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Optimal Substructures in Optimal and Approximate Circle Packings

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Abstract. This paper deals with the densest packing of equal circles in a square problem. Sharp bounds for the density of optimal circle packings have given. Several known optimal and approximate circle packings contain optimal substructures. Based on this observation it is sometimes easy to determine the minimal polynomials of the arrangements.

Keywords: circle packing, minimal polynomials, structures

1. Four equivalent allocation problems

The paper deals with an unsolved allocation problem of the discrete geometry. First of all let us see some equivalent problem settings.

Definition 1. *$P(r_n, S) \in P_{r_n}$ is a circle packing with radius r_n in $[0, S]^2$, where $P_{r_n} = \{((x_1, y_1), \ldots, (x_n, y_n)) \in [0, S]^{2n} \mid (x_i - x_j)^2 + (y_i - y_j)^2 \geq 4r_n^2;\ x_i, y_i \in [r_n, S - r_n]\ (1 \leq i < j \leq n)\}$. $P(r_n, S) \in P_{\overline{r}_n}$ is an optimal circle packing, if $\overline{r}_n = \max\limits_{P_{r_n} \neq \emptyset} r_n$.*

Problem $\mathfrak{P}_\mathbf{1}^\mathbf{n}$. Determine the optimal circle packings for $n \geq 2$.

Definition 2. *$A(m_n, \Sigma) \in A_{m_n}$ is a point arrangement with minimal distance m_n in $[0, \Sigma]^2$, where $A_{m_n} = \{((x_1, y_1), \ldots, (x_n, y_n)) \in [0, \Sigma]^{2n} \mid (x_i - x_j)^2 + (y_i - y_j)^2 \geq m_n^2; (1 \leq i < j \leq n)\}$. $A(m_n, \Sigma) \in A_{\overline{m}_n}$ is an optimal point arrangement, if $\overline{m}_n = \max\limits_{A_{m_n} \neq \emptyset} m_n$.*

Problem $\mathfrak{P}_2^n$. Determine the optimal point arrangements for $n \geq 2$.

Definition 3. *$P'(R, s_n) \in P'_{s_n}$ is an associate circle packing with radius R in $[0, s_n]$, where $P'_{s_n} = \{((x_1, y_1), \ldots, (x_n, y_n)) \in [0, s_n]^{2n} \,|(x_i - x_j)^2 + (y_i - y_j)^2 \geq 4R^2;\ x_i, y_i \in [R, s_n - R]\ (1 \leq i < j \leq n)\}$. $P'(R, s_n) \in P'_{\overline{s}_n}$ is an optimal associate circle packing, if $\overline{s}_n = \min_{P'_{s_n} \neq \emptyset} s_n$.*

Problem $\mathfrak{P}_3^n$. Determine the optimal associate circle packings for $n \geq 2$.

Definition 4. *$A'(M, \sigma_n) \in A'_{\sigma_n}$ is an associate point arrangement with the minimal distance M in $[0, \sigma_n]$, where $A'_{\sigma_n} = \{((x_1, y_1), \ldots, (x_n, y_n)) \in [0, \sigma_n]^2 \mid (x_i - x_j)^2 + (y_i - y_j)^2 \geq M^2\ (1 \leq i < j \leq n)\}$. $A'(M, \sigma_n) \in A'_{\overline{\sigma}_n}$ is an optimal associate point arrangement, if $\overline{\sigma}_n = \min_{A'_{\sigma_n} \neq \emptyset} \sigma_n$.*

Problem $\mathfrak{P}_4^n$. Determine the optimal associate point arrangements for $n \geq 2$.

