Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 46 (2005), No. 2, 523-535.

Transformations for Hypersurfaces with Vanishing Gauss-Kronecker Curvature

A. V. Corro[1] **W. Ferreira**[1] **K. Tenenblat**[2]

[1]*Instituto de Matemática e Estatística, Universidade Federal de Goiás*
74001-970 Goiânia, GO, Brazil
e-mail: corro@mat.ufg.br walter@mat.ufg.br

[2]*Departamento de Matemática, Universidade de Brasília*
70910-900, Brasília, DF, Brazil
e-mail: keti@mat.unb.br

Abstract. We provide a method of constructing families of hypersurfaces of a space form with zero Gauss-Kronecker curvature, from a given such hypersurface, based on Ribaucour transformations. Applications provide a 1-parameter family of complete, non-cylindrical hypersurfaces of R^4, with zero Gauss-Kronecker curvature, a 5-parameter family of compact Dupin hypersurfaces of S^4, with vanishing Gauss-Kronecker curvature, infinite families of hypersurfaces of R^{n+1} and of the hyperbolic space H^4, with flat Gauss-Kronecker curvature.

Introduction

Surfaces with flat Gaussian curvature in the Euclidean space R^3 are ruled, developable surfaces. The only complete ones are planes and cylinders over plane curves [9]. In higher dimensions, the complete hypersurfaces $M^n \subset R^{n+1}$ with flat sectional curvature are hyperplanes and cylinders over plane curves [9]. This result does not hold if one considers complete hypersurfaces with zero Gauss-Kronecker curvature. However, imposing additional conditions such as nonnegative sectional curvature [7] and constant relative nullity $\nu > 0$, one can show that such a hypersurface is a cylinder over an $(n-\nu)$-dimensional submanifold.

[1]Partially supported by CAPES/PROCAD and CNPq/PADCT.
[2]Partially supported by CNPq, CAPES/PROCAD and CNPq/PADCT.

In general, complete hypersurfaces with zero Gauss-Kronecker curvature are not necessarily cylinders. Such an example was given by Sacksteder [11]. Results on complete minimal hypersurfaces in S^4 with vanishing Gauss-Kronecker curvature were obtained in [1], [9] and [10]. Since the classification of complete hypersurfaces with zero Gauss-Kronecker curvature is far from complete, it is important to study methods which produce such hypersurfaces. In this paper we introduce a method based on Ribaucour transformations.

Ribaucour transformations for surfaces of constant Gaussian curvature and constant mean curvature (including minimal) surfaces, were considered at the beginning of last century (see Bianchi [2]) and they were recently applied for the first time to obtain minimal surfaces [4]. These results were extended to linear Weingarten surfaces in [5]. Ribaucour transformations were also considered in [3] to produce Dupin hypersurfaces of the Euclidean space and submanifolds of constant sectional curvature in [6].

In this paper, we consider n-dimensional orientable hypersurfaces M^n of a space form, with flat Gauss-Kronecker curvature. By considering an integrable system of differential equations on M, we provide a method of construction of families of hypersurfaces $\tilde{M}^n$, locally associated by Ribaucour transformations to M, such that $\tilde{M}$ has also zero Gauss-Kronecker curvature (see Remark 1.4, Theorems 1.5 and 1.6).

We provide some applications of this method. We first obtain a 6-parameter family of hypersurfaces with zero Gauss-Kronecker curvature, contained in R^4, which are associated to a hypersurface given by Sacksteder. Generically, a hypersurface of this family has singularities. However, the family contains a 1-parameter family of complete, non-cylindrical hypersurfaces. Our second application provides a 5-parameter family of compact, Dupin hypersurfaces of S^4, with zero Gauss-Kronecker curvature, associated to a tube around the Veronese surface contained in S^4. We then obtain two infinite families of hypersurfaces of R^{n+1}, with zero Gauss-Kronecker curvature, associated, by Ribaucour transformations, to a hyperplane and to a cylinder, respectively. We conclude with an infinite family of 3-dimensional hypersurfaces of the hyperbolic space H^4, with zero Gauss-Kronecker curvature, associated to $H^3 \subset H^4$.

1. Ribaucour transformation for hypersurfaces

In this section, we recall the basic theory of Ribaucour transformation for hypersurfaces and provide its characterization as a system of differential equations. For the proofs and more details see [2], [3] and [5].

Let M^n be an orientable hypersurface of a Riemannian manifold $\bar{M}^{n+1}$. Suppose M has an orthonormal frame of principal directions e_i, $1 \leq i \leq n$. A submanifold $\tilde{M}^n \subset \bar{M}^{n+1}$ is *associated to M by a Ribaucour transformation with respect to* $e_1, \dots, e_n$ if there exist a diffeomorphism $\psi : M \to \tilde{M}$, a differentiable function $\ell : M \to R$ and unit vector fields N and $\tilde{N}$ normal to M and $\tilde{M}$ respectively, such that:

a) $\exp_q(\ell(q)N(q)) = \exp_{\psi(q)}(\ell(q)\tilde{N}(\psi(q)))$, $\forall q \in M$;

b) the subset $\exp_q(\ell(q)N(q))$, $q \in M$ is a hypersurface;

c) $d\psi(e_i)$ are orthogonal principal directions on $\tilde{M}$.

