Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 46 (2005), No. 2, 351-356.

An Arrangement of Pseudocircles Not Realizable With Circles

Johann Linhart **Ronald Ortner**

Institut für Mathematik, Universität Salzburg
Hellbrunner Straße 34, 5020 Salzburg, Austria
e-mail: johann.linhart@sbg.ac.at

Department Mathematik und Informationstechnolgie, Montanuniversität Leoben
Franz-Josef-Strasse 18, 8700 Leoben, Austria
e-mail: ronald.ortner@unileoben.ac.at

Abstract. We present an arrangement of five pseudocircles that cannot be realized with (proper) circles.

1. Introduction

By a *pseudocircle* we mean a simple closed Jordan curve in the plane.

Definition 1.1. *An arrangement of pseudocircles is a finite set* $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ *of simple closed Jordan curves in the plane such that*

- ▷ *no three curves meet each other at the same point,*
- ▷ *if two pseudocircles* γ_i, γ_j *have a point* P *in common, they cross each other in that point, i.e. every neighborhood of* P *contains points of* γ_i *in the interior of* γ_j *as well as in the exterior of* γ_j*,*
- ▷ *each pair of curves intersects at most* 2 *times.*

An arrangement is said to be complete if each two pseudocircles intersect.

Given an arrangement of pseudocircles, we may consider the intersection points of the pseudocircles as vertices and the curves between the intersections as edges. Thus we obtain in a natural way an embedding of a graph and hence a cell complex. Two arrangements are said to be isomorphic if they have the same associated cell complex.

Obviously, arrangements of pseudocircles are a generalization of arrangements of (proper) circles. However, from the combinatorial point of view it is not clear whether the class of arrangements of pseudocircles is a proper extension of the class of arrangements of circles. Put in another way, the question is whether for every arrangement of pseudocircles there is an isomorphic arrangement of circles. There is an analogous problem concerning the stretchability of arrangements of pseudolines. That is, given an arrangement of pseudolines, is there a combinatorially equivalent arrangement of straight lines? In 1980, Goodman and Pollack [2] proved Grünbaum's conjecture that all arrangements of at most eight pseudolines are stretchable, so that some known non-stretchable arrangements of nine pseudolines are minimal in that sense (cf. [1], p. 259ff). These arrangements also guarantee the existence of arrangements of (nine) pseudocircles on the sphere that cannot be realized as arrangements of great circles (cf. [1], p. 249, 259ff). However, the problem of "straightening" arrangements of pseudocircles in the plane is a different matter.

In this paper, we settle the question by showing that there is an arrangement of five pseudocircles that is not isomorphic to any arrangement of circles. We conjecture that this example is minimal as well. In [3], we have shown that all five complete arrangements of three and all 72 complete arrangements of four pseudocircles are realizable with proper circles. Thus, if there is a smaller example it is not complete.

2. The arrangement

Consider the arrangement of pseudocircles in Figure 1.

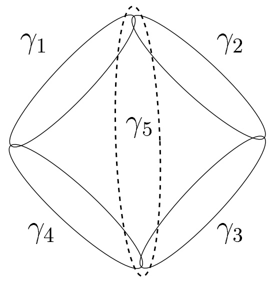

Figure 1. An arrangement of pseudocircles

Theorem 2.1. *The arrangement in Figure* 1 *cannot be realized with circles.*

For the proof of Theorem 2.1 we shall need some simple observations expressed in the following two lemmata.

Lemma 2.2. *Let* $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ *be an arrangement of four circles with centers* C_1, C_2, C_3, C_4 *that is isomorphic to the arrangement of the pseudocircles* $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ *in Figure* 1*. Let* S_i *and* S_i' *be the intersection points of* γ_i *and* γ_{i+1} *for each* $i \in \{1, 2, 3, 4\}$ modulo 4*, such that* S_i' *lies on the boundary of the unbounded region (see Figure* 2*). Then in the quadrangle* $S_1'S_2S_3'S_4$ *the sum of the angles at* S_2 *and* S_4 *is larger than the sum of the angles at* S_1' *and* S_3'*, i.e.* $\measuredangle(S_3'S_4S_1') + \measuredangle(S_1'S_2S_3') > \measuredangle(S_4S_1'S_2) + \measuredangle(S_2S_3'S_4)$.[1]

[1] In the following we consider all angles to be $\in [0°, 360°)$ and oriented counterclockwise.