Theorem 1. *Problems $\mathfrak{P}_1^n$, $\mathfrak{P}_2^n$, $\mathfrak{P}_3^n$ and $\mathfrak{P}_4^n$ are equivalent, in the sense that if Problem $\mathfrak{P}_i^n$ can be solved for a fixed n and i values, then the other Problems $\mathfrak{P}_i^n$ can be solved for all $1 \leq i \leq 4$ values.*

Proof. The centers of the circles in a packing $P(\overline{r}_n, S)$ determine an optimal point arrangement in a square of side length of $S - 2\overline{r}_n$ [19]. By scaling-up an optimal arrangement of n points in a square we obtain an optimal point arrangement in another square of arbitrary side length. By drawing circles by radius $\frac{\overline{m}_n}{2}$ around the points in a point arrangement $A(\overline{m}_n, \Sigma)$ the packing will give an optimal associate circle packing in a $\Sigma + \overline{m}_n$ side square. By scaling-up an optimal associate circle packing provides an optimal associate circle packing with any radius. The centers of the circles in a packing $P'(\overline{s}_n, R)$ determine an optimal associate point arrangement in an $\overline{s}_n - 2R$ side of square by a minimal distance of $2R$. By scaling-up this point arrangement gives an optimal associate point arrangement $A'(\overline{\sigma}_n, M)$. Drawing again circles around the points with radius $\frac{M}{2}$, the circle packing will be optimal in a $\overline{\sigma}_n + M$ side of square, hence we return to an optimal circle packing $P(\overline{r}_n, S)$. □

Proposition 1. *The relations between the parameters $\overline{m}_n$, $\overline{r}_n$, $\overline{s}_n$ and $\overline{\sigma}_n$ are given in the Tables* 1–2.

	$P(r_n, S)$	$A(m_n, \Sigma)$
$P(r_n, S)$	1	$\overline{r}_n = \frac{S\overline{m}_n}{2(\overline{m}_n + \Sigma)}$
$A(m_n, \Sigma)$	$\overline{m}_n = \frac{2\Sigma\overline{r}_n}{S - 2\overline{r}_n}$	1
$P'(R, s_n)$	$\overline{s}_n = \frac{RS}{\overline{r}_n}$	$\overline{s}_n = \frac{2R(\overline{m}_n + \Sigma)}{\overline{m}_n}$
$A'(M, \sigma_n)$	$\overline{\sigma}_n = \frac{M(S - 2\overline{r}_n)}{2\overline{r}_n}$	$\overline{\sigma}_n = \frac{M\Sigma}{\overline{m}_n}$

Table 1. Relations between the parameters of the problems

	$P'(R, s_n)$	$A'(M, \sigma_n)$
$P(r_n, S)$	$\overline{r}_n = \frac{RS}{\overline{s}_n}$	$\overline{r}_n = \frac{MS}{2(M+\overline{\sigma}_n)}$
$A(m_n, \Sigma)$	$\overline{m}_n = \frac{2R\Sigma}{\overline{s}_n-2R}$	$\overline{m}_n = \frac{M\Sigma}{\overline{\sigma}_n}$
$P'(R, s_n)$	1	$\overline{s}_n = \frac{2R(M+\overline{\sigma}_n)}{M}$
$A'(M, \sigma_n)$	$\overline{\sigma}_n = \frac{M(\overline{s}_n-2R)}{2R}$	1

Table 2. Relations between the parameters of the problems

Proof. It follows from suitable scaling based on the technique described in [19]. □

2. Some historical comments

To find $P(\overline{r}_n, 1)$ for a large n value is a great challenge in mathematics and computer sciences. From 1960 [11] until nowadays several researchers tried to solve this problem in the traditional way "by hand" and using computers too. As the structures of optimal packings are changing step by step, the determination of optimal packings is hard. There are repeated pattern classes among the structures of optimal packings but they do not cover every possible optimal structures [5, 12, 19].

It is clear that the circle packing problem is at one hand a discrete geometrical problem and on the other hand a global optimization problem. The earlier optimization models (as a continuous, constrained global optimization problem, DC programming problem, all-quadratic optimization problem, etc.) and other approaches (elimination methods "by hand" and based on computer-aided methods, energy function minimization, SA and TA techniques, billiard simulation, LP-relaxation, etc.) have given many approximate packings and some proofs for the optimality [1, 2, 5, 7–10, 12–13, 15, 21].

