [3, Theorem 1.1] that either $(\mathrm{ann}M)_P = 0_P$ or $(\mathrm{ann}M)_P = R_P$ from which we obtain that $M_P = 0_P$. Both cases show that the equality $\bigcap\limits_{\lambda\in\Lambda} I_\lambda M = \left(\bigcap\limits_{\lambda\in\Lambda} I_\lambda\right) M$ is true locally and hence globally. □

The following lemma gives some properties of pure submodules of multiplication modules.

Lemma 9. *Let R be a ring and N a submodule of a multiplication R-module M such that* $\mathrm{ann}\, M$ *is a pure ideal of R. Let I be an ideal of R.*

(1) *If I is a pure ideal of R and N is pure in M then IN is pure in M.*
(2) *If $[N : M]$ is a pure ideal of R then N is pure in M.*
(3) *Assuming further that M is finitely generated and faithful. If N is pure in M then $[N : M]$ is a pure ideal of R.*

Proof. Let A be an ideal of R.

(1) By Lemma 8 and [3, p. 69] we have that $A\,(IN) = I\,(AN) = I\,(AM \cap N) = I\,[AM : M]\,M \cap IN = (I \cap [AM : M])\,M \cap IN = IM \cap [AM : M]\,M \cap IN = AM \cap IN$, and IN is pure in M.

(2) Since $[N : M]$ is a pure ideal of R, $A\,[N : M] = A \cap [N : M]$, and by Lemma 8 , we get that $AN = A\,[N : M]\,M = (A \cap [N : M])\,M = AM \cap N$, and N is pure in M.

(3) Since N is pure in M, $AN = AM \cap N$. As M is multiplication, it follows by [8, Theorem 2] that $A\,[N : M]\,M = (A \cap [N : M])\,M$. But M is finitely generated faithful multiplication and hence cancellation. Thus $A\,[N : M] = A \cap [N : M]$ and $[N : M]$ is a pure ideal of R. □

In [3], we introduced idempotent submodules: A submodule N of M is an *idempotent in M* if $N = [N : M]\,N$. We proved that a submodule N of a multiplication module M with pure annihilator is pure if and only if N is multiplication and idempotent, [3, Theorem 1.1].

Proposition 10. *Let R be a ring and M_1, M_2 faithful multiplication R-modules. Let K be a submodule of M_1 and N a submodule of M_2.*

(1) *If K is pure in M_1 and N is pure in M_2 then $K \otimes N$ is pure in $M_1 \otimes M_2$.*
(2) *Suppose that M_1 is finitely generated and N is finitely generated, faithful and multiplication. If $K \otimes N$ is pure in $M_1 \otimes M_2$ then K is pure in M_1.*

Proof. (1) Let A be an ideal of R. Since K and N are pure submodules of faithful multiplication modules, K and N are flat R-modules. Hence

$$A\,(K\otimes N)\cong AK\otimes N=(AM_1\cap K)\otimes N,$$

and by [18, Theorem 7.4] and [16, p. 32], we obtain that

$$\begin{aligned}(AM_1\cap K)\otimes N &\cong (AM_1\otimes N)\cap(K\otimes N)\\ &\cong (M_1\otimes AN)\cap(K\otimes N)\cong(M_1\otimes(AM_2\cap N))\cap(K\otimes N)\,.\end{aligned}$$

Again M_1 is faithful multiplication (hence flat). So

$$M_1\otimes(AM_2\cap N)\cong(M_1\otimes AM_2)\cap(M_1\otimes N)\cong A\,(M_1\otimes M_2)\cap(M_1\otimes N)\,,$$

and this finally gives that $A\,(K\otimes N)\cong A(M_1\otimes M_2)\cap(K\otimes N)$. Since $M_1\otimes M_2$ is faithful multiplication (and hence flat), $K\otimes N$ is pure in $M_1\otimes M_2$. Alternatively, K and N are multiplication. Hence by Theorem 2, $K\otimes N$ is multiplication. Also K is idempotent in M_1 and N is idempotent in M_2. It follows that $K\otimes N=[K:M_1]\,K\otimes[N:M_2]\,N\cong([K:M_1]\otimes[N:M_2])\,(K\otimes N)\subseteq[K\otimes N:M_1\otimes M_2]\,(K\otimes N)\subseteq K\otimes N$, so that $K\otimes N\cong[K\otimes N:M_1\otimes M_2]\,(K\otimes N)$, and $K\otimes N$ is idempotent in $M_1\otimes M_2$. By [3, Theorem 1.1], $K\otimes N$ is pure in $M_1\otimes M_2$.

(2) As we have seen in the proof of Proposition 5, M_2 is finitely generated and by Theorem 2, $M_1\otimes M_2$ is a finitely generated faithful and multiplication R-module. If $K\otimes N$ is pure in $M_1\otimes M_2$ it follows by Lemma 9 and Corollary 3 that

$$[K:M_1]\,[N:M_2]\cong[K\otimes N:M_1\otimes M_2]$$

is a pure ideal of R. Let A be an ideal of R. Then $A\,[K:M_1]\,[N:M_2]=A\cap[K:M_1]\,[N:M_2]\supseteq A\,[N:M_2]\cap[K:M_1]\,[N:M_2]=(A\cap[K:M_1])\,[N:M_2]$. Now, by Lemma 1, $[N:M_2]$ is a finitely generated faithful multiplication ideal of R (and hence cancellation). It follows that $A\,[K:M_1]=A\cap[K:M_1]$, and hence $[K:M_1]$ is a pure ideal of R. By Lemma 9, K is a pure submodule of M_1. Alternatively, by Lemma 9, $[K\otimes N:M_1\otimes M_2]\cong[K:M_1]\,[N:M_2]$ is a pure ideal of R. Since $[N:M_2]$ is a finitely generated faithful multiplication module, we infer from Lemma 9 again that