This transformation is invertible in the sense that there exist orthonormal principal direction vector fields $\tilde{e}_1, \dots, \tilde{e}_n$ on $\tilde{M}$ such that M is associated to $\tilde{M}$ by a Ribaucour transformation with respect to these vector fields. One may consider the analogue definition for locally associated submanifolds or for immersions. The definition considered above differs slightly from the classical notion of a Ribaucour transformation. This is due to the fact that if M is a hypersurface with a principal curvature whose multiplicity is bigger than one, then Ribaucour transformations with respect to distinct sets of principal directions may provide distinct families of hypersurfaces associated to M. For example, any oriented hypersurface of the Euclidean space R^{n+1} is locally associated to a hyperplane, by a Ribaucour transformation with respect to a set of conveniently chosen orthonormal vector fields of R^{n+1} (see [5], [12]).

In what follows $\bar{M}(\bar{K})^{n+1}$ will be a space form of constant sectional curvature $\bar{K} \in \{-1, 0, 1\}$, i.e.

$$\bar{M}(\bar{K})^{n+1} = \begin{cases} S^{n+1} \subset R^{n+2} \quad \text{if } \bar{K} = 1, \\ R^{n+1} \text{ if } \bar{K} = 0, \\ H^{n+1} \subset L^{n+2} \quad \text{if } \bar{K} = -1, \end{cases}$$

where L^{n+2} is the Lorentzian space.

Let M^n be a hypersurface of $\bar{M}(\bar{K})^{n+1}$. Let e_i, $1 \leq i \leq n$, be an orthonormal frame of principal directions on M and let N be unit vector field normal to M. We denote by ω_i the one forms dual to the vector fields e_i and by ω_{ij}, $1 \leq i, j \leq n$, the connection forms determined by $d\omega_i = \sum_{j \neq i} \omega_j \wedge \omega_{ji}$, $\omega_{ij} + \omega_{ji} = 0$. The normal connection is given by $\omega_{in+1} =< \bar{\nabla} e_i, N >$, where $\bar{\nabla}$ is the connection of the space form $\bar{M}$. The Gauss equation is

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} + \omega_{in+1} \wedge \omega_{n+1j} - \bar{K} \omega_i \wedge \omega_j$$

and the Codazzi equations are

$$d\omega_{in+1} = \sum_j \omega_{ij} \wedge \omega_{jn+1}.$$

Since e_i are orthonormal principal directions, we have

$$\bar{\nabla}_{e_i} N = \lambda^i e_i, \qquad \omega_{in+1} = -\lambda^i \omega_i. \tag{1}$$

For each integer r, $1 \leq r \leq n$, the r-mean curvature, H_r, of M is given by

$$H_r = \frac{1}{\begin{pmatrix} n \\ r \end{pmatrix}} \sum_{1 \leq i_1 < \cdots < i_r \leq n} \lambda^{i_1} \lambda^{i_2} \cdots \lambda^{i_r}$$

and the n-mean curvature of M, $H_n = \lambda^1 \lambda^2 \cdots \lambda^n$, is called the *Gauss-Kronecker curvature of* M.

Whenever λ^i , $\forall i$, $1 \leq i \leq n$, are constant along the integral curves of e_i, i.e. $d\lambda^i(e_i) = 0$, M is said to be a *Dupin hypersurface*.

In what follows, we will provide a characterization, for the hypersurfaces which are locally associated to a given hypersurface, by Ribaucour transformations, by means of a system of differential equations for a function $h : M \to R$, where

$$h = \begin{cases} \tan \ell & \text{if} \quad \bar{K} = 1, \\ \ell & \text{if} \quad \bar{K} = 0, \\ \tanh \ell & \text{if} \quad \bar{K} = -1, \end{cases} \tag{2}$$

and ℓ is the function of the definition of a Ribaucour transformation. We observe that condition a) of the definition is equivalent to saying that

$$\tilde{X} = X + h(N - \tilde{N}), \tag{3}$$

where X and $\tilde{X}$ are local parametrizations of M and $\tilde{M}$.

The proofs of the results of this section, using differential forms, can be found in [3]. See [6] for a different proof in the holonomic case.

Theorem 1.1. *Let M^n be an orientable hypersurface of $\bar{M}^{n+1}(\bar{K})$. Let e_i, $1 \leq i \leq n$, be orthonormal principal directions of M, and λ^i the corresponding principal curvatures, i.e. $dN(e_i) = \lambda^i e_i$. A hypersurface $\tilde{M}$ is associated to M by a Ribaucour transformation with respect to $\{e_i\}$, if and only if, the function $h : M \to R$, described in (2), satisfies $1 + h\lambda^i \neq 0$ and*

$$dZ^j(e_i) + \sum_{k=1}^{n} Z^k \omega_{kj}(e_i) - Z^i Z^j \lambda^i = 0, \quad 1 \leq i \neq j \leq n, \tag{4}$$

where ω_{ij} are the connection forms of the frame e_i and $Z^i = dh(e_i)/(1 + h\lambda^i)$.

Equation (4) is a second order differential equation for h, which is equivalent to a first order linear system given in the following result.