Table 3 summarizes the known optimal packings with their authors. The optimal packings are known up to $n = 27$ and the $n = 36$ case.

Year	Authors	Results for n
1965	J. Schaer and A. Meir [16, 17]	$8, 9$
1970	B. L. Schwartz [18]	6
1983	G. Wengerodt [22, 23, 24]	$14, 16, 25$
1987	K. Kirchner and G. Wengerodt [6]	36
1992	R. Peikert et al. [15]	10–20
1999	K. J. Nurmela and P. R. J. Östergård [13]	7,21–27

Table 3. The authors of the known optimal packings

To find optimal packings and to prove the optimality of packings is a hard problem. Recently several papers have published not only optimal packings but approximate packings too. Table 4 contains the most important improvements in the last decade. A more detailed history of Problem $\mathfrak{P}_\mathbf{i}^\mathbf{n}$ $(1 \leq i \leq 4)$ is in [8, 15, 20, 21].

Year	Authors	Results for n
1995	C. A. Maranas et al. [8]	up to 30
1996	R. Graham and B. D. Lubachevsky [5]	up to 61
1997	K. J. Nurmela and P. R. J. Östergård [12]	up to 50
2000	D. W. Boll et al. [1]	$32, 37, 48, 50$
2001	L. G. Casado et al. [2]	up to 100
2002	M. Locatelli and U. Raber [7]	up to 40
Sub.	E. Specht and P. G. Szabó [21]	up to 200

Table 4. The authors of approximate packings

3. The density of packings

Definition 5. *Let X be a compact convex subset of $[0,1]^2$. The density of a circle packing $P(r_n, 1)$ in X is*

$$d(X, n') = \frac{n' r_n^2 \pi}{V(X)} \quad \left(= \frac{n' m_n^2 \pi}{4(m_n+1)^2 V(X)} \right),$$

where n' denotes the number of the circles (points) in X and $V(X)$ is the area of X. Let us denote by $\overline{d}([0,1]^2, n)$ the density of $P(\overline{r}_n, 1)$.

Remark 1. The finding of $P(\overline{r}_n, 1)$ is equivalent to the determination of the densest packing of n equal circles in $[0,1]^2$.

Theorem 2. *For every $n \geq 2$*

$$(3 - 2\sqrt{2})\pi \leq \overline{d}([0,1]^2, n) < \frac{\pi}{\sqrt{12}},$$

where the bounds are sharp.

Proof. It is known that $\sqrt{\frac{2}{\sqrt{3}n}} < \overline{m}_n$ [21]. This lower bound implies a lower bound of the density:

$$\frac{n\pi}{(2 + \sqrt[4]{12n^2})^2} < \overline{d}([0,1]^2, n).$$

As the densities of optimal packings are known up to $n = 27$, it easy to check that up to $n = 13$ circles the density of an optimal packing is greater or equal to $\overline{d}([0,1]^2, 2) = (3 - 2\sqrt{2})\pi \approx 0.539$ (Table 5).

n	approximate d_n	n	approximate d_n
2	0.5390120845	8	0.7309638253
3	0.6096448087	9	0.7853981634
4	0.7853981634	10	0.6900357853
5	0.6737651056	11	0.7007415778
6	0.6639569095	12	0.7384682239
7	0.6693108268	13	0.7332646949

Table 5. The density of packings up to $n = 13$ circles

If $n > 13$ then after a short calculation the following inequality can be proved:

$$(3-2\sqrt{2})\pi < \frac{n\pi}{(2+\sqrt[4]{12n^2})^2} < \overline{d}([0,1]^2, n).$$

