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Semipolarized Nonruled Surfaces with Sectional Genus Two

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Abstract. Complex projective nonruled surfaces S endowed with a numerically effective line bundle L of arithmetic genus $g(S,L)=2$ are investigated. In view of existing results on elliptic surfaces we focus on surfaces of Kodaira dimension $\kappa(S)=0$ and 2. Structure results for (S,L) are provided in both cases, according to the values of L^2. When S is not minimal we describe explicitly the structure of any birational morphism from S to its minimal model S_0, reducing the study of (S,L) to that of (S_0,L_0), where L_0 is a numerically effective line bundle with $g(S_0,L_0)=2$ or 3. Our description of (S,L) when S is minimal, as well as that of the pair (S_0,L_0) when $g(S_0,L_0)=3$, relies on several results concerning linear systems, mainly on surfaces of Kodaira dimension 0. Moreover, several examples are provided, especially to enlighten the case in which S is a minimal surface of general type, (S,L) having Iitaka dimension 1.

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1. Introduction

Polarized smooth surfaces with sectional genus $g = 2$ have been studied and classified in [5], [10], [26], and [12]. There is also a classification result for normal Gorenstein surfaces endowed with a nef and big line bundle of arithmetic genus 2, with special regard to singularities [4]. On the other hand Maeda [18] classified surfaces with a merely nef line bundle of arithmetic genus $g \leq 1$. All these results led us to study semipolarized surfaces with sectional genus $g = 2$. The case in which the surface is elliptic has been considered in [16]. In this paper we consider the same situation for surfaces of non-negative Kodaira dimension. In fact, studying semipolarized normal surfaces with $g = 2$ turns out to be equivalent, from the birational point of view, to study smooth semipolarized surfaces with $g = 2$. So, let S be a smooth projective surface and let L be a nef line bundle on S with $g(S, L) = 2$.

Our analysis relies on the relation $L^2 + LK_S = 2$, given by the definition of sectional genus. Of course $L^2 \geq 0$ since L is nef, and also $LK_S \geq 0$ provided that S is nonruled. This restricts the two summands to three possibilities, and we define the type of (S, L) according to them. Of course this approach cannot longer work if S is ruled: actually, in this case LK_S can be very negative. For instance, on the Segre-Hirzebruch surface $S = \mathbb{F}_2$ the line bundle $L = -K_S + f$, where f is a fibre of the ruling projection, is nef, $L^2 = 12$ and $LK_S = -10$. Another point is the following. If S is ruled and L is nef and big then the irregularity of S satisfies the same bound $q(S) \leq 2$ as for polarized surfaces (e. g., see [13], Lemma 1.2 (3)). However if L is simply nef, we can only say that $q(S) \leq 2 + h^1(K_S + L)$, although we have no examples with $q(S) \geq 3$. For these reasons in this paper we confine our study to nonruled surfaces.

Since we work up to birational equivalence, we can also assume that the pair (S, L) is L-minimal, i. e., $LE > 0$ for every (-1)-curve $E \subset S$. Moreover, in view of [16], we need only to analyze surfaces with $\kappa(S) = 2$ and 0.

For surfaces of Kodaira dimension zero, our general result is as follows.

Theorem 1. *Let (S, L) be a smooth L-minimal semipolarized surface with $g(S, L) = 2$ and $\kappa(S) = 0$. Then one of the following cases holds:*

- I) $L^2 = 2, LK_S = 0$, *S is minimal and either L is ample or S is Enriques or* K3*;*
- II) $L^2 = LK_S = 1$ *and (S, L) has a simple reduction as in case* I)*;*
- III) $L^2 = 0, LK_S = 2$ *and either*
  - i) *there is a morphism $\sigma : S \to S_1$ contracting a single (-1)-curve E, $L = \sigma^* L_1 - 2E$, where S_1 is a minimal surface and L_1 is a nef line bundle with $g(L_1) = 3$, or*
  - ii) *(S, L) has a simple reduction as in case* II)*.*

For each case we supply more details along the paper. In particular, in case III-i) we have that L_1 has to be ample if S_1 is abelian or bielliptic.

Case I) with L ample has been studied in [5] (see also [10]). So here we determine the structure of Enriques and K3 surfaces carrying a nef non-ample line bundle of genus two. As to cases II) and III) there are some restrictions on the point p where a birational morphism contracts the (-1)-curve of S. In general they can be expressed in terms of suitable Seshadri constants, but, sometimes, they can be made very explicit and this allows us to rule out some possibilities. More specifically, in case III-i) we combine these restrictions with several results from [23], [6], [7], [1], and [25] to describe all admissible line bundles L_1 of genus 3 on minimal surfaces of Kodaira dimension zero. One of the most delicate point is that of K3 surfaces when the linear system $|L_1|$ has a fixed part. In fact, due to a result of Nikulin, we show that this case does not occur in connection with case III-i). The form of L_1 as well as the restrictions on p can be easily determined also for bielliptic surfaces, thanks to the basis of $\mathrm{Num}(S)$ described by Serrano [25], according to the seven types occurring in the Bagnera-De Franchis classification.

As to surfaces of general type the result we get is the following.

Theorem 2. *Let (S, L) be a smooth L-minimal semipolarized surface with $g(S, L) = 2$ and $\kappa(S) = 2$. Then one of the following cases holds:*

II) $L^2 = LK_S = 1$, *S is minimal, and $L \equiv K_S$;*

III) $L^2 = 0, LK_S = 2$ *and either*

- i) (S, L) *has a simple reduction as in case* II), *or*
- ii) S *is minimal.*

Cases have been enumerated according to the types defined in Section 2. Note that type (I) cannot occur for $\kappa(S) \geq 1$.

Case II) with L ample has been studied in [5], so here we provide examples where L is nef but not ample. As to case III-i), some restrictions on the reduction (S_1, L_1) are needed: e. g., the point at which the reduction morphism contracts the (-1)-curve of S cannot lie on any (-2)-curve of S_1. However the more interesting case is III-ii). Here, different situations may correspond to the same type. The standard situation is that of fibrations having a multiple fibre whose support is an irreducible curve of genus 2, which of course includes the case of genus two fibrations. This corresponds to a line bundle L with Iitaka dimension $\kappa(S, L) = 1$. In this setting we produce several examples, by constructing some double covers. However, also $\kappa(S, L) = -\infty$ can occur. In fact we can easily relate our L to the "numerical type" studied by Sakai [24], but, unfortunately we have not been able to confirm or to exclude the possibility that $\kappa(S, L) = 0$. We are grateful to L. Bădescu for several discussions on this last point.

2. Notations and preliminaries

We work over the complex number field $\mathbb{C}$. We use standard notation and terminology in algebraic geometry: in particular we denote additively the tensor products of line bundles and we use the symbol $\equiv$ to denote the numerical equivalence. Following a current abuse of notation we do not distinguish between line bundles and the corresponding invertible sheaves.

We are interested in the classification of nonruled projective surfaces endowed with a nef line bundle of arithmetic genus 2. To make our set-up clear let us fix some more terminology.