$$[K:M_1]=[[K:M_1]\,[N:M_2]:[N:M_2]]\,,$$

is a pure ideal of R. The result follows by Lemma 9(2). □

Let R be a commutative ring with identity. Let S be the set of non-zero divisors of R and R_S the total quotient ring of R. For a non-zero ideal I of R, let

$$I^{-1}=\{x\in R_S:xI\subseteq R\}\,.$$

I is an invertible ideal of R if $II^{-1}=R$. Let M be an R-module and

$$T=\{t\in S:\ \text{ for all } m\in M,\ tm=0 \text{ implies } m=0\}\,.$$

T is a multiplicatively closed subset of S, and if M is torsion free then $T = S$. In particular, $T = S$ if M is a faithful multiplication module, see [10, Lemma 4.1]. Also $T = S$ if M is an ideal of R. Let N be a non-zero submodule of M, and let

$$N^{-1} = \{x \in R_T : xN \subseteq M\}.$$

N^{-1} is an R_S-submodule of $R_T, R \subseteq N^{-1}$, and $NN^{-1} \subseteq M$. Following [21], N is *invertible in* M if $NN^{-1} = M$. If N is an invertible submodule of M then $\mathrm{ann}N = \mathrm{ann}M$. For if $r \in \mathrm{ann}N$, then $rN = 0$, and hence $rM = rNN^{-1} = 0$. Then $\mathrm{ann}N \subseteq \mathrm{ann}M$. The other inclusion is always true.

Naoum and Al-Alwan, [21], introduced invertibility of submodules generalizing the concept for ideals and gave several properties and examples of such submodules. It is shown, [21, Lemma 3.2], that if N is a non-zero submodule of a multiplication R-module M such that $[N : M]$ is an invertible ideal of R then N is invertible in M. The converse is true if we assume further that M is finitely generated and faithful, [21, Lemma 3.3]. It is well-known that if I and J are ideals of a ring R then IJ is invertible if and only if I and J are invertible. Suppose K and N are submodules of finitely generated faithful multiplication R-modules M_1 and M_2 respectively. Suppose K is invertible in M_1 and N is invertible in M_2. Then $[K : M_1]$ and $[N : M_2]$ are invertible ideals of R, and hence

$$[K : M_1]\,[N : M_2] \cong [K : M_1] \otimes [N : M_2] \cong [K \otimes N : M_1 \otimes M_2]$$

is an invertible ideal of R. This implies that $K \otimes N$ is an invertible submodule of $M_1 \otimes M_2$. Conversely, let $K \otimes N$ be invertible in $M_1 \otimes M_2$ such that N is flat. Then $[K \otimes N : M_1 \otimes M_2] \cong [K : M_1]\,[N : M_2]$ is an invertible ideal of R. Hence $[K : M_1]$ and $[N : M_2]$ are invertible ideals of R and this shows that K is invertible in M_1 and N is invertible in M_2.

Following [21], an R-module M is called *Dedekind* (resp. *Prüfer*) if and only if every non-zero (resp. non-zero finitely generated) submodule of M is invertible. Faithful multiplication Dedekind (resp. Prüfer) modules are finitely generated. For let $0 \neq m \in M$. Then Rm is invertible in M. By [4, Corollary 1.2], $R = \theta(M) + \mathrm{ann}(m) = \theta(M) + \mathrm{ann}M = \theta(M)$, and M is finitely generated, [8, Theorem 1].

The next result gives necessary and sufficient conditions for the tensor product of Dedekind (resp. Prüfer) modules to be Dedekind (resp. Prüfer).

Theorem 11. *Let R be a ring and M_1, M_2 faithful multiplication R-modules. If $M_1 \otimes M_2$ is a Dedekind (resp. Prüfer) R-module then so too are M_1 and M_2. The converse is true if either M_1 or M_2 is Dedekind (resp. Prüfer).*

Proof. Suppose $M_1 \otimes M_2$ is Dedekind (resp. Prüfer). By Theorem 2, $M_1 \otimes M_2$ is faithful multiplication and by the remark made before the theorem, $M_1 \otimes M_2$ is finitely generated. Suppose N is a non-zero (resp. non-zero finitely generated) submodule of M_1. Then $N = [N : M_1]\,M_1$, where $[N : M_1]$ is a non-zero (resp. non-zero finitely generated) ideal of R, see [21, Note 3.7]. It follows that $N \otimes M_2$

is a non-zero (resp. non-zero finitely generated) submodule of $M_1 \otimes M_2$. For, if $N \otimes M_2 = 0$, we obtain from Corollary 3 that

$$0 = [0 : M_1 \otimes M_2] = [N \otimes M_2 : M_1 \otimes M_2] \cong [N : M_1],$$

and hence $N = [N : M_2]\, M_2 = 0$, a contradiction. Since $N \otimes M_2$ is invertible in $M_1 \otimes M_2$. we infer from Corollary 3 that $[N : M_1] \cong [N \otimes M_2 : M_1 \otimes M_2]$ is an invertible ideal of R, and hence N is invertible in M_1. This shows that M_1 is a Dedekind (resp. Prüfer) module. Similarly, M_2 is Dedekind (resp. Prüfer). Conversely, suppose M_1 is a Dedekind (resp. Prüfer) module. Since M_1 is faithful multiplication, M_1 is finitely generated. Let K be a non-zero (resp. non-zero finitely generated) submodule of $M_1 \otimes M_2$. Then $K = I\,(M_1 \otimes M_2) \cong IM_1 \otimes M_2$ for some non-zero (resp. non-zero finitely generated) ideal I of R. Since IM_1 is a non-zero (resp. non-zero finitely generated) submodule of M_1, IM_1 is invertible in M_1 and hence $I = [IM_1 : M_1]$ is an invertible ideal of R. Since $M_1 \otimes M_2$ is a faithful (and hence non-zero) R-module, $K = I\,(M_1 \otimes M_2)$ is invertible in $M_1 \otimes M_2$, [21, Remark 3.2], and hence $M_1 \otimes M_2$ is a Dedekind (resp. Prüfer) R-module. □

2. Large and small submodules

If I is a faithful ideal of a ring R then I is a large ideal of R. In particular, every non-zero ideal of an integral domain R is large. For all submodules K and N of M with $K \subseteq N$. If K is large in M then so too is N and if N is small in M then so too is K. Let I be an ideal of R and K, N submodules of an R-module M. Then $K \cap N$ is large in M if and only if K and N are large in M. If IN is large in M then N is large in M and if we assume further that M is faithful multiplication then I is a large ideal of R. Moreover, $K + N$ is small in M if and only if K and N are small in M and if N is small in M then IN is small in M. In case that M is finitely generated, faithful and multiplication and I is a small ideal of R then IN is small in M.