The lower bound is sharp, because $\overline{d}([0,1]^2, 2) = (3-2\sqrt{2})\pi$.
Let us study the upper bound. First we prove that for every $n \geq 2$

$$d([0,1]^2, n) < \frac{\pi}{\sqrt{12}}.$$

This statement is equivalent with

$$\overline{m}_n < f_1(n) = \frac{2+\sqrt{2\sqrt{3}n}}{\sqrt{3}n-2}.$$

It is not to hard to prove this inequality using a corollary of Oler's theorem [4]:
If X is a compact convex subset (with a perimeter of $S(X)$) of the plane, then the number of points with mutual distance of at least 1 is at most

$$\frac{2}{\sqrt{3}}V(X) + \frac{1}{2}S(X) + 1.$$

This statement gives the following upper bound for $\overline{m}_n$:

$$\overline{m}_n \leq f_2(n) = \frac{1+\sqrt{1+(n-1)\frac{2}{\sqrt{3}}}}{n-1}.$$

After a calculation it can be proved that $f_2(n) < f_1(n)$, for $n \geq 2$.
Secondly, we show that there is a point arrangement series $\{S_i\}_{i=1}^{\infty}$, for which $\lim_{i\to\infty} d(S_i, n_i) = \frac{\pi}{\sqrt{12}}$, thus the upper bound of the density is also sharp.
The proof is constructive. Let us denote by $[[p,q]]$ (where $p^2 \leq 3q^2, q^2 \leq 3p^2$) the following lattice point arrangement class: Divide the parallel sides of the square for p and q equal parts, to obtain pq rectangulars (see Figure 1 for $p = 3,\ q = 5,\ n = 12$). Put the first point in the lower left edge of square and put the others in every second gridpoint [14].

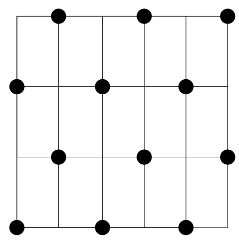


Figure 1. The [[3,5]] lattice arrangement

Let us consider the following packing series $\{S_i\}_{i=1}^{\infty}$:

$$S_1 = [[1,1]],\ \ S_2 = [[3,5]],$$

$$S_i = 4S_{i-1} - S_{i-2},$$

using the operations

$$[[p_1, q_1]] \pm [[p_2, q_2]] = [[p_1 \pm p_2, q_1 \pm q_2]]$$
$$\lambda[[p, q]] = [[\lambda p, \lambda q]] \quad (\lambda \in Z^+)$$

(it is easy to prove that these operations are well-defined).
The limit density of the packing series $\{S_i\}_{n=1}^{\infty}$ is $\frac{\pi}{\sqrt{12}}$, because $S_i = [[p_i, q_i]]$, $n(S_i) = \frac{(p_i+1)(q_i+1)}{2}$, $m(S_i) = \frac{\sqrt{p_i^2+q_i^2}}{p_i q_i}$, therefore

$$\begin{aligned}
\lim_{i\to\infty} d(S_i, n(S_i)) &= \lim_{i\to\infty} n(S_i)\pi \frac{m(S_i)^2}{4(m(S_i)+1)^2} \\
&= \lim_{i\to\infty} \frac{\pi}{4} \frac{\left(\frac{1}{p_i}+1\right)\left(\frac{1}{q_i}+1\right)}{2} \frac{\frac{p_i}{q_i}+\frac{q_i}{p_i}}{(1+m(S_i))^2} \\
&= \frac{\pi}{4}\frac{1}{2}\frac{4\sqrt{3}}{3} = \frac{\pi}{\sqrt{12}},
\end{aligned}$$

where $n(S_i)$ denotes the number of the points in S_i, and $m(S_i)$ is the minimum distance between the points in S_i. □

Remark 2. It is easy to prove on the previous way that for every $n \geq 4$

$$\frac{\pi}{4} \leq \overline{d}([0,1]^2, n),$$

and the density of square-lattice packings is always $\frac{\pi}{4}$.

4. Optimal substructures

Definition 6. *A circle packing/point arrangement in $X \subset [0,1]^2$ is an optimal substructure if the density $d(X, n')$ is maximal in X, where n' denotes the number of the circles/points in X.*

Figures 2 and 3 show two examples for optimal substructures where X is a square or a circle. The optimality of packing of 19 equal circles in a circle was proved in [3].