Let X be a projective surface and let $\mathcal{L}$ be a line bundle on X. If $\mathcal{L}$ is nef, i.e., $\mathcal{L}C \geq 0$ for all integral curves C on X, we call the pair $(X, \mathcal{L})$ a *semipolarized surface*. Two semipolarized surfaces $(X_1, \mathcal{L}_1)$, $(X_2, \mathcal{L}_2)$ are said to be *birationally equivalent* if there exist a projective surface Y and birational morphisms $f_i : Y \to X_i$ $(i = 1, 2)$ such that $f_1^*\mathcal{L}_1 = f_2^*\mathcal{L}_2$. If X is normal, the *sectional genus* $g(X, \mathcal{L})$ of the semipolarized normal surface $(X, \mathcal{L})$ is defined by the formula $2g(X, \mathcal{L}) - 2 = (\omega_X + \mathcal{L})\mathcal{L}$, where ω_X denotes the canonical sheaf of X. Let $(X, \mathcal{L})$ be a semipolarized normal surface with $g(X, \mathcal{L}) > 0$. Then, by [18] Theorem 1 there exist a smooth surface S and a nef line bundle L on S such that the semipolarized surface (S, L) is birationally equivalent to $(X, \mathcal{L})$ and $K_S + L$ is nef, where K_S is the canonical bundle of S. Moreover, by [11], Lemma 1.8, (see also [18], Lemma 3.1) we have $g(S, L) = g(X, \mathcal{L})$. In particular this shows that to study semipolarized normal surfaces of sectional genus 2 up to birational equivalence it is enough to consider semipolarized smooth surfaces (S, L) where

$$2 = 2g(S, L) - 2 = (K_S + L)L = K_S L + L^2. \tag{1}$$

So from now on the word 'surface' will mean smooth projective surface. In particular, in this paper we are interested in nonruled surfaces.

Since L is nef and a positive multiple of K_S is effective or trivial, both summands appearing in the right hand of (1) are non-negative. Hence we get the following three possibilities for the pair (L^2, LK_S): $(2, 0), (1, 1)$ and $(0, 2)$. We will call them *types* (I), (II), and (III) respectively.

Let (S, L) be a semipolarized surface. If S is not relatively minimal there exists a (-1)-curve $E \subset S$, let $\sigma : S \longrightarrow S'$ be a contraction of E and let $p = \sigma(E)$. Then there exists a line bundle L' on S' such that $L = \sigma^*L' - rE$, where $r = LE \geq 0$ since L is nef. It is immediate to check that L' is nef. Since $K_S = \sigma^*K_{S'} + E$, we have $K_S + L = \sigma^*(K_{S'} + L') - (r - 1)E$, so that

$$\begin{aligned} 2 = 2g(S, L) - 2 &= L(K_S + L) = L'(K_{S'} + L') - r(r - 1) \\ &= 2g(S', L') - 2 - r(r - 1). \end{aligned}$$

If $r = 1$, by using the adjunction theoretic terminology, we say that (S', L') is a *simple reduction of* (S, L). In this case note that $g(S, L) = g(S', L')$.

Let (S, L), E, σ and (S', L') be as above. Checking the nefness of L' we see in fact much more. Actually, if $C \subset S'$ is an irreducible curve having a point of multiplicity $m (\geq 0)$ at $p = \sigma(E)$, we have

$$0 \leq L\sigma^{-1}(C) = (\sigma^*L' - rE)(\sigma^*C - mE) = L'C - mr.$$

Conversely, starting from the semipolarized surface (S', L') and blowing-up p, we see that the line bundle $L := \sigma^* L' - rE$ is nef if and only if

$$\varepsilon(L', p) \geq r,$$

where

$$\varepsilon(L', p) = \inf_{C \ni p} \frac{L'C}{\mathrm{mult}_p(C)}$$

is the Seshadri constant of L' at p. So we get the following conclusion.

Fact 2.1. Let S be a smooth surface and let $\sigma : S \to S'$ be the contraction of a (-1)-curve E to a point $p \in S'$. Let L' be a nef line bundle on S' and let $L := \sigma^* L' - rE$, with $r \geq 0$. Then L is nef if and only if $\varepsilon(L', p) \geq r$.

As a consequence, if σ is a reduction morphism, for instance (S', L') is a simple reduction of (S, L) as in cases II) and III-ii) of Theorem 1 and in case III-i) of Theorem 2, then the nefness of L is equivalent to the condition that $\varepsilon(L', p) \geq 1$. In particular:

(1) no curve $C \subset S'$ with $L'C = 0$ can pass through p, and

(2) p cannot be a singular point of any irreducible curve $E \subset S'$ with $L'E = 1$; otherwise $\varepsilon(L', p) \leq \frac{L'E}{\mathrm{mult}_p(E)} \leq \frac{1}{2}$.

Similarly, in case III-i) of Theorem 1 it must be $\varepsilon(L_1, p) \geq 2$. In particular this says that p cannot lie on any curve $C \subset S_1$ such that $L_1 C = 1$; moreover it cannot be a singular point of any irreducible curve $E \subset S_1$ with $L_1 E = 2$. These conditions will be relevant in Sections 3 and 4.

We want to use birational equivalence to reduce the study of our pairs (S, L) to the case when S is a minimal surface. It is useful to recall the following definition ([13], Definition 1.9, see also [4], Definition 0.9).

Definition 2.2. *(S, L) is said to be L-minimal if $LE > 0$ for every (-1)-curve $E \subset S$.*

Recall that up to a birational morphism we can always replace our semipolarized pair (S, L) with another $(S^\sharp, L^\sharp)$, which is $L^\sharp$-minimal and has the same type as (S, L) ([16], Fact 2.4). Moreover we have

Lemma 2.3. *Let (S, L) be an L-minimal semipolarized nonruled pair with $LK_S = 1$. If S is non-minimal then there exists a simple reduction (S_1, L_1) of (S, L) which is L_1-minimal.*

Proof. Since S is not minimal consider a birational morphism $\eta : S \to S_0$ onto the minimal model S_0. Then $K_S = \eta^* K_{S_0} + \mathcal{E}$, where $\mathcal{E}$ is an effective divisor contracted by η. We have

$$1 = LK_S = L\eta^* K_{S_0} + L\mathcal{E} \geq L\mathcal{E}.$$

Since $\mathcal{E}$ contains at least a (-1)-curve, say E, as a component and (S, L) is L-minimal this gives $LE = 1$. Let $\sigma : S \to S_1$ be the contraction of E (note that η factors through σ and in fact $\eta = \sigma$, "a posteriori"). Let $L_1 \in \mathrm{Pic}(S_1)$ be the nef line bundle such that $L = \sigma^* L_1 - E$. Moreover (S_1, L_1) is L_1-minimal. To see this, let $E_1 \subset S_1$ be a (-1)-curve and let $p \in S_1$ be the point blown-up by σ. Then $\sigma^* E_1 = \tilde{E}_1 + \nu E$, where $\tilde{E}_1 = \sigma^{-1}(E_1)$ and $\nu = \tilde{E}_1 E = \mathrm{mult}_p(E_1)$ is 0 or 1 according to whether $p \notin E_1$ or $p \in E_1$. Since $L = \sigma^* L_1 - E$ we have

$$\begin{aligned} L_1 E_1 &= \sigma^* L_1 \sigma^* E_1 = \sigma^* L_1 (\tilde{E}_1 + \nu E) = \sigma^* L_1 \tilde{E}_1 = (L + E)\tilde{E}_1 \\ &= L\tilde{E}_1 + E\tilde{E}_1. \end{aligned}$$

Now both summands on the right hand are non-negative since L is nef. Moreover, if $\nu = 0$, then $\tilde{E}_1$ is a (-1)-curve in S, hence the first summand is positive, since (S, L) is L-minimal. On the other hand, if $\nu = 1$ then the second summand is positive. This shows that $L_1 E_1 > 0$ in every case. □

We conclude this section by recalling some technical facts on K3, bielliptic, and abelian surfaces, which will be used later on in the paper. Linear systems on K3 surfaces have been studied by Saint-Donat [23]. The fixed part of a nef and big complete linear system on a K3 surface is much simpler than expected, as shown by Nikulin [20]. We rediscovered this result in the course of our study (see also [28]).

Proposition 2.4. (see [20], Proposition 0.1) *Let L be a nef and big line bundle on a* K3 *surface S, such that $|L|$ has a fixed part. Then $|L| = |kE| + \Gamma$, where the fixed part Γ is an irreducible (-2)-curve, E is a curve with $E^2 = 0$, $E\Gamma = 1$, and $k = g(S, L)$.*

A bielliptic surface S can be of seven types, according to a classical result of Bagnera-de Franchis ([3], p. 87). Explicit bases of $\mathrm{Num}(S)$ have been found by Serrano ([25], Theorem 1.4) according to the seven types. For the convenience of the reader we recall this result in the form we need it.