Proposition 1.2. *If h is a solution of (4) which does not vanish on a simply connected domain, then $h = \Omega/W$, where W is a nonvanishing function and the functions Ω, Ω_i, W satisfy*

$$d\Omega_i(e_j) = \sum_{k=1}^{n} \Omega_k \omega_{ik}(e_j), \qquad \text{for } i \neq j, \tag{5}$$

$$d\Omega = \sum_{i=1}^{n} \Omega_i \omega_i, \tag{6}$$

$$dW = -\sum_{i=1}^{n} \Omega_i \lambda^i \omega_i. \tag{7}$$

Conversely, suppose (5)–(7) are satisfied and $W(W + \lambda^i \Omega) \neq 0$, then $h = \Omega/W$ is a solution of (4).

It is a straightforward computation to verify that equation (5) is the integrability condition of equations (6) and (7). The proof of the following result can be found in [3] or [5], in the case $\bar{K} = 0$. For $\bar{K} \neq 0$, the proof is entirely analogous (see also [6]).

Theorem 1.3. *Let M^n be an orientable hypersurface of $\bar{M}^{n+1}(\bar{K})$ parametrized by $X : U \subset R^n \to M$. Assume e_i, $1 \leq i \leq n$, are orthogonal principal directions, λ^i the corresponding principal curvatures and N is a unit vector field normal to M. A hypersurface $\tilde{M}$ is locally associated to M, by a Ribaucour transformation w.r.t. to $\{e_i\}$, if and only if, there exist differentiable functions $W, \Omega, \Omega_i : V \subset U \to R$, which satisfy (5)–(7) with*

$$WS(W + \lambda^i \Omega)(S - \Omega T^i) \neq 0, \quad 1 \leq i \leq n, \tag{8}$$

where

$$S=\sum_i(\Omega_i)^2+W^2+\bar{K}\Omega^2, \tag{9}$$

$$T^i=2\left(\sum_k\Omega_k\omega_{ki}(e_i)-W\lambda^i+d\Omega_i(e_i)+\bar{K}\Omega\right), \tag{10}$$

and $\tilde{X}:V\subset R^n\to\tilde{M}$*, is a parametrization of* $\tilde{M}$ *given by*

$$\tilde{X}=X-\frac{2\Omega}{S}\left(\sum_i\Omega_ie_i-WN+\bar{K}\Omega X\right). \tag{11}$$

Moreover, the normal map of $\tilde{X}$ *is given by*

$$\tilde{N}=N+\frac{2W}{S}\left(\sum_i\Omega_ie_i-WN+\bar{K}\Omega X\right) \tag{12}$$

and the principal curvatures of $\tilde{X}$ *are given by*

$$\tilde{\lambda}^i=\frac{WT^i+\lambda^iS}{S-\Omega T^i}. \tag{13}$$

In (8), we observe that the condition $W\neq 0$ is required by the expression $h=\Omega/W$, while $W+\lambda^i\Omega\neq 0$ corresponds to condition b) of the definition of Ribaucour transformation and $S\neq 0$ determines the domain of the hypersurface $\tilde{X}$. The regularity condition is given by $S-\Omega T^i\neq 0$. In fact, a straightforward computation shows that, using (11) and (5)–(7), we have

$$|d\tilde{X}(e_i)|^2=\frac{(S-\Omega T^i)^2}{S^2}.$$

Therefore, the parametrization $\tilde{X}$ given by (11) may extend regularly to points where $W(W+\lambda^i\Omega)=0$, whenever $S(S-\Omega T^i)\neq 0$.

Remark 1.4. As an immediate consequence of the above theorem we observe that if M is a hypersurface of $\bar{M}^{n+1}(\bar{K})$ with vanishing Gauss-Kronecker curvature and $\{e_i\}$, $1\leq i\leq n$, is an orthonormal frame of principal directions on M, such that $\lambda^{i_0}=0$, then for any solution of the system (5)–(7), satisfying $T^{i_0}=0$, the hypersurface $\tilde{M}$, locally associated to M as in Theorem 1.3, has also zero Gauss-Kronecker curvature. See Propositions 2.3–2.5 for families of such hypersurfaces, obtained by this procedure.

Our next result shows that if we consider solutions of (5)–(7), such that T^i is a multiple of λ^i, say $T^i=-2b\lambda^i$, $b\in R$, we get an integrable system. We observe that this condition together with (5) is equivalent to requiring

$$d\Omega_i=\sum_k\Omega_k\omega_{ik}-(W-b)\omega_{in+1}-\bar{K}\Omega\omega_i.$$

Theorem 1.5. *Let M be a hypersurface of $\bar{M}^{n+1}(\bar{K})$ and let $\{e_i\}$, $1 \leq i \leq n$, be an orthonormal frame of principal directions on M. Then the system*

$$\begin{aligned} d\Omega &= \sum_i \Omega_i \omega_i \\ dW &= \sum_i \Omega_i \omega_{in+1} \\ d\Omega_i &= \sum_k \Omega_k \omega_{ik} - (W-b)\omega_{in+1} - \bar{K}\Omega\omega_i \end{aligned} \tag{14}$$

is integrable. Such solutions determine a family of hypersurfaces $\tilde{M}$ of $\bar{M}(\bar{K})$, locally associated to M by a Ribaucour transformation with respect to $\{e_i\}$, which are regular on the subset satisfying

$$S(S + 2b\lambda^i\Omega) \neq 0, \tag{15}$$

where S is defined by (9) and λ^i are the principal curvatures corresponding to e_i. The function $S - 2bW = 2c$ is a constant, determined by the initial conditions. If $c = 0$, then $\tilde{M}$ is totally geodesic in $\bar{M}$. If $c \neq 0$, then the principal curvatures of M and $\tilde{M}$ have the same multiplicity and $\tilde{H}_n = 0$ if and only if $H_n = 0$.