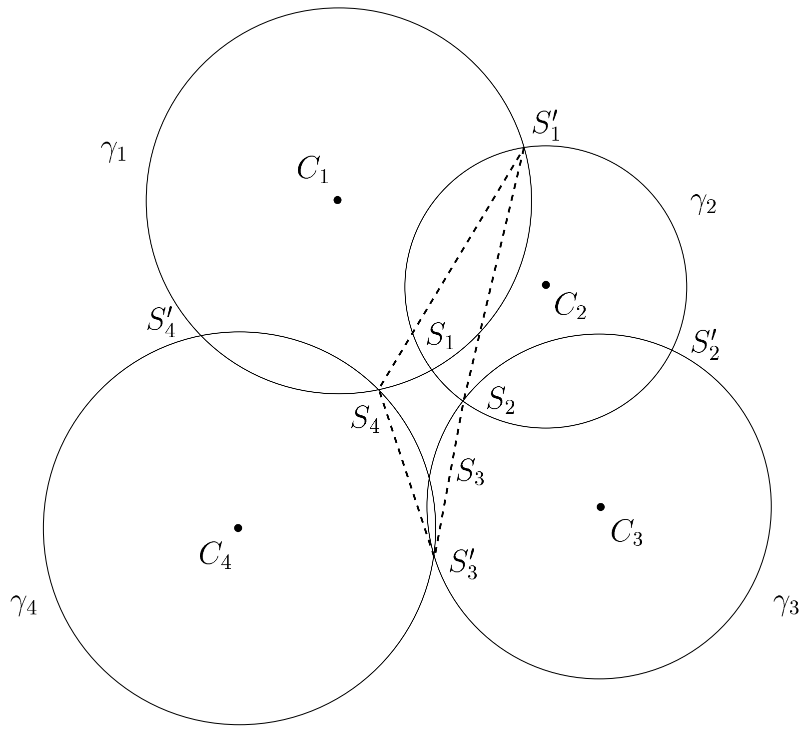

Figure 2. Illustration of Lemma 2.2

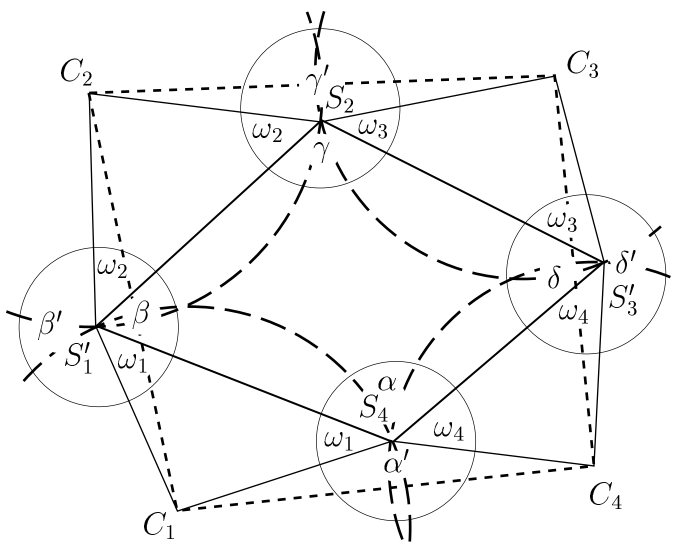

Figure 3. Illustration of proof of Lemma 2.2

Proof. Consider the points S_1', S_2, S_3', S_4 together with the centers C_1, C_2, C_3, C_4. Note that the triangles $\Delta_1 = S_4C_1S_1'$, $\Delta_2 = S_1'C_2S_2$, $\Delta_3 = S_2C_3S_3'$ and $\Delta_4 = S_3'C_4S_4$ are isosceles. Thus, let ω_i be the two angles in Δ_i lying opposite to C_i. Furthermore we set $\alpha := \measuredangle(S_3'S_4S_1')$, $\beta := \measuredangle(S_4S_1'S_2)$, $\gamma := \measuredangle(S_1'S_2S_3')$ and $\delta := \measuredangle(S_2S_3'S_4)$. Finally, the exterior angles at the points S_1', S_2, S_3', S_4 are denoted by $\alpha', \beta', \gamma', \delta'$ (cf. Figure 3). Note that the points

S_1, S_2, S_3, S_4 are always contained in the interior of the quadrangle $C_1C_2C_3C_4$, while the points S'_1, S'_2, S'_3, S'_4 lie outside. Hence, the angles α', γ' are $< 180°$ while $\beta', \delta' > 180°$, so that $\alpha' + \gamma' < \beta' + \delta'$. Thus,

$$(360° - \alpha - \omega_1 - \omega_4) + (360° - \gamma - \omega_2 - \omega_3) = \alpha' + \gamma'$$
$$< \beta' + \delta' = (360° - \beta - \omega_1 - \omega_2) + (360° - \delta - \omega_3 - \omega_4),$$

whence $\alpha + \gamma > \beta + \delta$. □

Lemma 2.3. *Let P_1, P_2, P_3, P_4, Q be five points in the plane such that the angles $\measuredangle(P_1QP_2)$, $\measuredangle(P_2QP_3)$, $\measuredangle(P_3QP_4)$, $\measuredangle(P_4QP_1)$ are all $\leq 180°$. Then $Q \in \operatorname{conv}(P_1, P_2, P_3, P_4)$.*[2]