Proposition 2.5. (see [25], Theorem 1.4 and Proposition 1.2) *Let S be a bielliptic surface, $S = (A \times B)/G$, where A, B, are elliptic curves and G is an abelian group acting on A and on B such that A/G is elliptic and $B/G \cong \mathbb{P}^1$. Then $\{\frac{1}{h}A, \frac{1}{k}B\}$ is a basis of* $\mathrm{Num}(S)$*, with*

$$(h, k) = (2, 1), (2, 2), (4, 1), (4, 2), (3, 1), (3, 3), (6, 1),$$

according to the seven types. Moreover the classes induced by the two factors in $\mathrm{Num}(S)$*, still denoted by A, B, intersect as follows: $A^2 = B^2 = 0$; $AB = hk = \gamma$, where γ is the order of G.*

We also recall the following fact.

Lemma 2.6. *Let S be a smooth minimal surface which is either abelian or bielliptic. Let L be a nef and big line bundle on S. Then L is ample.*

Proof. Since L is nef and big, if it is not ample, there exists an irreducible curve $C \subset S$ such that $LC = 0$. By the Hodge index theorem this implies that $C^2 < 0$. This combined with the fact that K_S is numerically trivial says that C is a (-2)-curve. In particular C is a smooth rational curve. Recall that abelian surfaces cannot contain rational curves. The same fact is true for bielliptic surfaces. Actually, if $C \subset S$ is any effective divisor on a bielliptic surface, by using Proposition 2.5 it follows that $C^2 \geq 0$. Hence S cannot contain smooth curves of genus 0. □

3. L nef and big

In this section we will consider the case in which the line bundle L is nef and big. Hence the semipolarized surface (S, L) is either of type (I): $(L^2, LK_S) = (2, 0)$ or of type (II): $(L^2, LK_S) = (1, 1)$.

Lemma 3.1. *If (S, L) is L-minimal of type* (I) *with $\kappa(S) \geq 0$, then S is a minimal surface with $\kappa(S) = 0$.*

Proof. Assume that S is not minimal. Then there exists a birational morphism $\eta : S \to S_0$ onto the minimal model S_0, and $K_S = \eta^* K_{S_0} + \mathcal{E}$, where $\mathcal{E}$ is an effective divisor contracted by η. Note that since η is nontrivial, $\mathcal{E}$ contains at least one (-1)-curve as a component; hence $L\mathcal{E} > 0$, because (S, L) is L-minimal. On the other hand $L\eta^* K_{S_0} \geq 0$, mK_{S_0} being effective or trivial for $m >> 0$. Thus, since (S, L) is of type (I), we get

$$0 = LK_S = \eta^* K_{S_0} L + L\mathcal{E} \geq L\mathcal{E} > 0,$$

a contradiction. This shows that S is minimal. Now, by the Hodge index theorem, we conclude from $LK_S = 0$ that either $K_S \equiv 0$, or $K_S^2 < 0$. But S is a nonruled minimal surface, hence $K_S^2 \geq 0$, so that the second possibility cannot occur. Thus $K_S \equiv 0$, which proves the assertion. □

Let us continue to study type (I). If L is ample, then the classification of pairs (S, L) where S is a minimal surface with $\kappa(S) = 0$ can be found in [5], Theorem 2.7. Note that, by Proposition 2.5, for all seven types of bielliptic surfaces occurring in case (d) of the table in [5], p. 202 we have $L = \frac{1}{h}A + \frac{1}{k}B$. So we can assume that L is not ample.

Recalling Lemma 2.6 we have the following

Proposition 3.2. *If (S, L) is L-minimal of type* (I) *with $\kappa(S) \geq 0$ and L is not ample, then S can only be either Enriques or* K3.

This gives case I) in Theorem 1. Now we investigate further the structure of our pairs (S, L) when L is not ample.

K3 surfaces

If $|L|$ has no fixed part then $|L|$ is base point free and $\varphi_L : S \longrightarrow \mathbb{P}^2$ is a generically finite morphism of degree two. If L is ample this is a double cover branched along a smooth sextic, while if L is nef but not ample then (S, L) appears as the minimal desingularization of a double plane branched along a singular sextic. A concrete example of the second case is as follows.

Example 3.3. Let $\pi : Y \longrightarrow \mathbb{P}^2$ be a double cover of $\mathbb{P}^2$ branched along a sextic B with one node. Let $p : X \longrightarrow Y$ be a minimal resolution of Y and consider the map $f = \pi \circ p : X \longrightarrow \mathbb{P}^2$. Then X is a K3 surface. Set $L = f^*(\mathcal{O}_{\mathbb{P}^2}(1))$. We have $L^2 = 2$ and of course $LK_X = 0$, hence $g(X, L) = 2$. Let $\Gamma \subset X$ be the exceptional curve arising from the resolution. Note that Γ is a (-2)-curve, (see [2], III, 7) and $L\Gamma = 0$. Moreover we have $LC = f^*(\mathcal{O}_{\mathbb{P}^2}(1))C > 0$ for any curve $C \subset X$ with $C \neq \Gamma$, by the projection formula. Hence we conclude that L is nef and not ample.

If $|L|$ has a fixed part then by Proposition 2.4 we know that $|L| = |kE| + \Gamma$, where Γ is an irreducible (-2)-curve, E is a curve with $E^2 = 0$, $E\Gamma = 1$, and $k = 1 + \frac{1}{2}L^2$. Hence $L = 2E + \Gamma$. Note that this case is effective. It corresponds to the following example.

Example 3.4. Let X be the K3 surface considered in Example 3.3 and let ℓ be a line in $\mathbb{P}^2$ passing through the node of the branch divisor B. Let $E = f^{-1}(\ell)$ and let Γ be as in Example 3.3. Note that $|E|$ is an elliptic pencil and $E\Gamma = 1$. The line bundle $L' = 2E + \Gamma = E + f^*(\ell)$ is nef. In fact for every curve $C \subset X$ with $C \neq \Gamma$, we have $L'C = 2EC + \Gamma C \geq 0$. On the other hand $L'\Gamma = E\Gamma + (E + \Gamma)\Gamma = 0$, thus L' is nef and not ample being $L'\Gamma = 0$. Moreover L' satisfies the condition $L'^2 = 2$. Finally note that $|L'|$ has Γ as fixed part.

Enriques surfaces

As to Enriques surfaces the situation is slightly more involved than for K3's. Actually from [7] Proposition 2.4 if the nef and big line bundle L has a fixed part then it is of the form $|L| = |P + R|$, for an elliptic pencil $|P|$ and a nodal curve R such that $PR = 2$. This gives no contradiction for $L^2 = 2$. Moreover the adjoint linear system (which has the same numerical class as L) has no fixed part, [7], Lemma 2.5. So, in order to describe S, up to replace L with its adjoint line bundle, we can assume that $|L|$ has no fixed components. Then, by [7], 2.12, $|L|$ is a hyperelliptic linear system since $L^2 = 2$, and there are just two possibilities ([7], p. 590). For each of them there is an appropriate line bundle L such that the semipolarized pair (S, L) is of type (I). They are the following.

(1) (non-special pencil of genus two) Let S be a non-special Enriques surface and let L be a nef line bundle with $L^2 = 2$. Then $|L| = |E_1 + E_2|$ with $E_1E_2 = 1$, where $|2E_1|$, $|2E_2|$ are two elliptic pencils on S. The linear system $|2L|$ defines a morphism of degree 2, $f : S \longrightarrow V_1 \subset \mathbb{P}^4$ onto a 4-nodal quartic in $\mathbb{P}^4$.