Proof. We consider the ideal $\mathcal{I}$ generated by the 1-forms

$$\begin{aligned} \theta &= d\Omega - \sum_i \Omega_i \omega_i \\ \beta &= dW - \sum_i \Omega_i \omega_{in+1} \\ \theta_i &= d\Omega_i - \sum_k \Omega_k \omega_{ik} + (W-b)\omega_{in+1} + \bar{K}\Omega\omega_i. \end{aligned} \tag{16}$$

A straightforward computation shows that $d\theta = -\sum_k \theta_k \wedge \omega_k$ and $d\beta = -\sum_k \theta_k \wedge \omega_{kn+1}$. Similarly, using (16) we obtain that $d\theta_i = -\sum_k \theta_k \wedge \omega_{ik} + \beta \wedge \omega_{in+1} + \bar{K}\theta \wedge \omega_i$. It follows that $\mathcal{I}$ is closed under exterior differentiation, hence the system (14) is integrable and the solution is uniquely determined, on a simply connected domain, by the initial conditions given at a point. Moreover, since $dS - 2bdW = 2\sum_{i,j} \Omega_i \Omega_j \omega_{ij} = 0$, we conclude that $S - 2bW$ is a constant function.

Hence any such solution satisfies $S - 2bW = 2c \in R$ and it determines a hypersurface $\tilde{M}$ locally associated to M by a Ribaucour transformation with respect to $\{e_i\}$. The regularity condition requires that $S(S - \Omega T^i) \neq 0$. From (13) the principal curvatures of the associated hypersurfaces are given by

$$\tilde{\lambda}^i = \frac{c\lambda^i}{b(W + \lambda^i\Omega) + c}. \tag{17}$$

By choosing the initial condition such that $c \neq 0$, we conclude the proof of the theorem by using (17). □

In our next result, we obtain all hypersurfaces $\tilde{M}$ associated to a given hypersurface $M^n \subset \bar{M}^{n+1}(\bar{K})$ as in Theorem 1.5. We observe that for $\bar{K} = \pm 1$, we consider the unit sphere as a subset of R^{n+2} and the hyperbolic space as a subset of the Lorentzian space. Hence, $<,>$ will denote the usual metric on R^{n+1} or R^{n+2} if $\bar{K}=0$ or 1 and it will denote the Lorentzian metric on R^{n+2} if $\bar{K}=-1$. Moreover, we will denote $||Y||^2=<Y,Y>$.

Theorem 1.6. *Let $X : M^n \to \bar{M}^{n+1}(\bar{K})$ be a parametrized hypersurface. Then any hypersurface $\tilde{M}$, locally associated to M by a Ribaucour transformation, with respect to any orthonormal frame of principal directions $\{e_i\}$ on M, as in Theorem 1.5, is given by*

$$\tilde{X}_{brV} = X - \frac{2(<V,X>+r)}{||V-bN||^2}\,(V-bN)\,, \tag{18}$$

where N is a unit vector field normal to M, $b,r\in R$, $\bar{K}r=0$ and V is a vector of R^{n+1} (resp. $V\in R^{n+2}$) if $\bar{K}=0$, (resp. $\bar{K}=\pm1$), are such that $(||V||^2-b^2)(||V-bN||)\neq 0$.

Proof. It follows from a straightforward computation that the following functions are solutions to the system (14)

$$\begin{aligned} \Omega_i &= <V,e_i> \\ \Omega &= <V,X>+r \\ W &= -<V,N>+b, \end{aligned} \tag{19}$$

where $b,r\in R$, $\bar{K}r=0$ and V is a vector of R^{n+1} (resp. R^{n+2}), if $\bar{K}=0$ (resp. $\bar{K}=\pm1$). Moreover, $S=||V-bN||^2$ and $S-2bW=||V||^2-b^2$. Since for a fixed constant b, any solution of (14) depends on $n+2$ parameters, it follows that (19) provides all the solutions of (14). The expression of the parametrization of the associate hypersurface follows from (11). Observe that when $\bar{K}\neq 0$, the condition $\bar{K}r=0$ guaranties that the image of $\tilde{X}_{brV}$ is in $\bar{M}$. From Theorem 1.5, we conclude that (18) is a regular hypersurface defined on the subset where $||V-bN||(S+2b\lambda^i\Omega)\neq 0$, $1\leq i\leq n$, where Ω and W are given by (19). □