We start this section by the following result which gives several properties of large and small submodules.

Proposition 12. *Let R be a ring and M an R-module. Let I be an ideal of R and N a submodule of M.*

1. *If M is multiplication and N is faithful then N is large in M.*
2. *If M is faithful multiplication and I is a large ideal of R then IM is large in M.*
3. *Let M be faithful multiplication. If N is large in M and I is faithful then IN is large in M.*
4. *Let M be multiplication. If I is a large ideal of R and N is faithful then IN is large in M.*
5. *Let M be faithful multiplication. Then N is large in M if and only if $[N : M]$ is a large ideal of R.*

(6) *Let M be finitely generated faithful. If IM is small in M then I is a small ideal of R.*

(7) *Let M be finitely generated. If M is faithful and N is a small in M then $[N : M]$ is small ideal of R. The converse is true if M is multiplication.*

Proof. (1) Suppose K is a submodule of M such that $K \cap N = 0$. Since N is faithful, M is faithful and hence

$$[K : M]\,[N : M] \subseteq [K : M] \cap [N : M] = [(K \cap N) : M] = 0.$$

It follows that $[K : M] \subseteq \operatorname{ann}[N : M] = \operatorname{ann}N = 0$, so that $[K : M] = 0$, and hence $K = [K : M]\,M = 0$ and N is large in M.

(2) Suppose K is a submodule of M such that $K \cap IM = 0$. Since M is multiplication, it follows by, [8, Theorem 2], that $0 = K \cap IM = ([K : M] \cap I)\,M$, and hence $([K : M] \cap I) \subseteq \operatorname{ann}M = 0$. It follows that $[K : M] \cap I = 0$, and hence $[K : M] = 0$. This gives that $K = [K : M]M = 0$, and IM is large in M.

(3) Suppose K is a submodule of M such that $K \cap IN = 0$. Then $I\,(K \cap N) \subseteq IK \cap IN \subseteq K \cap IN = 0$, so that $I\,(K \cap N) = 0$. Hence

$$I\,[(K \cap N) : M] \subseteq [I\,(K \cap N) : M] = \operatorname{ann}M = 0,$$

and hence $[(K \cap N) : M] \subseteq \operatorname{ann}I = 0$. This implies that $K \cap N = [(K \cap N) : M]\,M = 0$. Since N is large in M, $K = 0$ and IN is large in M.

(4) Suppose K is a submodule of M such that $K \cap IN = 0$. Then $[N : M]\,(K \cap IM) \subseteq [N : M]\,K \cap I\,[N : M]\,M = [N : M]\,K \cap IN \subseteq K \cap IN = 0$, so that $[N : M]\,(K \cap IM) = 0$. Hence $K \cap IM \subseteq \operatorname{ann}[N : M] = \operatorname{ann}N = 0$. This implies that $K \cap IM = 0$. By (2), IM is large in M and hence $K = 0$ and IN is large in M.

(5) Assume N is large in M and I an ideal of R such that $I \cap [N : M] = 0$. It follows by, [11, Corollary 1.7] and [8, Theorem 2], that

$$0 = (I \cap [N : M])\,M = IM \cap [N : M]\,M = IM \cap N.$$

Hence $IM = 0$ and hence $I \subseteq \operatorname{ann}M = 0$. It follows that $[N : M]$ is a large ideal of R. The converse follows by (2).

(6) Suppose IM is small in M. Let J be an ideal of R such that $J + I = R$. Then $JM + IM = M$, and hence $JM = M$. It follows by, [14, Theorem 76], that $R = J + \operatorname{ann}M = J$, and hence I is a small ideal of R.

(7) Suppose M is finitely generated multiplication and $[N : M]$ is a small ideal of R. Let K be a submodule of M such that $K + N = M$. It follows by, [24, Corollary 3 to Theorem 1], that

$$R = [(K + N) : M] = [K : M] + [N : M].$$

Hence $R = [K : M]$ and then $K = M$. This shows that N is small in M. The converse follows by (6). $\square$

The next result gives necessary and sufficient conditions for the tensor product of large (resp. small) submodules of multiplication modules to be large (resp. small).

Corollary 13. *Let R be a ring and M_1, M_2 finitely generated faithful multiplication R-modules. Let K be a submodule of M_1 and N a flat submodule of M_2.*

1. *If $K \otimes N$ is large in $M_1 \otimes M_2$ then K is large in M_1 and N is large in M_2.*
2. *If K is faithful and N is large in M_2 then $K \otimes N$ is large in $M_1 \otimes M_2$.*
3. *If K is small in M_1 then $K \otimes N$ is small in $M_1 \otimes M_2$.*
4. *If $K \otimes M_2$ is small in $M_1 \otimes M_2$, then K is small in M_1.*

In case that M_1 and M_2 are faithful multiplication (not necessarily finitely generated) modules and K is large in M_1 then $K \otimes M_2$ is large in $M_1 \otimes M_2$.

Proof. (1) By Corollary 3 and Proposition 12, $[K : M_1]\,[N : M_2] \cong [K \otimes N : M_1 \otimes M_2]$ is a large ideal of R. Hence $[K : M_1]$ and $[N : M_2]$ are large ideals of R and by Proposition 12, K is large in M_1 and N is large in M_2.