(2) (special pencil of genus two) Let S be a special Enriques surface and let $|L|$ be a nef line bundle with $L^2 = 2$ on S. Let $|2E|$ be the elliptic pencil and R the nodal curve such that $ER = 1$. Then $|L| = |2E + R + K_S|$. The

linear system $|2L|$ defines a morphism of degree 2, $f : S \longrightarrow V_2 \subset \mathbb{P}^4$ onto a degenerate 4-nodal quartic V_2.

Note that in both cases $2L = f^*\mathcal{O}_{V_i}(1)$, hence S contains (-2)-curves C corresponding to the nodes of V_i, such that $LC = 0$. This shows that L cannot be ample.

Now let us consider type (II): $(L^2, LK_S) = (1, 1)$. First suppose that S is not minimal. Then, by Lemma 2.3 we know (S, L) has a simple reduction (S_1, L_1) which is L_1-minimal. Note that (S_1, L_1) is a semipolarized surface of sectional genus 2. Moreover, since $L^2 = L_1^2 - 1$ and $LK_S = L_1K_{S_1} + 1$, we see that (S_1, L_1) is of type (I). Recall that, according to the study of type (I) made before, S_1 is a minimal surface (hence $\eta = \sigma$); furthermore $\kappa(S_1) = 0$ and there are two possibilities: either L_1 is ample, or S_1 is Enriques or K3. This gives case II) in Theorem 1. If L_1 is ample then there are no evident restrictions deriving from (2.1). Let S_1 be a K3 surface. If $|L_1|$ has a fixed part, then $|L_1| = |2E| + \Gamma$, where E is a curve with $E^2 = 0$ and Γ is a (-2)-curve with $\Gamma E = 1$ by Proposition 2.4. Then the effect of (2.1) is that the point p blown-up by σ cannot lie on Γ, since $L_1\Gamma = 0$. Moreover, p cannot be a singular point of any singular element of the pencil $|E|$. On the other hand, if S_1 is an Enriques surface an obvious effect of (2.1) is that the point p blown-up by σ cannot lie on the (-2)-curves contracted by the morphism associated to $|2L_1|$.

Now suppose that S is minimal. We have $L(L - K_S) = 0$; hence by the Hodge index theorem there are two possibilities:

(a) $L \equiv K_S$, or

(b) $(L - K_S)^2 < 0$.

In case (a) we get $K_S^2 = L^2 = 1$, hence $\kappa(S) = 2$, since S is minimal. Moreover L is ample up to (-2)-curves, since it is numerically equivalent to K_S.

In case (b) we have $0 > L^2 - 2LK_S + K_S^2 = -1 + K_S^2$. Thus $K_S^2 \leq 0$. Since S is a minimal surface, this shows that S cannot be of general type, and then $\kappa(S) = 0$. But in this case $K_S \equiv 0$, since S is minimal, which clearly contradicts the fact that $LK_S = 1$. Thus case (b) cannot occur. This shows

Proposition 3.5. *If* (S, L) *is of type* (II) *with* S *minimal, then either* $\kappa(S) = 1$, *or* S *is of general type with* $K_S^2 = 1$ *and* $L \equiv K_S$.

The former case is studied in [16], Sections 3–5. The latter one can be studied as in [5], Theorem 1.4, leading to the same result as in the ample case, simply removing the condition that S cannot contain (-2)-curves. This gives case II) in Theorem 2.

We like to remark that the Campedelli-Kulikov-Oort surface S in [9], pp. 167–171 falls exactly in the latter class. In fact its canonical class is nef and not ample since on S there are (-2)-curves, and $K_S^2 = 1$. Recall that such a surface is the minimal non singular double plane branched along the curve $W = C_1 \cup C_2 \cup D_1 \cup D_2$, where C_1 and C_2 are conics, D_1 and D_2 are cubics such that one of the cubics has a double point, both conics pass through this point and touch both cubics at the other points. For details see [9].

4. L nef and not big

In this section we consider type (III): $(L^2, LK_S) = (0, 2)$, which is the most intricate.

First suppose that S is not minimal and let $\eta : S \to S_0$ and $\mathcal{E}$ be as in the proof of Lemma 3.1. Then, from

$$2 = LK_S = L\eta^* K_{S_0} + L\mathcal{E},$$

we see that either $L\mathcal{E} = 2$ and $L\eta^* K_{S_0} = 0$, or $L\mathcal{E} = L\eta^* K_{S_0} = 1$.

a) If $L\mathcal{E} = 2$, then either

- (a_1) $\mathcal{E}$ is irreducible and $L\mathcal{E} = 2$, or
- (a_2) $\mathcal{E}$ is reducible, $\mathcal{E}$ contains at least a (-1)-curve, say E, as a component, with $LE = 1$, or 2, but $\mathcal{E} \neq E$.

Let $\mathcal{E}$ be as in (a_1). Since η is non trivial, $\mathcal{E} = E$ is a (-1)-curve. Let $\sigma : S \to S_1$ be the contraction of E. Let $L_1 \in \mathrm{Pic(S_1)}$ be the nef line bundle such that $L = \sigma^* L_1 - 2E$. Recall from Section 2 that (S_1, L_1) is a semipolarized surface of sectional genus 3. Moreover S_1 is minimal (note that in case (a_1) $\eta = \sigma$). Since $L^2 = L_1^2 - 4$ and $LK_S = L_1 K_{S_1} + 2$ we see that $(L_1^2, L_1 K_{S_1})$=$(4, 0)$. Now, by the Hodge index theorem, we conclude from $L_1 K_{S_1} = 0$ that either $K_{S_1} \equiv 0$, or $K_{S_1}^2 < 0$. But S_1 is a nonruled minimal surface, hence $K_{S_1}^2 \geq 0$, so that the second possibility cannot occur. Thus $K_{S_1} \equiv 0$. This is case III-i) of Theorem 1. We will say more on this case later.

Let $\mathcal{E}$ be as in (a_2). Let $\sigma : S \to S_1$ be the contraction of E. Let $L_1 \in \mathrm{Pic(S_1)}$ be the nef line bundle such that $L = \sigma^* L_1 - mE$, for $m = LE = 1$, or 2.

By Section 2 we have that (S_1, L_1) is a semipolarized surface of sectional genus 3 or 2 according to whether $m = 2$ or 1. The same reasoning as in Lemma 2.3 shows that (S_1, L_1) is L_1-minimal. Moreover, since $L^2 = L_1^2 - m^2$ and $LK_S = L_1 K_{S_1} + m$ we see that $(L_1^2, L_1 K_{S_1})$=$(4, 0)$, or $(1, 1)$ according to whether $m = 2$ or 1. Thus, either

- (i) (S_1, L_1) is a semipolarized surface of sectional genus 3, L_1-minimal, with $(L_1^2, L_1 K_{S_1}) = (4, 0)$, this does not occur (see Claim 4.1), or
- (ii) (S_1, L_1) is a semipolarized surface of sectional genus 2, L_1-minimal, with $(L_1^2, L_1 K_{S_1}) = (1, 1)$, i.e. of type (II).

Let (S_1, L_1) be as in (ii). The final condition in (a_2) implies that $S_1 \neq S_0$. So S_1 is not minimal. Hence, the study done in Section 3 says that (S_1, L_1) has a simple reduction (S_2, L_2) which is a semipolarized surface of sectional genus 2, L_2-minimal, of type (I). In conclusion (S, L) has a reduction (S_2, L_2) with $\kappa(S_2) = 0$ and either L_2 is ample or S_2 is Enriques or K3 (by Proposition 3.2). This leads to III-ii) in Theorem 1.