We conclude this section by providing a geometric interpretation of the family of hypersurfaces described by (18). For fixed b, r, $\in R$ such that $\bar{K}r=0$ and V_1 a unit vector, consider the set of hypersurfaces in $\bar{M}(\bar{K})$ given by

$$Y^t_{brV_1} = X - \frac{2(t<V_1,X>+(1-t)r)}{||tV_1-(1-t)bN||^2}[tV_1-(1-t)bN]$$

where $t\in R$. The family $Y^t_{brV_1}$ contains the parallel surface $(t=0)$ and the reflection of X with respect to a hyperplane orthogonal to V_1 passing through the origin $(t=1)$. This family is associated to the solution of (5)–(7) given by

$$\begin{aligned} \Omega^t_i &= t<V_1,e_i>, \\ \Omega^t &= t<V_1,X>+(1-t)r, \\ W^t &= -t<V_1,N>+(1-t)b. \end{aligned}$$

It is easy to see that the family $\tilde{X}_{brV}$ given by (18) coincides with $Y^t_{brV_1}$.

2. Applications

In this section, we provide some applications of Remark 1.4 and Theorem 1.6. We first obtain a 6-parameter family of hypersurfaces with zero Gauss-Kronecker curvature contained in R^4, which are associated to a hypersurface given by Sacksteder [12] which has zero Gauss-Kronecker curvature. Generically, a hypersurface of this family will have singularities. However, we will show that the family contains a 1-parameter family of complete, non-cylindrical hypersurfaces. Our second application will provide a 5-parameter family of compact, Dupin hypersurfaces of S^4, with zero Gauss-Kronecker curvature. This family is associated to a tube around the Veronese surface contained in S^4. We then obtain two infinite families of hypersurfaces of R^{n+1}, with zero Gauss-Kronecker curvature, associated, by Ribaucour transformations, to a hyperplane and to a cylinder, respectively. We conclude this section with an infinite family of 3-dimensional hypersurfaces of the hyperbolic space H^4, with zero Gauss-Kronecker curvature, associated to $H^3 \subset H^4$.

Proposition 2.1. *Consider the hypersurface of R^4 defined by $X(x,y,z)=(x,\ y,\ z,\ f(x,y,z))$, where $f = x\cos z + y\sin z$, and its Gauss map $N = (\cos z,\ \sin z,\ f_z,\ -1)/\sqrt{2+f_z^2}$.*

i) *For any vector V of R^4 and real numbers b, r such that $|V|^2 - b^2 \neq 0$,*

$$\tilde{X}_{brV} = X - \frac{2(<V,X> + r)}{|V-bN|^2}\,(V - bN)\,, \tag{20}$$

is a hypersurface with zero Gauss-Kronecker curvature, which is locally associated to X by a Ribaucour transformation.

ii) *If $r = 0$ and $V = (0,0,0,\varepsilon)$, where $\varepsilon = \pm 1$, then for any constant b such that $\varepsilon b < 0$ and $b^2 + 2\sqrt{2}\varepsilon b + 1 > 0$, $\tilde{X}_b$, defined by (20), is a complete, non-cylindrical hypersurface, not congruent to X, with zero Gauss-Kronecker curvature.*

Proof. i) The principal curvatures of the hypersurface X are

$$\begin{aligned}
\lambda_1 &= 0,\\
\lambda_2 &= -(2+f_z^2)^{-3/2}\left(f + \sqrt{f^2 + 2(2+f_z^2)}\right),\\
\lambda_3 &= -(2+f_z^2)^{-3/2}\left(f - \sqrt{f^2 + 2(2+f_z^2)}\right)
\end{aligned}$$

and the corresponding principal directions are $e_i = dX(v_i)/|dX(v_i)|$, where

$$\begin{aligned}
v_1 &= \cos z\frac{\partial}{\partial x} + \sin z\frac{\partial}{\partial y},\\
v_2 &= (-y + Q\sin z)\frac{\partial}{\partial x} + (x - Q\cos z)\frac{\partial}{\partial y} + 2\frac{\partial}{\partial z},\\
v_3 &= -(y + Q\sin z)\frac{\partial}{\partial x} + (x + Q\cos z)\frac{\partial}{\partial y} + 2\frac{\partial}{\partial z}
\end{aligned}$$

and

$$Q = \sqrt{f^2 + 2(2+f_z^2)}. \tag{21}$$

Since the Gauss-Kronecker curvature H_n of the hypersurface X vanishes, it follows from Theorems 1.5 and 1.6 that $\tilde{X}_{brV}$ is locally associated to X by a Ribaucour transformation and its curvature $\tilde{H}_n = 0$.

ii) If $r = 0$ and $V = (0, 0, 0, \varepsilon)$, with the hypothesis on the constants b we have $\tilde{X}_b$ globally defined on R^3. We will prove that $\tilde{X}_b$ is complete and non-cylindrical. We consider the orthogonal principal vector fields of $\tilde{X}_b$, $d\tilde{X}_b(e_i)$, $i = 1, 2, 3$. Then we will prove that there exists $\delta > 0$ such that $|d\tilde{X}_b(e_i)|^2 \geq \delta$, for all i. We first observe that

$$d\tilde{X}_b(v_i) = (1 + 2bL\lambda^i)v_i - 2dL(v_i)(V - bN) \quad \text{where} \quad L = \frac{\varepsilon f}{|V - bN|^2},$$

and

$$dL(v_i) = \varepsilon\left(\frac{df(v_i)}{|V - bN|^2} + \frac{2bf\lambda^i < v_i, V >}{|V - bN|^4}\right).$$