(2) Suppose K is faithful and N is large in M_2. Then $[K : M_1]$ is a faithful ideal of R and by Proposition 12, $[N : M_2]$ is a large ideal of R. By Proposition 12 and Corollary 3, $[K : M_1][N : M_2] \cong [K \otimes N : M_1 \otimes M_2]$ is a large ideal of R. The result follows again by Proposition 12.

The proofs of (3) and (4) are similar to that of (1) and (2) by using Lemma 1, Corollary 3 and Proposition12.

Finally, suppose M_1 and M_2 are faithful multiplication (not necessarily finitely generated) modules. Let K be large in M_1. Let L be a submodule of $M_1 \otimes M_2$ such that $L \cap (K \otimes M_2) = 0$. Then $L = I\,(M_1 \otimes M_2)$ for some ideal I of R. It follows that $(I \cap [K : M_1])\,(M_1 \otimes M_2) = I\,(M_1 \otimes M_2) \cap [K : M_1]\,(M_1 \otimes M_2) \cong L \cap ([K : M_1]\,M_1 \otimes M_2) = L \cap (K \otimes M_2) = 0$. Since $M_1 \otimes M_2$ is faithful, $I \cap [K : M_1] = 0$. As K is large in M_1, we obtain from Proposition 12, that $[K : M_1]$ is a large ideal of R and hence $I = 0$. This implies that $L = 0$ and $K \otimes M_2$ is large in $M_1 \otimes M_2$. □

An R-module M is called *uniform* if the intersection of any two non-zero submodules of M is non-zero, and M has *finite uniform dimension* if it does not contain an infinite direct sum of non-zero submodules.

The next result gives some properties of finitely cogenerated and uniform modules.

Proposition 14. *Let R be a ring and M a multiplication R-module with pure annihilator. Let N be a submodule of M such that* $\mathrm{ann}N = \mathrm{ann}\,[N : M]$.

1. *N is finitely cogenerated if and only if $[N : M]$ is finitely cogenerated.*
2. *N is uniform if and only if $[N : M]$ is uniform.*
3. *N has finite uniform dimension if and only if $[N : M]$ has finite uniform dimension.*

Proof. Let $I_\lambda\,(\lambda \in \Lambda)$ be a non-empty collection of ideals of R contained in $[N : M]$ such that $\bigcap_{\lambda\in\Lambda} I_\lambda = 0$. It follows by Lemma 8 that $\bigcap_{\lambda\in\Lambda} I_\lambda M = 0$. For all $\lambda \in \Lambda$, $I_\lambda M \subseteq [N : M]\,M = N$. Hence there exists a finite subset Λ' of

Λ such that $\bigcap_{\lambda\in\Lambda'} I_\lambda M = 0$. It follows that $\left(\bigcap_{\lambda\in\Lambda'} I_\lambda\right) M = \bigcap_{\lambda\in\Lambda'} I_\lambda M = 0$, and hence $\bigcap_{\lambda\in\Lambda'} I_\lambda \subseteq \mathrm{ann}M$. As $\mathrm{ann}M$ is a pure ideal of R, we infer that $\bigcap_{\lambda\in\Lambda'} I_\lambda = \left(\bigcap_{\lambda\in\Lambda'} I_\lambda\right) \mathrm{ann}M \subseteq [N : M]\,\mathrm{ann}N = [N : M]\,\mathrm{ann}\,[N : M] = 0$, so that $\bigcap_{\lambda\in\Lambda'} I_\lambda = 0$, and $[N : M]$ is finitely cogenerated. Conversely, suppose $K_\lambda\,(\lambda\in\Lambda)$ be a non-empty collection of submodules of N with $\bigcap_{\lambda\in\Lambda} K_\lambda = 0$. Then $\bigcap_{\lambda\in\Lambda} [K_\lambda : M] = \left[\left(\bigcap_{\lambda\in\Lambda} K_\lambda\right) : M\right] = \mathrm{ann}M$. Since $\mathrm{ann}M$ is pure and hence an idempotent, we obtain that
$\bigcap_{\lambda\in\Lambda} [K_\lambda : M] = \left(\bigcap_{\lambda\in\Lambda} [K_\lambda : M]\right) \mathrm{ann}M \subseteq [N : M]\,\mathrm{ann}N = [N : M]\,\mathrm{ann}\,[N : M] = 0$, so that $\bigcap_{\lambda\in\Lambda} [K_\lambda : M] = 0$. As $[N : M]$ is finitely cogenerated, there exists a finite subset Λ' of Λ such that $\bigcap_{\lambda\in\Lambda'} [K_\lambda : M] = 0$. It follows by Lemma 8 that $0 = \left(\bigcap_{\lambda\in\Lambda'} [K_\lambda : M]\right) M = \bigcap_{\lambda\in\Lambda'} [K_\lambda : M]\,M = \bigcap_{\lambda\in\Lambda'} K_\lambda$, and N is finitely cogenerated.

(2) Follows by (1).

(3) Suppose N has finite uniform dimension. If $[N : M]$ contains a direct sum of ideals $I_\lambda\,(\lambda\in\Lambda)$, then by Lemma 8 it follows that $\sum_{\lambda\in\Lambda} I_\lambda M$ is direct sum, and hence all but a finite number of the submodules $I_\lambda M$ of N are zero. If $I_\lambda M = 0$, then $I_\lambda \subseteq \mathrm{ann}M$ and hence $I_\lambda = I_\lambda \mathrm{ann}M \subseteq [N : M]\,\mathrm{ann}N = [N : M]\,\mathrm{ann}\,[N : M] = 0$. So that $I_\lambda = 0$ and $[N : M]$ has finite uniform dimension. Conversely, suppose $[N : M]$ has finite uniform dimension. Suppose N contains a direct sum of submodules $K_\lambda\,(\lambda\in\Lambda)$. Since $\bigcap_{\lambda\in\Lambda} [K_\lambda : M] \subseteq \mathrm{ann}M$, we have that

$$\bigcap_{\lambda\in\Lambda} [K_\lambda : M] = \left(\bigcap_{\lambda\in\Lambda} [K_\lambda : M]\right) \mathrm{ann}M \subseteq [N : M]\,\mathrm{ann}N = [N : M]\,\mathrm{ann}\,[N : M] = 0.$$