Claim 4.1. *The case* (i) *does not occur.*

Proof. Let (S_1, L_1) be as in (i). By the Hodge index theorem, from $L_1 K_{S_1} = 0$ we conclude that either $K_{S_1} \equiv 0$, or $K_{S_1}^2 < 0$. We show that the latter case cannot occur. Assume, by contradiction, that $K_{S_1}^2 < 0$. This implies, since $\kappa(S_1) \geq 0$, that S_1 is non minimal. So there exists a birational morphism $\eta_1 : S_1 \to S_0$ onto the minimal model S_0, and $K_{S_1} = \eta_1^* K_{S_0} + \mathcal{E}_1$, where $\mathcal{E}_1$ is an effective divisor contracted by η_1. We have

$$0 = L_1 K_{S_1} = L_1 \eta_1^* K_{S_0} + L_1 \mathcal{E}_1.$$

Being L_1 nef it follows that both members on the right hand side are zero. In particular $L_1 \mathcal{E}_1 = 0$. But $\mathcal{E}_1$ contains at least a (-1)-curve and thus $L_1 \mathcal{E}_1 = 0$ contradicts the fact that the pair (S_1, L_1) is L_1-minimal. Thus $K_{S_1} \equiv 0$. In particular S_1 is minimal, but this contradicts the final condition in (a_2). So the claim is proved. □

b) Consider now the case $L\mathcal{E} = L\eta^* K_{S_0} = 1$. Since $\mathcal{E}$ contains a (-1)-curve, say E, as a component, and (S, L) is L-minimal, we have $LE = 1$. Let $\sigma : S \to S_1$ be the contraction of E, and let $L_1 \in \mathrm{Pic}(S_1)$ be the nef line bundle such that $L = \sigma^* L_1 - E$. Then (S_1, L_1) is a semipolarized surface of genus 2 and of type (II) (a simple reduction of (S, L)). Moreover we have the following

Lemma 4.2. S_1 *is minimal.*

Proof. Suppose that S_1 is not minimal. What we have seen before shows that (S_1, L_1) has a simple reduction which is a semipolarized surface of sectional genus two, L_2-minimal of type (I). Let $\sigma_2 : S_1 \longrightarrow S_2$ be the reduction morphism contracting the exceptional curve E_1. Then $L_1 = \sigma_2^* L_2 - E_1$, $L_1 E_1 = 1$. Let $p = \sigma(E)$, where $\sigma : S \longrightarrow S_1$ is the contraction considered before. Then $\sigma^* E_1 = \tilde{E}_1 + \nu E$, where $\tilde{E}_1 = \sigma^{-1}(E_1)$ and $\nu = \tilde{E}_1 E = \mathrm{mult_p}(\mathrm{E_1})$ is 0 or 1 according to whether $p \notin E_1$ or $p \in E_1$.

If $\nu = 0$, then $\tilde{E}_1$ is a (-1)-curve in S, hence $\tilde{E}_1$ is also a component of $\mathcal{E}$ and thus $1 = L\mathcal{E} \geq LE + L\tilde{E}_1 = 2$, which is clearly impossible.

If $\nu = 1$, then $\sigma^* E_1 = \tilde{E}_1 + E$, where $\tilde{E}_1 = \sigma^{-1}(E_1)$ and $\tilde{E}_1 E = \mathrm{mult_p}(\mathrm{E_1}) = 1$ since $p \in E_1$. Moreover $L = \eta^* L_2 + \tilde{E}_1$. We have that

$$\begin{aligned} L\eta^* K_{S_2} &= (\eta^* L_2 + \tilde{E}_1)\eta^* K_{S_2} = L_2 K_{S_2} + \tilde{E}_1 \eta^* K_{S_2} \\ &= (\sigma^* E_1 - E)(\sigma^* K_{S_1} - \sigma^* E_1) = 0 \end{aligned}$$

and this contradicts the assumption $1 = L\eta^* K_{S_0} = L\eta^* K_{S_2}$. Hence S_1 is minimal. □

As a consequence of Lemma 4.2, $S_1 = S_0$ and (S_1, L_1) is as in Proposition 3.5. This leads to III-i) in Theorem 2. As to the effect of (2.1) note that $p = \sigma(E)$ cannot lie on any (-2)-curve C of S_1 since otherwise $L_1 C = K_{S_1} C = 0$, being $L_1 \equiv K_{S_1}$. Similarly, if $p_g(S) = 2$ (the maximum, since $K_{S_1}^2 = L_1^2 = 1$) then $|K_{S_1}|$

is a pencil of curves of genus 2, and some curve $C \in |K_{S_1}|$ can be singular. Then p cannot be a singular point of C, otherwise $L_1 C = K_{S_1} C = K_{S_1}^2 = 1$, contradicting (2) in (2.1).

Now we study further the case arising from (a_1).

For simplicity let $(\Sigma, M) = (S_1, L_1)$, where Σ is a minimal surface with $K_\Sigma \equiv 0$ and M is a nef line bundle on Σ, with $M^2 = 4$.

Note that the line bundle $M - K_\Sigma$ is nef and big, being numerically equivalent to M. Hence $h^i(M) = h^i(K_\Sigma + (M - K_\Sigma)) = 0$ for $i = 1, 2$ by the Kawamata-Viehweg vanishing theorem. So

$$h^0(M) = \chi(M) = \chi(\mathcal{O}_\Sigma) + M^2/2 = 2 + \chi(\mathcal{O}_\Sigma).$$

Therefore $|M|$ is a pencil when Σ is either abelian or bielliptic, a net when Σ is an Enriques surface, and a web when Σ is K3.

Let Σ be a K3 surface. Then we know the following: if $|M|$ has no fixed components, then $|M|$ is base-point-free ([23], Theorem 3.1). Moreover, since $M^2 = 4$, $|M|$ cannot be composed with a pencil by [23], Proposition 2.6; hence its general element is a smooth curve. Let φ_M be the morphism associated to $|M|$.

First suppose that $|M|$ is not hyperelliptic; then $\varphi_M : \Sigma \to \varphi_M(\Sigma) \subset \mathbb{P}^3$ is exactly the normalization of a quartic surface in $\mathbb{P}^3$ ([23], Theorem 6.1). In this case the effect of (2.1) is that the point p blown up by σ cannot lie on any curve $C \subset \Sigma$ contracted by φ_M.

Suppose now that $|M|$ is hyperelliptic, i.e., φ_M has degree 2; then according to [23] Proposition 5.6, either

(j) there exists an irreducible curve E with $E^2 = 0$ and $ME = 2$, or
(jj) there exists an irreducible curve B of arithmetic genus two and $M = \mathcal{O}_\Sigma(2B)$.

In case (jj) $\varphi_M = v_2 \circ \varphi_B$, where $v_2 : \mathbb{P}^2 \longrightarrow \mathbb{P}^5$ is the Veronese embedding and $\varphi_B : \Sigma \longrightarrow \mathbb{P}^2$ is a double cover branched along a sextic [23], Proposition 5.7.

In case (j) either $\varphi_M(\Sigma)$ is a smooth quadric and φ_M is branched along a divisor in $|\mathcal{O}(4,4)|$, or there exists an elliptic pencil $|E|$ and one of the following conditions holds:

(α) $M = \mathcal{O}_\Sigma(2E + \Gamma_0 + \Gamma_1)$, where Γ_0, Γ_1 are rational irreducible curves such that $\Gamma_0 E = \Gamma_1 E = 1, \Gamma_0\Gamma_1 = 0$;

(β) $M = \mathcal{O}_\Sigma(2E + \Delta)$, with $\Delta = 2\sum_{i=0}^{N} \Gamma_i + \Gamma_{N+1} + \Gamma_{N+2} (N \geq 0)$, where all Γ_i's, are rational irreducible curves giving rise to a Dynkin diagram of type D_{N+3} and $E\Gamma_0 = 1, E\Gamma_j = 0$ for $j \neq 0$, Γ_0 corresponding to the first vertex of the diagram.

In both cases (α) and (β) $\varphi_M(\Sigma)$ is the quadric cone and the pencil $|E|$ is the pull-back of the system of generators. Note that in both cases $M\Gamma_0 = 0$, hence M is not ample.