Therefore,

$$|d\tilde{X}_b(e_1)|^2 = 1 \quad \text{and} \quad |d\tilde{X}_b(e_j)|^2 = (1 + 2bL\lambda^j)^2, \quad j = 2, 3. \tag{22}$$

Since

$$L\lambda^2 = \frac{2\varepsilon f}{U(f - Q)} \quad \text{and} \quad L\lambda^3 = \frac{2\varepsilon f}{U(f + Q)}, \quad \text{where} \quad U = 2\varepsilon b + (1 + b^2)\sqrt{2 + f_z^2},$$

we have $|d\tilde{X}_b(e_j)|^2 = 1$ for $j = 2, 3$, wherever f vanishes. Otherwise, where $f \neq 0$, it follows from the hypothesis on b that

$$0 < \left|\frac{2b}{U}\right| \leq \frac{-2\varepsilon b}{2\varepsilon b + (1 + b^2)\sqrt{2}}.$$

Moreover, we get from (21) that $0 < 2f/(f - Q) < 1$ and $f/(f + Q) < 0$ (resp. $f/(f - Q) < 0$ and $0 < 2f/(f + Q) < 1$) where $f < 0$ (resp. $f > 0$). Hence, it follows from (22) that there exists a real number $0 < \delta < 1$, such that $|d\tilde{X}_b(e_j)|^2 > \delta$ for $j = 2, 3$. Therefore, we conclude that the submanifold is complete since any divergent curve has infinite length.

In order to prove that $\tilde{X}_b$ is not a cylinder, we observe that the only vanishing principal curvature of $\tilde{X}_b$ is $\tilde{\lambda}_1$. Since $d\tilde{X}_b(e_1)$ is not parallel to a fixed direction in R^4, we conclude that $\tilde{X}_b$ is not a cylinder. Moreover, none of these complete hypersurfaces is congruent to the original hypersurface X. In fact, one can easily see that the principal vector field of X corresponding to $\lambda^1 = 0$ is orthogonal to the vector $(0, 0, 1, 0)$. However, there is no constant vector of R^4, which is orthogonal to the principal vector field corresponding to the vanishing principal curvature of $\tilde{X}_b$. □

Our next application will provide a 5-parameter family of compact Dupin hypersurfaces in the unit sphere S^4, whose Gauss-Kronecker curvature vanishes. We start considering the Veronese surface described by $X : S^2_{\sqrt{3}} \to S^4 \subset R^5$,

$$X(x, y, z) = \frac{1}{\sqrt{3}}(xy,\, xz,\, yz,\, \frac{x^2 - y^2}{2},\, \frac{\sqrt{3}}{2}(1 - z^2)),$$

where $S^2_{\sqrt{3}} \subset R^3$ is the sphere of radius $\sqrt{3}$. We denote by $T_1X^\perp$ the unit normal bundle of X, i.e.

$$T_1X^\perp = \{(p,\xi);\ p \in S^2_{\sqrt{3}},\ \xi \in (TpX)^\perp \subset T_pS^4 \text{ and } |\xi| = 1\}.$$

The tube of geodesic ray $R = \pi/2$, around X is the hypersurface $Y : T_1X^\perp \to S^4$ given by

$$Y(p,\xi) = \exp_{X(p)}(\frac{\pi}{2}\xi) = \xi.$$

A vector field normal to Y (tangent to S^4 along Y) $N : T_1X^\perp \to S^4 \subset R^5$ is given by $N(p,\xi) = X(p)$.

One can show (see [1]) that Y is an isoparametric minimal hypersurface in S^4 whose principal curvatures are $\lambda^1 = 0,\ \lambda^2 = \sqrt{3},\ \lambda^3 = -\sqrt{3}$.

Proposition 2.2. *Let Y be the tube of geodesic ray $\pi/2$ around the Veronese surface X. Then the map $\tilde{Y}_{bV} : T_1X^\perp \to S^4 \subset R^5$ given by*

$$\tilde{Y}_{bV} = Y - \frac{2 < V, Y >}{|V - bX|^2}(V - bX)$$

is a regular, compact, Dupin hypersurface of S^4, with zero Gauss-Kronecker curvature, locally associated to Y by a Ribaucour transformation, $\forall b \in R$ and any unit vector $V \in R^5$ such that

$$b^2 + 1 - 2|b|(1 + \sqrt{3}) > 0. \tag{23}$$

Proof. It follows from Theorems 1.3 and 1.5 that $\tilde{Y}$ is a Dupin hypersurface with zero Gauss-Kronecker curvature, which is regular whenever (15) is satisfied. Observe that Ω_i, Ω and W are given by (19), where $r = 0$ i.e. $W = - < V, X > +b$, $\Omega_i =< V, e_i >$ and $\Omega =< V, Y >$. Moreover, $S - 2bW = 2c$ is a constant. Therefore, it follows from (9) that $S = 1 + b(-2 < V, X > +b)$ and $c = (1 - b^2)/2$. The hypothesis (23) implies that $c \neq 0$ and hence $S = 2(bW + c) \geq (|b| - 1)^2 > 0$. In order to conclude the regularity of $\tilde{Y}$, we need to show that $b(W + \lambda^i\Omega) + c \neq 0$ for $i = 1, 2, 3$. In fact, for $i = 1$ this follows from $S > 0$, and for $i = 2, 3$ we have that