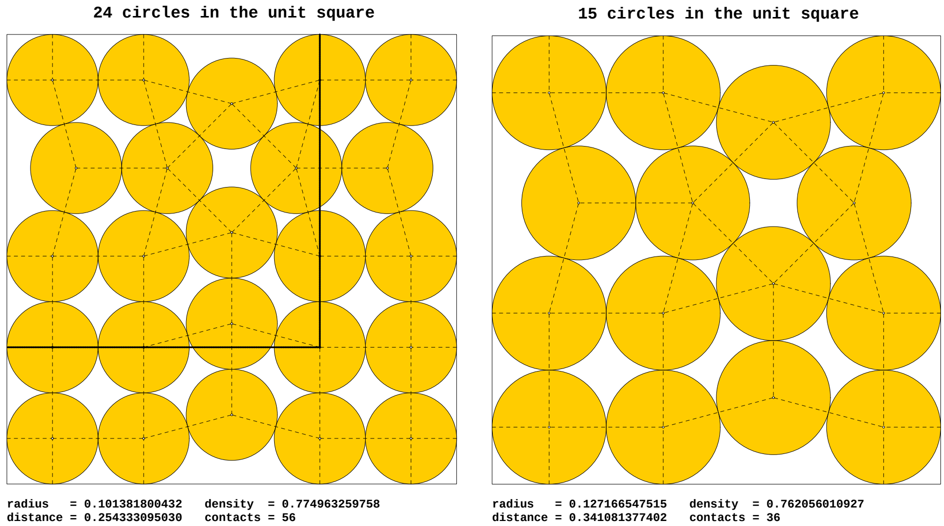


Figure 2. Optimal substructure in an optimal packing, where X is a square

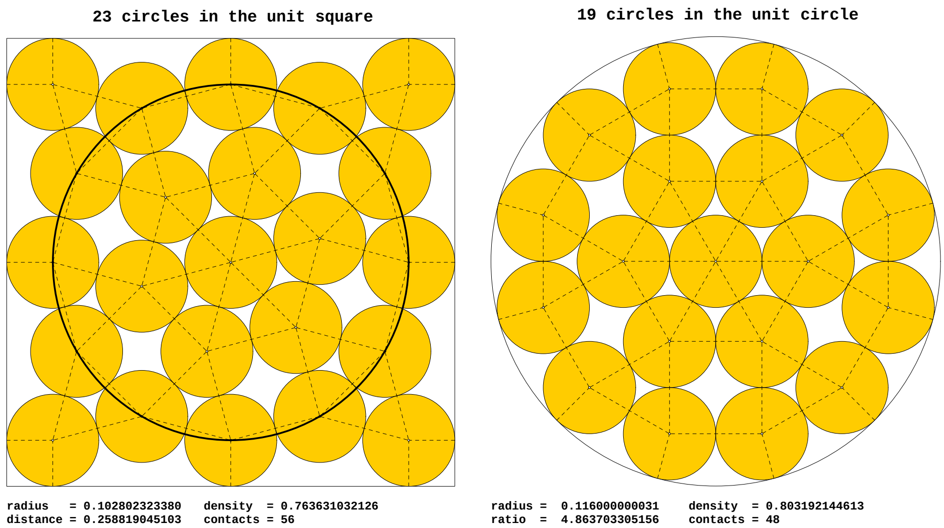


Figure 3. Optimal substructure in an optimal packing, where X is a circle

It is interesting that the known optimal packings (and many approximate packings) contain sometimes optimal substructures. For studying the connection between the packings a good concept is the containment graph.

Definition 7. *The containment graph for a fixed set X is a directed graph, where the nodes are circle packing instances. There is a directed edge from A to B, if A is an optimal substructure in B.*

There is an example of a containment graph in Figure 4.