As to the effect of (2.1) in connection with Theorem 1, III-i), in case (jj) there are no evident restrictions. In case (j) when $\varphi_M(\Sigma)$ is a smooth quadric, let $E_1 = \varphi_M^* \mathcal{O}(1,0), E_2 = \varphi_M^* \mathcal{O}(0,1)$. Since $ME_i = 2$ for $i = 1, 2$ it is clear that the point p blown-up by σ cannot be a singular point of any singular element of the pencils $|E_i|, i = 1, 2$. Let $\varphi_M(\Sigma)$ be a quadric cone. In both cases (α) and (β) we can note that the point p cannot lie on any Γ_j; moreover it cannot be a singular point of any singular element of the pencil $|E|$.

Finally, the case when M has a fixed part is easily settled by Proposition 2.4. Actually $|M| = |3E| + \Gamma$, where E is a curve with $E^2 = 0$, Γ is a (-2)-curve and $E\Gamma = 1$. So we have $ME = \Gamma E = 1$; on the other hand $|E|$ is a pencil sweeping out the whole Σ. Taking into account (2.1) this gives the following

Proposition 4.3. *Let* (S, L) *be as in case* III-i) *of Theorem* 1. *If* S_1 *is a* K3 *surface, then* $|L_1|$ *is base point free.*

Now let Σ be an Enriques surface. First of all it is easy to see that $|M|$ cannot have fixed components. Otherwise, by [7], Proposition 2.4 (see also [8], Proposition 3.1.6, p. 173) our linear system would be of the form $|2E + R|$, where $|2E|$ is a genus 1-pencil and R is a smooth rational curve with $ER = 1$. But then we would get $4 = M^2 = 4E^2 + 4ER + R^2 = 0 + 4 - 2 = 2$, a contradiction. Note also that $|M|$ is not composed with a pencil. Actually, $|M|$ itself is not a pencil, as we said; on the other hand, if $M \equiv kB$, with $k \geq 2$ from $4 = M^4 = k^2B^2$ we get $k = 2$ and $B^2 = 1$, but this is impossible by the genus formula. Now, the possibilities which can occur “a priori” are listed in [6], Lemma 3.3.3 (see also [8], Proposition 4.1.2, p. 227). However, recall that we are working over the field of complex numbers. So, if $|M|$ has base points, then it can have only two distinct base points and the map φ_M exhibits Σ as a double plane ([8], Theorem 4.4.1, p. 240). On the other hand, if $|M|$ is base point free, then the condition $M^2 = 4$ rules out all possibilities except the case in which φ_M exhibits Σ as a four-tuple cover of $\mathbb{P}^2$. A complete description of this case can be found in [29]. As to the effect of (2.1), in both cases it seems difficult to convert the condition $\epsilon(L_1, p) \geq 2$ into explicit restrictions on the point p.

For bielliptic surfaces we know that M is an ample line bundle, by Lemma 2.6. Recalling the explicit bases of $\mathrm{Num}(\Sigma)$, listed in Proposition 2.5, we can write $M = a\frac{1}{h}A + b\frac{1}{k}B$ in $\mathrm{Num}(\Sigma)$ for some positive integers a, b. Then from the condition $M^2 = 4$ we get $ab = 2$. Hence we get the following two possibilities for each of the seven types: $(a, b) = (2, 1), (1, 2)$. Let $\psi : \Sigma = (A \times B)/G \to B/G = \mathbb{P}^1$ be the elliptic fibration and let f be the reduced component of the fibre of maximal multiplicity. It is immediate to see that $f = \frac{1}{h}A$, hence if $(a, b) = (2, 1)$ we get $Mf = M\frac{1}{h}A = (2\frac{1}{h}A + \frac{1}{k}B)\frac{1}{h}A = 1$. Thus the effect of (2.1) for case III-i) of Theorem 1 is the following. If L_1 corresponds to $(a, b) = (2, 1)$ then the point p blown up by σ cannot belong to the fibre of maximal multiplicity of the elliptic fibration of S_1.

Finally consider abelian surfaces. In this case, note that M has to be ample by Lemma 2.6. Let (d_1, d_2) be the type of the polarization induced by M on Σ ([14], p. 47). From the equality $2 = h^0(M) = \frac{1}{2}M^2 = d_1d_2$ ([14], p. 289) we thus see that $(d_1, d_2) = (1, 2)$. Abelian surfaces with a polarization of type $(1, 2)$ have been studied in [1]. In particular it can be that $\Sigma = E \times F$, with E, F elliptic curves and $M = \mathcal{O}_\Sigma(E + 2F)$. Hower this case cannot occur by (2.1) since $MF = 1$

and the fibres F sweep out the whole Σ. Apart from this case, $|M|$ has no fixed components, its base locus consists of 4 distinct points, and the general element is a smooth curve [1], Section 1. Moreover Σ with the $(1,2)$ polarization given by M can be identified with the dual of the Prym variety associated with the elliptic involution carried by the general element $D \in |M|$, with the natural polarization [1], Theorem 1.12. Apparently, (2.1) gives no special restrictions in this case.

Now we suppose that S is minimal. Note that it cannot be $\kappa(S) = 0$; otherwise K_S would be numerically 0, but this is impossible, since $LK_S = 2$. Since the case $\kappa(S) = 1$ is studied in [16], we can suppose that S is minimal and of general type. This case will be treated in the next section.

5. More on L not big on a minimal surface of general type

In this section we give a structure theorem for pairs (S, L) where S is a minimal surface of general type, L is a nef line bundle on S with $L^2 = 0, LK_S = 2$, having Iitaka dimension $\kappa(S, L) = 1$. Before doing that we produce two examples of pairs (S, L) with Iitaka dimension $\kappa(S, L) = 1, -\infty$, respectively.

Example 5.1. Let C, Γ be two smooth curves with $g(\Gamma) = 2$ and $g(C) \geq 2$. Let $S = C \times \Gamma$ and let $p : S \longrightarrow C$ and $q : S \longrightarrow \Gamma$ be the projections on the first and second factor, respectively. Note that S is a minimal surface of general type endowed with a genus two fibration $p : S \longrightarrow C$. Let $L = p^*\mathcal{O}_C(x)$, for some point $x \in C$. Note that L is nef, $L^2 = 0$ and $LK_S = (p^*\mathcal{O}_C(x))(p^*K_G + q^*K_\Gamma) = 2$. Moreover, by the projection formula we have $H^0(S, mL) = H^0(C, \mathcal{O}_C(mx))$, hence $\dim \phi_{|mL|}(S) = 1$ for $m >> 0$. Therefore $\kappa(S, L) = 1$.

Example 5.2. Let S be as in the previous Example 5.1. Let $y_1, y_2 \in \Gamma$ be two distinct points such that $y_1 - y_2$ is not a torsion divisor and let $x \in C$. Set $L = p^*\mathcal{O}_C(x) \otimes q^*\mathcal{O}_\Gamma(y_1 - y_2)$. In this example L is nef and not big, since $L^2 = 2\deg(y_1 - y_2) = 0$, and $LK_S = 2$, as before. Moreover $H^0(S, mL) \cong H^0(C, \mathcal{O}_C(mx)) \otimes H^0(\Gamma, \mathcal{O}_\Gamma(m(y_1 - y_2))) = 0$, for every $m > 0$ since $\mathcal{O}_\Gamma(m(y_1 - y_2))$ has no nontrivial sections. Therefore $\kappa(S, L) = -\infty$.

The case in which $\kappa(S, L) = 1$, that is there exists an integer $t > 0$ such that $h^0(S, tL) \geq 2$, is settled by the following proposition.