So that $\bigcap_{\lambda\in\Lambda} [K_\lambda : M] = 0$. Hence $\sum_{\lambda\in\Lambda} [K_\lambda : M]$ is direct and hence all but a finite number of ideals $[K_\lambda : M] \subseteq [N : M]$ are zero. If $[K_\lambda : M] = 0$, then $K_\lambda = [K_\lambda : M]\,M = 0$, and N has a finite uniform dimension. □

As a consequence of the above result we state the following corollary.

Corollary 15. *Let R be a ring and N a submodule of a faithful multiplication R-module. Then N is finitely cogenerated (resp. uniform, has finite uniform dimension) if and only if $[N : M]$ is finitely cogenerated (resp. uniform, has finite uniform dimension).*

Let R be a ring and M a flat R-module. Let $N_i\,(1\le i\le n)$ be a finite collection of submodules of an R-module N. Then $\left(\bigcap_{i=1}^{n} N_i\right) \otimes M \cong \bigcap_{i=1}^{n} (N_i \otimes M)$, [16, p.

32] and [18, Theorem 7.4]. This property is not true for an arbitrary collection of submodules of N. The next lemma gives some conditions under which this property of flat modules is true in general. It will be useful for our next result.

Lemma 16. *Let R be a ring and M_1, M_2 faithful multiplication R-modules. Then for all non-empty collection $N_\lambda\,(\lambda \in \Lambda)$ of submodules of M_1,*

$$\left(\bigcap_{\lambda\in\Lambda} N_\lambda\right) \otimes M_2 \cong \bigcap_{\lambda\in\Lambda} \left(N_\lambda \otimes M_2\right).$$

Proof. By Theorem 2, $M_1 \otimes M_2$ is a faithful multiplication R-module. It follows by, [10, Corollary 1.7], that

$$\begin{aligned}\left(\bigcap_{\lambda\in\Lambda} N_\lambda\right) \otimes M_2 &= \left(\bigcap_{\lambda\in\Lambda} [N_\lambda : M_1]\, M_1\right) \otimes M_2 = \left(\bigcap_{\lambda\in\Lambda} [N_\lambda : M_1]\right) M_1 \otimes M_2 \\ &\cong \left(\bigcap_{\lambda\in\Lambda} [N_\lambda : M_1]\right) (M_1 \otimes M_2) = \bigcap_{\lambda\in\Lambda} \left([N_\lambda : M_1]\,(M_1 \otimes M_2)\right) \\ &\cong \bigcap_{\lambda\in\Lambda} \left(([N_\lambda : M_1]\, M_1) \otimes M_2\right) \cong \bigcap_{\lambda\in\Lambda} \left(N_\lambda \otimes M_2\right). \qquad \square\end{aligned}$$

The following result gives necessary and sufficient conditions for the tensor product of finitely cogenerated (resp. uniform, has finite uniform dimension) to be a finitely cogenerated (resp. uniform, has finite uniform dimension) module.

Proposition 17. *Let R be a ring and M_1, M_2 faithful multiplication R-modules.*

(1) *If $M_1 \otimes M_2$ is finitely cogenerated then so too are M_1 and M_2. The converse is true if either M_1 or M_2 is finitely cogenerated.*
(2) *If $M_1 \otimes M_2$ is uniform then so too are M_1 and M_2. The converse is true if either M_1 or M_2 is uniform.*
(3) *If $M_1 \otimes M_2$ has finite uniform dimension then so too have M_1 and M_2. The converse is true if either M_1 or M_2 has finite uniform dimension.*

Proof. (1) Suppose $M_1 \otimes M_2$ is finitely cogenerated. Let $N_\lambda\,(\lambda \in \Lambda)$ be a non-empty collection of submodules of M_1 such that $\bigcap_{\lambda\in\Lambda} N_\lambda = 0$. It follows by Lemma 16 that $0 = \left(\bigcap_{\lambda\in\Lambda} N_\lambda\right) \otimes M_2 \cong \bigcap_{\lambda\in\Lambda} (N_\lambda \otimes M_2)$, where $N_\lambda \otimes M_2$ are submodules of $M_1 \otimes M_2$. Hence there exists a finite subset Λ' of Λ such that

$$\begin{aligned}0 = \bigcap_{\lambda\in\Lambda'} (N_\lambda \otimes M_2) &\cong \left(\bigcap_{\lambda\in\Lambda'} N_\lambda\right) \otimes M_2 = \left(\bigcap_{\lambda\in\Lambda'} [N_\lambda : M_1]\, M_1\right) \otimes M_2 \\ &= \left(\bigcap_{\lambda\in\Lambda'} [N_\lambda : M_1]\right) M_1 \otimes M_2 \cong \left(\bigcap_{\lambda\in\Lambda'} [N_\lambda : M_1]\right) (M_1 \otimes M_2).\end{aligned}$$