$$\begin{aligned} 2b(W + \lambda^i\Omega) + 2c &= 1 + b^2 - 2b < V, N > \pm 2\sqrt{3}\, b < V, Y > \\ &\geq 1 + b^2 - 2(1 + \sqrt{3})|b| > 0, \end{aligned}$$

where the last inequality follows from (23). □

We observe that each hypersurface $\tilde{Y}_{bV}$ is a tube of geodesic ray $\pi/2$ over the image of its Gaussian normal map $\tilde{N}_{bV} : T_1X^\perp \to S^4$ given by

$$\tilde{N}_{bV} = X + \frac{2(- < V, X > +b)}{|V - bX|^2}(V - bX).$$

Our next results follow from the basic theorem on Ribaucour transformations.

Proposition 2.3. *Let X be the parametrized hyperplane $x_{n+1} = 0$ in R^{n+1} and let $e_1, \ldots, e_n$ be the canonical orthonormal basis of X. Consider arbitrary differentiable functions $f_i(x_i)$*

of x_i such that for some i_0, $1 \le i_0 \le n$, $f_{i_0} = ax_{i_0} + b$, $a, b \in R$ and $\gamma, \alpha \neq 0$ real numbers. Then

$$\tilde{X} = X - \frac{2(\sum_{i=1}^n f_i + \gamma)}{\sum_{i=1}^n (f_i')^2 + \alpha^2}(f_1', \dots, f_n', -\alpha)$$

is a family of hypersurfaces with zero Gauss-Kronecker curvature, locally associated to X by a Ribaucour transformation with respect to $\{e_i\}$.

Proof. The proof follows from the fact that the solutions of (5)–(7) are given by

$$\Omega_i = f_i'(x_i), \quad \Omega = \sum_{i=1}^n f_i(x_i) + \gamma \quad \text{and } W = \alpha \neq 0.$$

Since $S = \sum_{i=1}(f_i')^2 + \alpha^2$ and $T_i = 2f_i''$, it follows from (13) that the principal curvatures of $\tilde{X}$ are

$$\tilde{\lambda}^i = \frac{2\alpha f_i''}{\sum_{i=1}(f_i')^2 + \alpha^2 - 2f_i''(\sum_i f_i + \gamma)}.$$ □

Similarly one can show that

Proposition 2.4. *Let $X = (\cos x_1, \sin x_1, x_2, \dots, x_n)$ be a parametrized cylinder in R^{n+1} and let $e_i = X_{x_i}$, $1 \le i \le n$. Consider arbitrary differentiable functions $f_i(x_i)$ of x_i such that for some $i_0 \ge 2$, $f_{i_0} = ax_{i_0} + b$, $a, b \in R$ and $\gamma, \alpha \in R$. Then*

$$\tilde{X} = X - \frac{2(\sum_{i=1}^n f_i + \gamma)}{\sum_{i=1}^n (f_i')^2 + (f_1 - \alpha)^2}(-f_1' \sin x_1 - (\alpha - f_1)\cos x_1, f_1' \cos x_1 - (\alpha - f_1)\sin x_1, f_2', \dots, f_n')$$

provides a family of hypersurfaces with zero Gauss-Kronecker curvature in R^{n+1}, locally associated to the cylinder by a Ribaucour transformation with respect to $\{e_i\}$.

Proposition 2.5. *Consider a parametrization of the hyperbolic space H^3, as a hypersurface of H^4, contained in the Lorentzian space L^5, given by*

$$X = \sinh x_3(\cos x_2 \cos x_1, \cos x_2 \sin x_1, \sin x_2, 0, 0) + (0, 0, 0, 0, \cosh x_3),$$

where $-\pi/2 < x_2 < \pi/2$ and $x_3 > 0$. Let $e_i = X_{x_i}/|X_{x_i}|$, $i = 1, 2, 3$ and let $N = (0, 0, 0, 1, 0)$ be the normal map. Then the hypersurfaces of H^4, locally associated to X by a Ribaucour transformation with respect to e_i, are given by

$$\tilde{X} = X - \frac{2\Omega}{S}(\sum_i \Omega_i e_i - WN - \Omega X),$$

where

$$S = \sum_i \Omega_i^2 + W^2 - \Omega^2, \quad \Omega_1 = f_1', \quad \Omega_2 = -f_1 \sin x_2 + f_2', \quad W = b \neq 0,\ b \in R, \tag{24}$$

$$\Omega_3 = (f_1 \cos x_2 + f_2)\cosh x_3 + f_3', \qquad \Omega = (f_1 \cos x_2 + f_2)\sinh x_3 + f_3, \tag{25}$$

and f_i is an arbitrary differentiable real function of x_i. Moreover, if

$$f_3 = c_1 \sinh x_3 + c_2 \cosh x_3, \quad c_1, c_2 \in R, \tag{26}$$

then $\tilde{X}$ has zero Gauss-Kronecker curvature.