Proof. Let P_1, P_2, P_3, P_4, Q be five points in the plane such that Q is not contained in $\operatorname{conv}(P_1, P_2, P_3, P_4)$. We show that one of the angles $\measuredangle(P_1QP_2)$, $\measuredangle(P_2QP_3)$, $\measuredangle(P_3QP_4)$, $\measuredangle(P_4QP_1)$ is $> 180°$. Since $Q \notin \operatorname{conv}(P_1, P_2, P_3, P_4)$, there is a straight line h that separates Q from $\operatorname{conv}(P_1, P_2, P_3, P_4)$. Let P'_1, P'_2, P'_3, P'_4 be the central projections of P_1, P_2, P_3, P_4 from Q onto h. Now, if the counterclockwise order of the points P'_i on h relative to Q is P'_1, P'_2, P'_3, P'_4, then $\measuredangle(P_4QP_1) = \measuredangle(P'_4QP'_1) > 180°$. Otherwise, there is an $i \in \{1, 2, 3\}$ such that P'_{i+1} occurs before P'_i in the considered order. In this case, $\measuredangle(P_iQP_{i+1}) = \measuredangle(P'_iQP'_{i+1}) > 180°$. □

Finally, we will also make use of the following well-known result of elementary geometry.

Proposition 2.4. *Two opposite angles in a chord quadrangle*[3] *add up to* 180°.

Proof of Theorem 2.1. Let $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ be an arrangement of four circles as described in Lemma 2.2. We show that it is not possible to add another circle γ_5 such that:[4]

$$\begin{aligned} \operatorname{cl}(\operatorname{int}\gamma_1 \cap \operatorname{int}\gamma_2) &\subseteq \operatorname{int}\gamma_5, \\ \operatorname{cl}(\operatorname{int}\gamma_3 \cap \operatorname{int}\gamma_4) &\subseteq \operatorname{int}\gamma_5, \\ \operatorname{cl}(\operatorname{int}\gamma_2 \cap \operatorname{int}\gamma_3) \cap \operatorname{int}\gamma_5 &= \varnothing, \\ \operatorname{cl}(\operatorname{int}\gamma_4 \cap \operatorname{int}\gamma_1) \cap \operatorname{int}\gamma_5 &= \varnothing. \end{aligned} \tag{1}$$

Let the points S_i, S'_i $(i = 1, 2, 3, 4)$ and the angles $\alpha, \beta, \gamma, \delta$ be defined as in Lemma 2.2. The conditions (1) imply

$$\begin{aligned} S_1, S'_1, S_3, S'_3 &\in \operatorname{int}\gamma_5, \\ S_2, S'_2, S_4, S'_4 &\notin \operatorname{int}\gamma_5. \end{aligned} \tag{2}$$

Case 1: All angles in the quadrangle $S'_1S_2S'_3S_4$ are $< 180°$.

[2] $\operatorname{conv} A$ denotes the convex hull of A.
[3] A *chord quadrangle* is a quadrangle with all four vertices lying on a circle.
[4] $\operatorname{cl} A$ denotes the closure of A, $\operatorname{int}\gamma$ the interior of γ.

Assume that there is a circle γ_5 that contains S_1', S_3' but not S_2, S_4 in its interior. We show that $\measuredangle(S_3'S_4S_1') + \measuredangle(S_1'S_2S_3') < \measuredangle(S_4S_1'S_2) + \measuredangle(S_2S_3'S_4)$, which contradicts Lemma 2.2.

Let $\widetilde{S}_1', \widetilde{S}_3'$ be the intersection points of the straight line $S_1'S_3'$ with γ_5, such that $\widetilde{S}_1'$ is nearer to S_1' than to S_3'. Note that due to the convexity of the quadrangle $S_1'S_2S_3'S_4$ there are also two intersection points $\widetilde{S}_2, \widetilde{S}_4$ of the straight line S_2S_4 with γ_5 (again, we assume that $\widetilde{S}_2$ is nearer to S_2 than to S_4). Evidently, $\measuredangle(\widetilde{S}_1'S_2\widetilde{S}_3') > \measuredangle(S_1'S_2S_3')$ and $\measuredangle(\widetilde{S}_3'S_4\widetilde{S}_1') > \measuredangle(S_3'S_4S_1')$. Furthermore we have $\measuredangle(\widetilde{S}_1'\widetilde{S}_2\widetilde{S}_3') > \measuredangle(\widetilde{S}_1'S_2\widetilde{S}_3')$ and $\measuredangle(\widetilde{S}_3'\widetilde{S}_4\widetilde{S}_1') > \measuredangle(\widetilde{S}_3'S_4\widetilde{S}_1')$. Applying Proposition 2.4 yields