Since $M_1 \otimes M_2$ is faithful, $\bigcap_{\lambda\in\Lambda'} [N_\lambda : M_1] = 0$, and hence

$$\bigcap_{\lambda\in\Lambda'} N_\lambda = \bigcap_{\lambda\in\Lambda'} [N_\lambda : M_1]\, M_1 = \left(\bigcap_{\lambda\in\Lambda'} [N_\lambda : M_1]\right) M_1 = 0,$$

and M_1 is finitely cogenerated. Similarly, M_2 is finitely cogenerated. Conversely, suppose M_1 is finitely cogenerated and let $K_\lambda\,(\lambda\in\Lambda)$ be a non-empty collection of submodules of $M_1\otimes M_2$ such that $\bigcap_{\lambda\in\Lambda} K_\lambda=0$. As $M_1\otimes M_2$ is a multiplication R-module, $K_\lambda=I_\lambda\,(M_1\otimes M_2)$ for some ideals I_λ of R. Since $M_1\otimes M_2$ is faithful multiplication, it follows that

$$0=\bigcap_{\lambda\in\Lambda} K_\lambda=\bigcap_{\lambda\in\Lambda} I_\lambda\,(M_1\otimes M_2)\cong\left(\bigcap_{\lambda\in\Lambda} I_\lambda\right)(M_1\otimes M_2)\,,$$

and hence $\bigcap_{\lambda\in\Lambda} I_\lambda=0$. Since M_1 is faithful multiplication, we infer that $0=\left(\bigcap_{\lambda\in\Lambda} I_\lambda\right)M_1=\bigcap_{\lambda\in\Lambda} I_\lambda M_1$, and hence there exists a finite subset Λ' of Λ such that $\bigcap_{\lambda\in\Lambda'} I_\lambda M_1=0$. It follows that

$$0=\left(\bigcap_{\lambda\in\Lambda'} I_\lambda M_1\right)\otimes M_2\cong\bigcap_{\lambda\in\Lambda'}(I_\lambda M_1\otimes M_2)\cong\bigcap_{\lambda\in\Lambda'} I_\lambda\,(M_1\otimes M_2)=\bigcap_{\lambda\in\Lambda'} K_\lambda,$$

and $M_1\otimes M_2$ is finitely cogenerated.

(2) Follows by (1).

(3) Suppose $M_1\otimes M_2$ has finite uniform dimension. Suppose M_1 contains a direct sum of submodules $N_\lambda\,(\lambda\in\Lambda)$. Since M_2 is faithful multiplication and hence flat, it follows that $\bigoplus_{\lambda\in\Lambda} N_\lambda\otimes M_2=\bigoplus_{\lambda\in\Lambda}(N_\lambda\otimes M_2)$, [18, p. 267], and hence $\sum_{\lambda\in\Lambda} N_\lambda\otimes M_2$ is direct sum (see also the proof of part (1)). Hence all but a finite number of $N_\lambda\otimes M_2$ are zero. If $N_\lambda\otimes M_2=0$, then $0=[N_\lambda:M_1]\,M_1\otimes M_2\cong([N_\lambda:M_1])\,(M_1\otimes M_2)$. Since $M_1\otimes M_2$ is faithful, $[N_\lambda:M_1]=0$, and hence $0=[N_\lambda:M_1]\,M_1=N_\lambda$, and M_1 has finite uniform dimension. Similarly, M_2 has finite uniform dimension. Conversely, suppose M_1 has finite uniform dimension and suppose $M_1\otimes M_2$ contains a direct sum of submodules $K_\lambda\,(\lambda\in\Lambda)$. As $M_1\otimes M_2$ is multiplication, $K_\lambda=I_\lambda(M_1\otimes M_2)$ for some ideals I_λ of R. As we have seen in the proof of the first part, $\sum_{\lambda\in\Lambda} I_\lambda M_1$ is direct sum of submodules of M_1. Since M_1 has finite uniform dimension, all but a finite number of $I_\lambda M$ are zero. If $I_\lambda M=0$, $I_\lambda=0$ and hence $K_\lambda=0$ and $M_1\otimes M_2$ has finite uniform dimension. $\square$

3. Join principal submodules

Let R be a ring and M an R-module. A submodule N of M is called join principal if for all ideals A of R and all submodules K of M, $[(AN+K):N]=A+[K:N]$, [6] and [10]. It is easy to see that the following conditions are equivalent for a submodule N of M:

1. N is a join principal submodule of M.
2. For all ideals A and B of R and all submodules K of M, $[(AN+K):BN]=[(A+[K:N]):B]$.

(3) For all ideals A and B of R and all submodules K and L of M, $AN+K=BN+L$ implies $A+[K:N]=B+[L:N]$.

It is obvious from the above statements that join principal submodules are weak cancellation. The converse is not true. Let R be an almost Dedekind domain but not Dedekind. Hence R has a maximal ideal P that is not finitely generated. So P is a cancellation ideal (hence weak cancellation), but P is not join principal, [6].

We start this section by the following result which gives necessary and sufficient conditions for the product of join principal submodules (ideals) to be join principal.

Proposition 18. *Let R be a ring and M an R-module. Let I be an ideal of R and N a submodule of M.*

(1) *If I is join principal and N is join principal (resp. weak cancellation) then IN is join principal (resp. weak cancellation).*
(2) *Let M be finitely generated, faithful and multiplication. If I is weak cancellation and N is join principal then IN is weak cancellation.*
(3) *If N is cancellation and IN is join principal (resp. weak cancellation) then I is join principal (resp. weak cancellation).*
(4) *Let M be finitely generated, faithful and multiplication. If I is cancellation and IN is join principal (resp. weak cancellation) then N is join principal (resp. weak cancellation).*

Proof. Let A and B be ideals of R and K a submodule of M.
(1)

$$\begin{gathered}[(A\,(IN)+K):IN]=[[((AI)\,N+K):N]:I]\\=[(AI+[K:N]):I]=A+[[K:N]:I]=A+[K:IN]\end{gathered}$$

and IN is join principal. The proof of weak cancellation submodules case follows by letting $K=0$.
(2) Let M be finitely generated, faithful and multiplication. Then

$$\begin{aligned}[AIN:IN]&=[[AIN:M]\,M:[IN:M]\,M]=[[AIN:M]:[IN:M]]\\&=[IA\,[N:M]:I\,[N:M]]=[[IA\,[N:M]:I]:[N:M]]\\&=[(A\,[N:M]+\mathrm{ann}I):[N:M]]\subseteq[(AN+(\mathrm{ann}I)\,M):N]\\&=A+[(\mathrm{ann}I)\,M:N]\subseteq A+\mathrm{ann}\,(IN)\,.\end{aligned}$$

The reverse inclusion is clear and IN is weak cancellation.
(3) $[(AI+B):I]\subseteq[(AIN+BN):IN]=A+[BN:IN]=A+[B:I]$. The reverse inclusion is always true and I is join principal. The proof of the weak cancellation ideal case is obvious by assuming $B=0$.