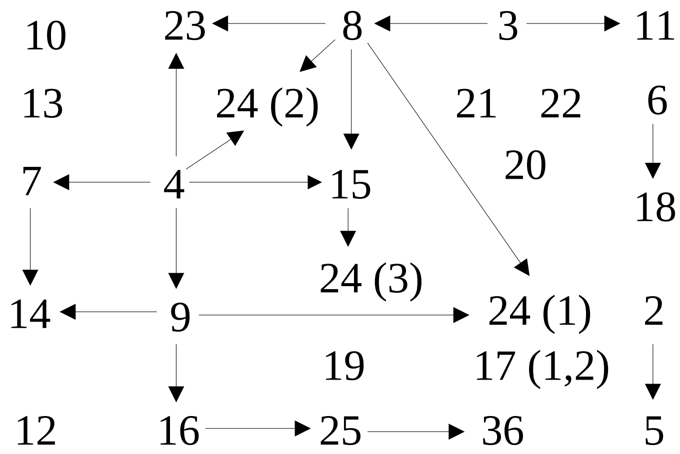


Figure 4. The containment graph, where X is a square with parallel sides with the unit square for the known optimal packings. There are two and three different included optimal packings for n=17 and 24, respectively.

Sometimes, when a packing contains optimal substructures, it is easy to calculate the minimal polynomial based on the minimal polynomial of the substructures. In the following section we introduce the concept of the generalized minimal polynomial of packings and we use it to calculate the traditional minimal polynomials of the arrangements.

5. Generalized minimal polynomials

Definition 8. *$p_n^I(x)$ is a generalized minimal polynomial, where $x \in \{r, m, s, \sigma\}$ and $I \in \{S, \Sigma, R, M\}$ respectively, and the first positive root of the polynomial $p_n^I(x)$ is $\overline{x}_n$, and the degree of $p_n^I(x)$ is minimal. We use the $P_n(x) = p_n^1(x)$ notation too.*

Remark 3. If $p_n^I(x)$ is a generalized minimal polynomial, then $cp_n^I(x)$ is also a minimal polynomial, where $c \neq 0$ real number.

Proposition 2. *The relations between the minimal polynomials are described in Table 6.*

$$
\begin{array}{rc|rc}
\mathbf{p_n^S}(r) = & p_n^{\Sigma:=S-2r}(m := 2r) & \mathbf{p_n^\Sigma}(m) = & p_n^{R:=\Sigma+m}(s := \frac{m}{2}) \\
 & p_n^{R:=S}(s := r) & & p_n^{M:=\Sigma}(\sigma := m) \\
 & p_n^{M:=S-2r}(\sigma := 2r) & & p_n^{S:=\Sigma+m}(r := \frac{m}{2}) \\
\hline
\mathbf{p_n^R}(s) = & p_n^{M:=R-2s}(\sigma := 2r) & \mathbf{p_n^M}(\sigma) = & p_n^{S:=M+\sigma}(r := \frac{\sigma}{2}) \\
 & p_n^{S:=R}(r := s) & & p_n^{\Sigma:=M}(m := \sigma) \\
 & p_n^{\Sigma:=R-2s}(m := 2s) & & p_n^{R:=M+m}(s := \frac{\sigma}{2})
\end{array}
$$

Table 6. Relationships between the minimal polynomials

Proof. It is based on Proposition 1, with a short calculation. □

Example 1. Let us calculate $p^S_{11}(r)$ if we know that

$$P_{11}(m) = m^8 + 8m^7 - 22m^6 + 20m^5 + 18m^4 - 24m^3 - 24m^2 + 32m - 8.$$

It is easy to check that $p^{\Sigma}_n(m) = P_n(m)\Sigma^{\deg P_n}$, so $p^{\Sigma}_{11}(m) = m^8 + 8m^7\Sigma - 22m^6\Sigma^2 + 20m^5\Sigma^3 + 18m^4\Sigma^4 - 24m^3\Sigma^5 - 24m^2\Sigma^6 + 32m\Sigma^7 - 8\Sigma^8$.