Proposition 5.3. *Let (S, L) be a semipolarized surface with $g(S, L) = 2$ and $L^2 = 0$, with S a minimal surface of general type. If $\kappa(S, L) = 1$, then S admits a fibration over a smooth curve B, whose general fibre F has genus $g(F) \geq 2$ and $L \equiv \frac{1}{g(F)-1}F$.*

Proof. Let $t > 0$ be an integer such that $h^0(tL) \geq 2$, let $D \in |tL|$ and write $|D| = Z + |M|$, where Z and $|M|$ are the fixed and the moving part of $|D|$ respectively. The fact that D is nef along with $0 = t^2L^2 = D^2 = D(Z + M) = DZ + DM$ gives that $DM = 0, DZ = 0$. From $0 = DM$ and the fact that M moves, we have $ZM \geq 0$ and $M^2 \geq 0$, hence $M^2 = ZM = 0$. Combining this with $DZ = 0$

we get $Z^2 = 0$. Therefore $|M|$ defines a morphism Φ from S to $\mathbb{P}^N$ whose image is a curve Γ. Moreover the map associated to D coincides with Φ. Consider the Remmert-Stein factorization of Φ:

$$\Phi : S \xrightarrow{\phi} B \xrightarrow{\psi} \Gamma \subset \mathbb{P}^N,$$

where B is a smooth curve and ϕ has connected fibres. Let F be a general fibre of ϕ. Then

$$tL = \phi^*\psi^*\mathcal{O}_\Gamma(1) \equiv (\deg \Gamma)(\deg \psi)\phi^{-1}(b),$$

where $b \in B$. So $tL \equiv mF$, where $m = (\deg \Gamma)(\deg \psi)$. Then

$$2g(F) - 2 = F^2 + FK_S = FK_S = \frac{t}{m}\, LK_S = \frac{2t}{m}.$$

This says that $(g(F) - 1)L \equiv F$. $\square$

Here are several examples of pairs (S, L) as in Proposition 5.3. The most obvious is that of genus two fibrations.

Example 5.4. Let $Y = \mathbb{P}_C(\mathcal{E})$ be a $\mathbb{P}^1$-bundle over a smooth curve C of genus $q \geq 0$, defined by a rank-2 vector bundle $\mathcal{E}$ normalized as in [15], p. 373, let E be the section of minimal self-intersection $E^2 = -e = \deg \mathcal{E}$, and let f be a fibre. Recall that the classes of E and f generate $\mathrm{Num}(Y)$. Let $B \equiv 3E + kf$, for some integer k. For k sufficiently large, the line bundle $2B$ is ample and spanned so that its linear system contains a smooth curve Δ. Let $p : S \to Y$ be the double cover of Y, branched along $\Delta \in |2B|$. Then the line bundle $L := p^*[f]$ is nef and $L^2 = (p^*f)^2 = 2f^2 = 0$. Moreover, recalling that $K_Y \equiv -2E + (2q - 2 - e)f$, we get $LK_S = (p^*f)(p^*(K_Y + B)) = 2f(E + (k + 2q - 2 - e)f) = 2$. Let t be a positive integer. Since $p_*\mathcal{O}_S = \mathcal{O}_Y \oplus [-B]$, we get $h^0(tL) = h^0(p_*tL) = h^0(tf)) + h^0((tf - B))$ by projection formula. But $tf - B$ cannot be effective, because $(tf - B)f < 0$. Hence $h^0(tL) = h^0(tf) \geq 2$ for $t >> 0$. So (S, L) is a semipolarized surface of sectional genus 2 endowed with a genus two fibration over C and L corresponds to a fibre. Now we have

$$K_S^2 = (p^*(K_Y + B))^2 = 2(K_Y + B)^2 = 2(2k + 4q - 4 - 3e).$$

Furthermore, recalling that $\chi(\mathcal{O}_S) = 2\chi(\mathcal{O}_Y) + g(B) - 1$ (e. g., see [22], Corollary 2.2), we get

$$\chi(\mathcal{O}_S) = q + 2k - 3e - 1.$$

For k large enough we thus see that $K_S^2 > 0$ and $\chi(\mathcal{O}_S) \geq 2$, which implies that S is of general type. Finally note that S is minimal, due to the construction.

This gives a concrete family. Surfaces of general type endowed with a genus two fibration are widely discussed in [30].

Example 5.5. Let $\psi : X = (E \times B)/G \to C = \mathbb{P}^1$ be an elliptic quasi bundle as in [16], (1.2), p. 364. Here E is a smooth elliptic curve, B is a smooth curve of genus ≥ 1 and G is an abelian group of type $\mathbb{Z}_r \times \mathbb{Z}_s$ (possibly cyclic, i.e., $r = 1$) acting on B with $C = B/G$, acting on E by translations, and acting on X with the induced diagonal action. Denote again by E the general fibre of ψ and by B the general fibre of the morphism $X \to E/G$. Then $E^2 = B^2 = 0$ while $EB = \gamma$, the order of G. Let μ be the least common multiple of the multiplicities $m_1, \dots, m_l$ of the multiple fibres of ψ. By [26], Proposition 1.4 we know that $\frac{1}{\mu}E \in \mathrm{Num}(X)$. Recall that a multiple fibre of multiplicity m_i of ψ corresponds to a branch point of the finite morphism (of degree γ) $\pi : B \to C = B/G$, at which m_i branches collapse together. Then, by combining the Riemann-Hurwitz formula for π with the canonical bundle formula for elliptic surfaces, one can easily see that $\frac{\mu}{\gamma}(2g(B) - 2)$ is an integer. Let $\delta = 0$ or 1 denote its class mod 2. By an unpublished result of Palleschi, Serrano and the third author [17] (see [19], Theorem 2.1) $\mathrm{Num}(X)$ is generated by the classes of $\frac{1}{\mu}E$ and $\frac{\mu}{\gamma}B + \frac{\delta}{2\mu}E$. Now let $\mathcal{B} \in \mathrm{Pic(X)}$ be a line bundle, whose numerical class is

$$a\frac{1}{\mu}E + b(\frac{\mu}{\gamma}B + \frac{\delta}{2\mu}E)$$

for some integers a and b. According to [19], Proposition 2.5 and Proposition 2.8 we know that $2\mathcal{B}$ is spanned for $b \geq 1$ and either $a \geq g(B)\frac{\mu}{\gamma}$ if $\delta = 0$, or $2a + b \geq \frac{\mu}{\gamma}(2g(B) - 2) + 2$ if $\delta = 1$. Note that for $b = 1$ these conditions simply require that a is sufficiently large. Suppose that a and b satisfy these conditions. Then by Bertini's theorem there exists a smooth divisor $\Delta \in |2\mathcal{B}|$. Let $p : S \to X$ be the double cover branched along Δ. Then S is a smooth surface endowed with a fibration $\phi = \psi \circ p : S \to C$, whose general fibre, say F, is a nef divisor. Let $M \in \mathrm{Pic(X)}$ be a line bundle whose numerical class is that of $\frac{1}{\mu}E$ and set $L := p^*M$. Then L is nef and $L^2 = 0$, being $\mu L \equiv F$. Since $K_S = p^*(K_X + \mathcal{B})$ and K_X is a rational multiple of E by the canonical bundle formula for elliptic surfaces, we get

$$K_S L = p^*(K_X + \mathcal{B})p^*(\frac{1}{\mu}\ E) = 2\mathcal{B}\frac{1}{\mu}E = \frac{2}{\mu}b\frac{\mu}{\gamma}BE = 2b.$$

Thus

$$K_S L = 2 \quad \text{if and only if} \quad b = 1.$$

So for $b = 1$ and a large enough we get a semipolarized surface (S, L) of sectional genus 2 fibered over C. By looking at the invariants K_S^2 and $\chi(\mathcal{O}_S)$ it is easy to check that S is of general type. As to the genus of the general fibre F of $\phi : S \to C$, since $b = 1$, we immediately see that

$$2g(F) - 2 = F^2 + FK_S = (p^*E)(p^*(K_X + \mathcal{B})) = 2E\mathcal{B} = 2b\ \frac{\mu}{\gamma}BE = 2\mu.$$

Hence $g(F) = \mu + 1$.