(4) Let M be finitely generated, faithful and multiplication. Then

$$\begin{aligned}[(AN+K):N] \subseteq [(AIN+IK):IN] &= A+[IK:IN] \\ &= A+[[IK:M]M:[IN:M]M] = A+[[IK:M]:[IN:M]] \\ &= A+[I[K:M]:I[N:M]] = A+[[K:M]:[N:M]] \\ &= A+[[K:M]M:[N:M]M] = A+[K:N],\end{aligned}$$

and N is join principal. The proof of the weak cancellation submodule case is clear by setting $K=0$. □

The next result gives necessary and sufficient conditions for the intersection and sum of join principal submodules to be join principal.

Proposition 19. *Let R be a ring and M an R-module. Let $N_i\,(1 \le i \le n)$ be a finite collection of submodules of M such that*

$$[N_i : N_j] + [N_j : N_i] = R \text{ for all } i < j.$$

Let $N = \bigcap_{l=1}^{n} N_l$ and $S = \sum_{l=1}^{n} N_l$.

1. *If each N_i is a join principal (resp. weak cancellation) submodule of M then N is join principal (resp. weak cancellation).*
2. *If each N_i+N_j is a finitely generated join principal (resp. weak cancellation) submodule of M then S is join principal (resp. weak cancellation).*
3. *Suppose N_i are finitely generated and N is join principal. If N_i+N_j are join principal (resp. weak cancellation) then N_i are join principal (resp. weak cancellation).*
4. *Suppose N_i are finitely generated and S is join principal. If $N_i \cap N_j$ are join principal (resp. weak cancellation) then N_i are join principal (resp. weak cancellation).*

Proof. We only do the proof of the join principal submodules case.

(1) Only one implication requires proof. Let A be an ideal of R and K a submodule of M. It follows by [5, Corollary 1.2] that

$$\begin{aligned}[(AN+K):N] &= \sum_{i=1}^{n}[(AN+K):N_i] \subseteq \sum_{i=1}^{n}[(AN_i+K):N_i] \\ &= \sum_{i=1}^{n} A+[K:N_i] = A+\sum_{i=1}^{n}[K:N_i] = A+[K:N].\end{aligned}$$

(2) By [5, Corollary 1.2] we have that $\sum_{i=1}^{n}[N_i : S] = R$. Hence

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{n} [(N_i+N_j):S] = R.$$

Using the fact that a finitely generated submodule N is join principal (resp. weak cancellation) if and only if N is locally join principal (resp. locally weak cancellation), [20, Proposition 2.3], it is enough to prove the result locally. Thus, we assume that R is a local ring. It follows that $S = (N_{i_0} + N_{j_0})$ for some $1 \le i_0, j_0 \le n, i_0 \ne j_0$, and S is join principal.

(3) It is also enough to assume that R is a local ring. It follows by [5, Corollary 1.2], that $R = \sum_{i=1}^{n} [N : N_{i_0}]$. There exists $i_0 \in \{1, \ldots, n\}$ such that $[N : N_{i_0}] = R$ and hence $N_{i_0} = N$ is join principal. Let $j \ne i_0$. By [5, Corollary 2.4], [22, Corollary 3.4] and [24, Proposition 12], $N_{i_0} \cap N_j$ is finitely generated. Since $[N : N_{i_0}] = R, [N : N_{i_0} \cap N_j] = R$. Hence $N_{i_0} \cap N_j = N$ is join principal. Since $[N_{i_0} : N_j] + [N_j : N_{i_0}] = R$, we infer that $[(N_{i_0} \cap N_j) : N_j] + [N_j : (N_{i_0} + N_j)] = R$. Hence either $[N_{i_0} \cap N_j : N_j] = R$ and hence $N_j = N_{i_0} \cap N_j$ or $[N_j : N_{i_0} + N_j] = R$ and hence $N_j = N_{i_0} + N_j$. Both cases give that N_j is join principal.

(4) Again it is enough to prove the result locally. Thus we assume that R is local. By [5, Corollary 1.2], $R = \sum_{i=1}^{n} [N_i : S]$ and hence there exists $i_0 \in \{1, \ldots, n\}$ such that $[N_{i_0} : S] = R$. Hence $N_{i_0} = S$ is join principal. Let $j \ne i_0$. Then $[N_{i_0} + N_j : S] = R$ and hence $N_{i_0} + N_j = S$ is join principal. Since $N_{i_0} \cap N_j$ (which is finitely generated) is join principal, the result follows. □

We conjecture that the last three parts of the above result are true even if N_i are not necessarily finitely generated. It is proved by Anderson, [6, Theorem 5.3], that if R is a one-dimensional integral domain then an R-module M is cancellation if and only if it is locally cancellation. Hence the last three parts of the above result are true for cancellation modules (not necessarily finitely generated) over one-dimensional integral domains.

Before we give our results on the tensor product of join principal submodules, we need the following lemma.