Using the $p^S_n(r) = p^{\Sigma:=S-2r}_n(m := 2r)$ relation
$p^S_{11}(r) = p^{\Sigma:=S-2r}_{11}(m := 2r) = (2r)^8 + 8(2r)^7(S - 2r) - 22(2r)^6(S - 2r)^2 + 20(2r)^5(S - 2r)^3 + 18(2r)^4(S - 2r)^4 - 24(2r)^3(S - 2r)^5 - 24(2r)^2(S - 2r)^6 + 32(2r)(S - 2r)^7 - 8(S - 2r)^8 = -18176r^8 + 45056r^7S - 63360r^6S^2 + 56192r^5S^3 - 30432r^4S^4 + 9920r^3S^5 - 1888r^2S^6 + 192rS^7 - 8S^8$.
Divided by -8 the previous generalized minimal polynomial is
$p^S_{11}(r) = 2272r^8 - 5632r^7S + 7920r^6S^2 - 7024r^5S^3 + 3804r^4S^4 - 1240r^3S^5 + 236r^2S^6 - 24rS^7 + S^8$.

5.1. Calculation of minimal polynomials from the minimal polynomials of substructures

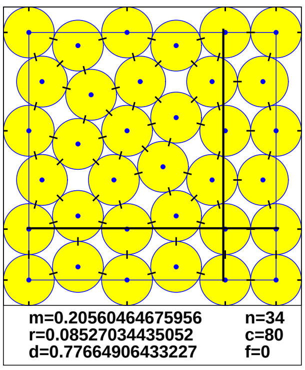
Proposition 3. *Let us consider a point arrangement in* $[0, 1]^2$. *Let us suppose, there are* $N \geq 2$ *optimal substructures of the previous arrangement in a square of sides* $\Sigma_1, \Sigma_2, \ldots, \Sigma_N$. *If* $f_\Sigma(x)$ *is a polynomial and there exist* $1 \leq i, j \leq N$ *such that* $\Sigma_j = f_\Sigma(\Sigma_i)$, *then the minimal polynomial* $p^{\Sigma}_n(m)$ *can be calculated from the minimal polynomials of the optimal substructures in the following way:*

$$p^{\Sigma}_n(m) = \mathrm{Res}(p^{\Sigma_j}_{n_1}(m), p^{f(\Sigma_j)}_{n_2}(m), \Sigma_j) =$$

$$\det(\mathrm{Syl}(p^{\Sigma_j}_{n_1}(m), p^{f(\Sigma_j)}_{n_2}(m), \Sigma_j)).$$

Proof. It follows immediately from the definition of the resultant. □

Example 2. Determine $P_{34}(m)$ based on $p^{\Sigma_1}_{23}(m)$ and $p^{\Sigma_2}_4(m)$.



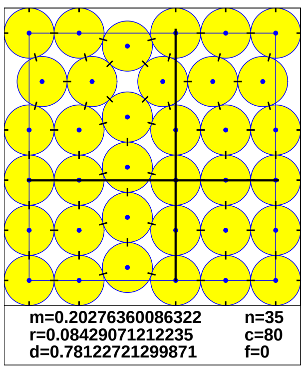


Figure 5. Approximate circle packings for $n = 34$ and $n = 35$

In this example

$$f_\Sigma(x) = \Sigma - x \quad \text{and} \quad \Sigma = 1,$$

$$p_{23}^{\Sigma_1}(m) = 16m^4 - 16m^2\Sigma_1^2 + \Sigma_1^4 \qquad p_4^{\Sigma_2}(m) = m - \Sigma_2 = m - 1 + \Sigma_1$$

$$P_{34}(m) = \mathrm{Res}(p_{23}^{\Sigma_1}(m), p_4^{1-\Sigma_1}(m), \Sigma_1) = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ m-1 & 1 & 0 & 0 & 0 \\ 0 & m-1 & 1 & 0 & -16m^2 \\ 0 & 0 & m-1 & 1 & 0 \\ 0 & 0 & 0 & m-1 & 16m^4 \end{vmatrix}$$