Proof. From the expression of X, we have that $a_i = |X_{x_i}|$ are given by $a_1 = \sinh x_3 \cos x_2$, $a_2 = \sinh x_3$ and $a_3 = 1$. It follows from (5) and a straightforward computation that the functions Ω_i are given by (24), (25). From (6), we obtain the expression of Ω and this concludes the proof of the first part of the theorem.

Using these expressions into (10), we obtain

$$T^i = \frac{2}{a_i}\left(\sum_{k \neq i} \frac{\Omega_k}{a_k}\frac{\partial a_i}{\partial x_k} + \frac{\partial \Omega_i}{\partial x_i} - a_i \Omega\right). \tag{27}$$

Since $\lambda^i = 0$, for all i, it follows from (13) that $T^{i_0} = 0$, if and only if, the principal curvature of $\tilde{X}$, $\tilde{\lambda}^{i_0} = 0$. We will now obtain the conditions on the functions f_i for the vanishing of some T^i.

It follows from (27), that $T^3 = 0$ if and only if $f_3'' - f_3 = 0$. Hence, f_1, f_2 are arbitrary differentiable functions of x_1 and x_2 respectively, f_3 is given by (26) and $\tilde{\lambda}^3 = 0$.

Similarly, $T^2 = 0$ if and only if $f_2'' + f_2 = -c_1$ and $f_3' \cosh x_3 - f_3 \sinh x_3 = c_1$, where $c_1 \in R$. Hence, $f_2 = -c_1 + b_1 \cos x_2 + b_2 \sin x_2$, f_3 is given by (26) and f_1 is an arbitrary function. In this case, $\tilde{\lambda}^2 = \tilde{\lambda}^3 = 0$.

Finally, one can show that $T^1 = 0$, if and only if, $f_1 = -b_1 + c_3 \cos x_2 + c_4 \sin x_2$, $f_2 = -c_1 + b_1 \cos x_2 + b_2 \sin x_2$ and f_3 is given by (26), i.e. $\tilde{X}$ is a totally geodesic submanifold of H^3.

Therefore, we conclude that if f_3 is given by (26), then $\tilde{X}$ has vanishing Gauss-Kronecker curvature. □

In Proposition 2.5, we observe that if the functions f_i for $i = 2, 3$ are of the form $f_i = \alpha_i + \beta_i \cos x_i + \gamma_i \sin x_i$, then $\tilde{X}$ is a Dupin hypersurface with zero Gauss-Kronecker curvature.

References

[1] Almeida; S. C.; Brito, F. G.: *Minimal hypersurfaces of S^4 with constant Gauss-Kronecker curvature*. Math. Z. **195** (1987), 99–107. Zbl 0602.53040

[2] Bianchi, L.: *Lezioni di Geometria Differenziale*. N. Zanichelli Ed., Bologna 1927. JFM 53.0659.03

[3] Corro, A. V.; Ferreira, W.; Tenenblat, K.: *On Ribaucour transformations for hypersurfaces*. Mat. Contemp. **17** (1999), 137–160. Zbl 1018.53004

[4] Corro, A. V.; Ferreira, W.; Tenenblat, K.: *Minimal surfaces obtained by Ribaucour transformations*, Geom. Dedicata **97** (2003), 117–150. Zbl pre01901490

[5] Corro, A. V.; Ferreira, W.; Tenenblat, K.: *Ribaucour transformations for constant mean curvature and linear Weingarten surfaces*. Pac. J. Math. **212** (2004), 265–296. Zbl pre02101133

[6] Corro, A. V.; Tenenblat, K.: *Ribaucour transformations revisited*. Commun. Anal. Geom. **12** (2004) 1055–1082. Zbl pre02147074

[7] Dajczer, M.; Tojeiro, R.: *An extension of the classical Ribaucour transformation*. Proc. Lond. Math. Soc., III. Ser. **85** (2002), 211–232. Zbl 1028.53057

[8] Hartmann, P.: *On the isometric immersions in Euclidian space of manifolds with non-negative sectional curvatures. II.* Trans. Am. Math. Soc. **147** (1970), 529–540. Zbl 0194.22702

[9] Hartman, P.; Nirenberg, L.: *On spherical image maps whose Jacobians do not change sign*. Am. J. Math. **81** (1959), 901–920. Zbl 0094.16303

[10] Lusala, T.: *Non-closed minimal hypersurfaces of* $S^4(1)$ *with identically zero Gauss-Kronecker curvature*. Preprint.

[11] Ramanathan, J.: *Minimal hypersurfaces in* S^4 *with vanishing Gauss-Kronecker curvature*. Math. Z. **205** (1990), 645–658. Zbl 0694.53054

[12] Sacksteder, R.: *On hypersurfaces with no negative sectional curvatures*. Am. J. Math. **82** (1960), 609–630. Zbl 0194.22701

[13] Tenenblat, K.: *On Ribaucour transformations and applications to linear Weingarten surfaces*. An. Acad. Bras. Ciênc. **74** (2002), 559–575. Zbl pre01880868

Received October 1, 2003