$$\begin{aligned}
180^\circ &= \measuredangle(\widetilde{S}_1'\widetilde{S}_2\widetilde{S}_3') + \measuredangle(\widetilde{S}_3'\widetilde{S}_4\widetilde{S}_1') > \measuredangle(\widetilde{S}_1'S_2\widetilde{S}_3') + \measuredangle(\widetilde{S}_3'S_4\widetilde{S}_1') \\
&> \measuredangle(S_1'S_2S_3') + \measuredangle(S_3'S_4S_1').
\end{aligned}$$

Since the angles in the quadrangle $S_1'S_2S_3'S_4$ add up to 360°, it follows that $\measuredangle(S_4S_1'S_2) + \measuredangle(S_2S_3'S_4) > 180^\circ$, which leads to the desired contradiction.

Case 2: There is an angle $\geq 180^\circ$ in the quadrangle $S_1'S_2S_3'S_4$.

Since β and δ are $< 180^\circ$ we may assume without loss of generality that $\alpha \geq 180^\circ$. If $\alpha = 180^\circ$, then S_4 lies in the convex hull of the points S_1', S_3', so that any circle with S_1' and S_3' in its interior also contains S_4, which violates (2). Thus let us assume that $\alpha > 180^\circ$. We distinguish two cases.

(a) The angle $\measuredangle(S_3'S_4'S_1')$ is $< 180^\circ$:

In this case, the points S_3', S_4', S_1', S_4 form a quadrangle with all angles $< 180^\circ$ (note that the straight line through $S_4'S_4$ separates the points S_3', S_1'), so that its diagonals $S_3'S_1'$ and $S_4'S_4$ cut each other.

By convexity, it follows that any circle γ_5 with S_1', S_3' in its interior also contains a point $\in \mathrm{cl}(\mathrm{int}\,\gamma_1 \cap \mathrm{int}\,\gamma_4)$, which violates the last condition of (1).

(b) The angle $\measuredangle(S_3'S_4'S_1')$ is $\geq 180^\circ$:

We are going to show that the angles $\measuredangle(S_1'S_4'S_3')$, $\measuredangle(S_3'S_4'S_3)$, $\measuredangle(S_3S_4'S_1)$, $\measuredangle(S_1S_4'S_1')$ are all $\leq 180^\circ$. Then by Lemma 2.3, we may conclude that S_4' lies in the convex hull of the points S_1, S_1', S_3, S_3'. It follows that any circle γ_5 with S_1, S_1', S_3, S_3' in its interior by convexity also contains S_4', which contradicts (2).

Now let us take a look at the aforementioned angles:

- $\measuredangle(S_1'S_4'S_3')$: This angle is by assumption $\leq 180^\circ$.
- $\measuredangle(S_3S_4'S_1)$: By Lemma 2.2,

$$\measuredangle(S_3S_4'S_1) + \measuredangle(S_1S_2'S_3) < 180^\circ < \measuredangle(S_4'S_1S_2') + \measuredangle(S_2'S_3S_4'),$$

  so that $\measuredangle(S_3S_4'S_1) < 180^\circ$.
- $\measuredangle(S_3'S_4'S_3)$: Since S_4' is not contained in any circle, when walking counterclockwise on γ_4 starting in S_4', we first pass S_3' and then S_3. Hence, $\measuredangle(S_3'S_4'S_3)$ is the angle of a triangle inscribed in γ_4. It follows that $\measuredangle(S_3'S_4'S_3) < 180^\circ$.
- $\measuredangle(S_1S_4'S_1')$: An analogous argument as in the previous case shows that $\measuredangle(S_1S_4'S_1') < 180^\circ$. □

References

[1] Björner, A.; Las Vergnas, M.; Sturmfels, B.; White, N.; Ziegler, G.: *Oriented Matroids*. Encyclopedia of Mathematics and its Applications **46**, Cambridge University Press, Cambridge, second edition, 1999. Zbl 0944.52006

[2] Goodman, J. E.; Pollack, R.: *Proof of Grünbaum's conjecture on the stretchability of certain arrangements of pseudolines*. J. Comb. Theory, Ser. A **29**(3) (1980), 385–390. Zbl 0457.51006

[3] Ortner, R.: *Arrangements of Pseudocircles*. PhD Thesis, University of Salzburg 2001.

Received October 10, 2002