We would like to stress that pairs (S, L) as in Example 5.5 arise from every elliptic quasi bundle over $\mathbb{P}^1$. In particular, we can apply the conclusion above to the special case of bielliptic surfaces, mentioned in Proposition 2.5. For types 1, 3, 5 and 7 (see [25], Table I at p. 528) we have $\mu = 2, 4, 3, 6$, respectively. Hence they lead to pairs (S, L) with S of general type endowed with a fibration of genus $\mu + 1 = 3, 5, 4, 7$. More general quasi bundles in the sense of [27] can give other examples.

Example 5.6. Let S be as in [21], Theorem 2.2, case i). Then $S = (F \times C)/G$, where C is a smooth curve of genus ≥ 2, F is a hyperelliptic curve of genus 3, G is a finite group acting faithfully on C and F, with C/G and F/G both rational and with the induced diagonal action on $F \times C$ being free. Moreover, the projection $p : S \to C/G = \mathbb{P}^1$ has exactly 6 double fibres, each having a smooth support [21], Proof of Theorem 2.2, case i). So S is a quasi bundle over $\mathbb{P}^1$. Let $F_0 = 2f$ be one of the double fibres and denote again by F the general fibre of p. Of course $L = [f]$ is nef and $L^2 = 0$. We have

$$4 = 2g(F) - 2 = F^2 + FK_S = FK_S = 2fK_S = 2f^2 + 2fK_S = 2(2g(f) - 2).$$

Hence $g(f) = 2$. Thus (S, L) is an example as in Proposition 5.3. In [21], Theorem 1.2 one can find the precise list (namely, cases Ia–Id) of the possibilities for the invariants $g(C)$ and G for a surface S as above.

References

[1] Barth, W.: *Abelian surfaces with* $(1, 2)$*-polarization.* Algebraic geometry, Proc. Symp., Sendai/Jap. 1985, Adv. Stud. Pure Math. **10** (1987), 41–84. Zbl 0639.14023

[2] Barth, W.; Peters, C.; Van de Ven, A.: *Compact complex surfaces.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, **4**. Springer-Verlag Berlin etc. 1984. Zbl 0718.14023

[3] Beauville, A.: *Complex algebraic surfaces.* London Mathematical Society Lecture Notes **68**, Cambridge University Press, Cambridge etc. 1983. Zbl 0512.14020

[4] Beltrametti, M.; Sommese, A. J.: *On generically polarized Gorenstein surfaces of sectional genus two.* J. Reine Angew. Math. **386** (1988), 172–186. Zbl 0652.14014

[5] Beltrametti, M.; Lanteri, A.; Palleschi, M.: *Algebraic surfaces containing an ample divisor of arithmetic genus two.* Ark. Mat. **25** (1987), 189–210. Zbl 0645.14015

[6] Cossec, F. R.: *Projective models of Enriques surfaces.* Math. Ann. **265** (1983), 283–334. Zbl 0501.14021

[7] Cossec, F. R.: *On the Picard group of Enriques surfaces.* Math. Ann. **271** (1985), 577–600. Zbl 0541.14031

[8] Cossec, F. R.; Dolgachev, I. V.: *Enriques surfaces. I.* Progress in Mathematics **76**. Boston, MA etc.: Birkhäuser Verlag, Boston, MA etc. 1989. Zbl 0665.14017

[9] Dolgachev, I.: *Algebraic surfaces with $q = p_g = 0$.* In: Algebraic Surfaces. C.I.M.E., Cortona 1977, Liguori, Napoli 1981, 97–215.

[10] Fujita, T.: *Classification of polarized manifolds of sectional genus two.* Algebraic geometry and commutative algebra **I** (1988), 73–98. Zbl 0695.14019

[11] Fujita, T.: *Remarks on quasi-polarized varieties.* Nagoya Math. J. **115** (1989), 105–123. Zbl 0699.14002

[12] Fujita, T.: *On certain polarized elliptic surfaces.* In: Lanteri, A. (ed.) et al.: Geometry of complex projective varieties. Proceedings of the conference, Cetraro, Italy, May 28–June 2, 1990. Rende: Mediterranean Press, Semin. Conf. **9** (1993), 153–163. Zbl 0937.14023

[13] Fukuma, Y.: *A lower bound for the sectional genus of quasi-polarized surfaces.* Geom. Dedicata **64**(2) (1997), 229–251. Zbl 0897.14008

[14] Lange, H.; Birkenhake, Ch.: *Complex abelian varieties.* Grundlehren der Mathematischen Wissenschaften **302**, Springer-Verlag, Berlin 1992. Zbl 0779.14012

[15] Hartshorne, R.: *Algebraic geometry.* Graduate Texts in Mathematics **52**. Springer-Verlag, New York-Heidelberg-Berlin 1977. Zbl 0367.14001

[16] Lanteri, A.; Turrini, C.: *Elliptic surfaces with a nef line bundle of genus two.* Collect. Math. **49**(2–3) (1998), 361–381. Zbl 0928.14027

[17] Lanteri, A.; Palleschi, M.; Serrano, F.: *Elliptic surfaces with low genus polarizations.* Preliminary notes 1992.

[18] Maeda, H.: *Nef line bundles on algebraic surfaces.* Kodai Math. J. **18**(1) (1995), 187–197. Zbl 0854.14005

[19] Mella, M.; Palleschi, M.: *The k-very ampleness on an elliptic quasi bundle.* Abh. Math. Semin. Univ. Hamb. **63** (1993), 215–226. Zbl 0806.14016

[20] Nikulin, V. V.: *Weil linear systems on singular* K3 *surfaces.* Algebraic geometry and analytic geometry, Proc. Conf., Tokyo/Jap. 1990, ICM-90 Satell. Conf. Proc. 1991, 138–164. Zbl 0785.14021

[21] Pardini, R.: *The classification of double planes of general type with $K^2 = 8$ and $p_g = 0$.* J. Algebra **259**(1) (2003), 95–118. Zbl 1066.14046

[22] Persson, U.: *Double coverings and surfaces of general type.* Algebr. Geom., Proc., Tromsø Symp. 1977, Lect. Notes Math. **687** (1978), 168–195. Zbl 0396.14003

[23] Saint-Donat, B.: *Projective models of* K-3 *surfaces.* Am. J. Math. **96** (1974), 602–639. Zbl 0301.14011

[24] Sakai, F.: *D-dimensions of algebraic surfaces and numerically effective divisors.* Compos. Math. **48** (1983), 101–118. Zbl 0509.14038

[25] Serrano, F.: *Divisors of bielliptic surfaces and embeddings in $\mathbb{P}^4$.* Math. Z. **203**(3) (1990), 527–533. Zbl 0722.14023

[26] Serrano, F.: *Elliptic surfaces with an ample divisor of genus two.* Pac. J. Math. **152**(1) (1992), 187–199. Zbl 0756.14026

[27] Serrano, F.: *Fibrations on algebraic surfaces.* In: Lanteri, A. (ed.) et al.: Geometry of complex projective varieties. Proceedings of the conference, Cetraro, Italy, May 28–June 2, 1990. Rende: Mediterranean Press, Semin. Conf. **9** (1993), 289–301. Zbl 0937.14021

[28] Urabe, T.: *On Nikulin's theorem on fixed components of linear systems on* K3 *surfaces.* J. Math. Soc. Japan **45**(2) (1993), 185–195. Zbl 0808.14005

[29] Verra, A.: *On the Enriques' surface as a fourfold cover of* $\mathbb{P}^2$. Math. Ann. **266** (1983), 241–250. Zbl 0506.14032

[30] Xiao, G.:*Surfaces fibrées en courbes de genre deux.* (French) Lecture Notes in Mathematics **1137**. Springer-Verlag, Berlin etc. 1985. Zbl 0579.14028

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