Lemma 20. *Let R be a ring and N a submodule of a finitely generated faithful multiplication R-module M.*

1. *N is join principal (resp. weak cancellation) if and only if $[N : M]$ is a join principal (resp. weak cancellation) ideal of R.*
2. *N is cancellation if and only if $[N : M]$ is a cancellation ideal of R.*
3. *N is restricted cancellation if and only if $[N : M]$ is a restricted cancellation ideal of R.*

Proof. (1) We prove the join principal submodules case. Let N be join principal. Let A and B be ideals of R. Then

$$\begin{aligned}[(A\,[N : M] + B) : [N : M]] &= [(A\,[N : M]\,M + BM) : [N : M]\,M] \\ &= [(AN + BM) : N] = A + [BM : N] = A + [B : [N : M]]\,,\end{aligned}$$

and $[N:M]$ is join principal. Conversely, suppose $[N:M]$ is a join principal ideal of R. Let A be an ideal of R and K a submodule of M. Then

$$\begin{aligned}[(AN+K):N] &= [(A\,[N:M]+[K:M])\,M:[N:M]\,M]\\ &= [(A\,[N:M]+[K:M]):[N:M]]\\ &= A+[[K:M]:[N:M]] = A+[K:N]\,.\end{aligned}$$

(2) Using the fact that $\mathrm{ann}N = \mathrm{ann}\,[N:M] = 0$, the result follows by (1) since a submodule N of M is cancellation if and only if it is faithful weak cancellation.

(3) Again, using $\mathrm{ann}N = \mathrm{ann}\,[N:M]$ and the fact that a submodule N of M is restricted cancellation if and only if N is weak cancellation and $\mathrm{ann}N$ is comparable to every ideal of R, [6, Theorem 2.5], the result follows by (2). □

Proposition 21. *Let R be a ring and M_1, M_2 finitely generated faithful multiplication R-modules. Let K be a submodule of M_1 and N a flat submodule of M_2.*

1. *If K and N are join principal then $K\otimes N$ is a join principal submodule of $M_1\otimes M_2$.*
2. *If K is join principal and N is weak cancellation then $N\otimes K$ is a weak cancellation submodule of $M_1\otimes M_2$.*
3. *If K is weak cancellation and N is cancellation then $K\otimes N$ is a weak cancellation submodule of $M_1\otimes M_2$.*
4. *If K and N are cancellation then $K\otimes N$ is a cancellation submodule of $M_1\otimes M_2$.*

Proof. (1) Let A be an ideal of R and L a submodule of $M_1\otimes M_2$. It follows by Corollary 3 that

$$\begin{aligned}&[(A(K\otimes N)+L):K\otimes N] = [(A[K\otimes N:M_1\otimes M_2]+\\ &[L:M_1\otimes M_2])\,(M_1\otimes M_2):[K\otimes N:M_1\otimes M_2](M_1\otimes M_2)]\\ &= [(A\,[K\otimes N:M_1\otimes M_2]+[L:M_1\otimes M_2]):[K\otimes N:M_1\otimes M_2]]\\ &\cong [(A\,[K:M_1]\,[N:M_2]+[L:M_1\otimes M_2]):[K:M_1]\,[N:M_2]]\,.\end{aligned}$$

By Lemma 20, $[K:M_1]$ and $[N:M_2]$ are join principal and by Proposition 18, $[K:M_1]\,[N:M_2]$ is join principal. It follows that

$$\begin{aligned}[(A(K\otimes N)+L):N\otimes K] &= [A+[[L:M_1\otimes M_2]:[K:M_1]\,[N:M_2]]]\\ &\cong A+[[L:M_1\otimes M_2]:[K\otimes N:M_1\otimes M_2]]\\ &= A+[[L:M_1\otimes M_2]\,(M_1\otimes M_2):[K\otimes N:M_1\otimes M_2]\,(M_1\otimes M_2)]\\ &= A+[L:K\otimes N]\,,\end{aligned}$$

and $K\otimes N$ is join principal.

(2) Follows from (1) by letting $L=0$.

(3) Let A be an ideal of R. Then

$$[A(K \otimes N) : K \otimes N] \cong [A[K : M_1][N : M_2] : [K : M_1][N : M_2]].$$

Since $[K : M_1][N : M_2]$ is a weak cancellation ideal of R, we infer that

$$\begin{aligned}[A(K \otimes N) : K \otimes N] &\cong A + \operatorname{ann}([K : M_1][N : M_2]) \\ &\cong A + \operatorname{ann}[K \otimes N : M_1 \otimes M_2] = A + \operatorname{ann}(K \otimes N),\end{aligned}$$

and $K \otimes N$ is weak cancellation.

(4) Follows by (3). □

The converse of Proposition 21 is not true in general. Let M be the maximal ideal of a non-discrete rank one valuation ring R. Then M is not finitely generated, not weak cancellation (and hence not join principal). Otherwise, since $M = M^2$, we would have $R = [M^2 : M] = M + \operatorname{ann}M$, and hence M is finitely generated (in fact M is principal and generated by idempotent), a contradiction. Now, M is not join principal but $R/M \otimes M \cong M/M^2 = 0$ is join principal. Our final result gives conditions under which the converse of Proposition 21 is true.

Proposition 22. *Let R be a ring and M_1 a finitely generated faithful multiplication R-module. Let N be a finitely generated faithful multiplication submodule of a multiplication R-module M_2.*

(1) *For all submodules K of M_1, if $K \otimes N$ is a join principal submodule of $M_1 \otimes M_2$ then K is join principal.*
(2) *For all submodules K of M_1, if $K \otimes N$ is a weak cancellation submodule of $M_1 \otimes M_2$ then K is weak cancellation.*
(3) *For all submodules K of M_1, if $K \otimes N$ is a cancellation submodule of $M_1 \otimes M_2$ then K is cancellation.*
(4) *For all submodules K of M_1, if $K \otimes N$ is a restricted cancellation submodule of $M_1 \otimes M_2$ then K is restricted cancellation.*

Proof. M_2 is finitely generated and faithful and hence $M_1 \otimes M_2$ is a finitely generated faithful multiplication R-module. See the proof of Proposition 5. It follows by Corollary 3 and Lemma 20 that $[K : M_1][N : M_2] \cong [K \otimes N : M_1 \otimes M_2]$ is join principal and hence by Proposition 18 we have that $[K : M_1]$ is join principal. By Lemma 20, K is join principal and the first part is proved. The proofs of the other parts of the result follow by the same argument by using Proposition 18 and Lemma 20. □

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