$$= m^4 + 28m^3 - 10m^2 - 4m + 1.$$

Proposition 4. *Let us consider the minimal polynomial $P_n(m)$ and suppose that*

$$m_n = \frac{am_{n'} + b}{cm_{n'} + d} \quad \text{and} \quad m_{n'} = \frac{b - dm_n}{cm_n - a},$$

where $a, b, c,$ and d are real numbers. The minimal polynomial $P_{n'}(m)$ can be calculated in the following way:

$$P_{n'}(m) = P_n\left(\frac{am + b}{cm + d}\right)(cm + d)^{\deg P_n}.$$

Proof. It is easy too see that $P_n\left(\frac{am+b}{cm+d}\right)(cm+d)^{\deg P_n}$ is a polynomial and $m_{n'}$ is a root of this polynomial. It is a minimal polynomial because if it would not be the case then there would be another polynomial R, with $R(m_{n'}) = 0$ and

$$\deg R < \deg P_n\left(\frac{am + b}{cm + d}\right)(cm + d)^{\deg P_n}.$$

But this is impossible since in this case

$$(\deg R =) \deg R\left(\frac{b - dm}{cm - a}\right)(cm - a)^{\deg R} < \deg P_n,$$

which contradicts that $P_n(m)$ is a minimal polynomial. □

Example 3. Let us determine $P_{35}(m)$.
a) Based on Proposition 3 using $p_{15}^{\Sigma_1}(m)$ and $p_9^{\Sigma_2}(m)$, we have

$$f_\Sigma(x) = \Sigma - x \quad \text{and} \quad \Sigma = 1,$$

$$p_{15}^{\Sigma_1}(m) = 2m^4 - 4m^3\Sigma_1 - 2m^2\Sigma_1^2 + 4m\Sigma_1^3 - \Sigma_1^4, \quad p_9^{\Sigma_2}(m) = 2m - \Sigma_2 = 2m - 1 + \Sigma_1,$$

$$P_{35}(m) = \mathrm{Res}(p_{15}^{\Sigma_1}(m), p_9^{1-\Sigma_1}(m), \Sigma_1) = \begin{vmatrix} 1 & 0 & 0 & 0 & -1 \\ 2m-1 & 1 & 0 & 0 & 4m \\ 0 & 2m-1 & 1 & 0 & -2m^2 \\ 0 & 0 & 2m-1 & 1 & -4m^3 \\ 0 & 0 & 0 & 2m-1 & 2m^4 \end{vmatrix}$$

$$= 46m^4 - 84m^3 + 50m^2 - 12m + 1.$$

b) Based on Proposition 4 using

$$P_{24}(m) = m^4 - 16m^3 + 20m^2 - 8m + 1$$

and the $m_{35} = 2r_{24}$ relationship,

$$m_{35} = 2r_{24} = \frac{m_{24}}{m_{24}+1}, \quad \text{so} \quad m_{24} = \frac{m_{35}}{1-m_{35}} \quad \text{and}$$

$$P_{35}(m) = P_{24}\left(\frac{m}{1-m}\right)(1-m)^4 = 46m^4 - 84m^3 + 50m^2 - 12m + 1.$$

5.2. Determining minimal polynomials in a different way

Sometimes the structure of an optimal packing is not symmetric and it does not contain an optimal substructure. In this case a possible way to calculate the minimal polynomial is the following: Let us define a quadratical system of equations to the packing where an equation reflects the fact that distance of two points is m_n. To determine the minimal polynomial we have to eliminate all variables without m_n. Using Buchberger's algorithm (Gröbner basis) or another technique based on the resultant and a symbolic algebra system (e.g. Maple, Mathematica, CoCoA, Macaulay2, Singular, etc.) this can be done, but sometimes this is also hard [15].

Example 4. Let us determine $P_{10}(m)$.

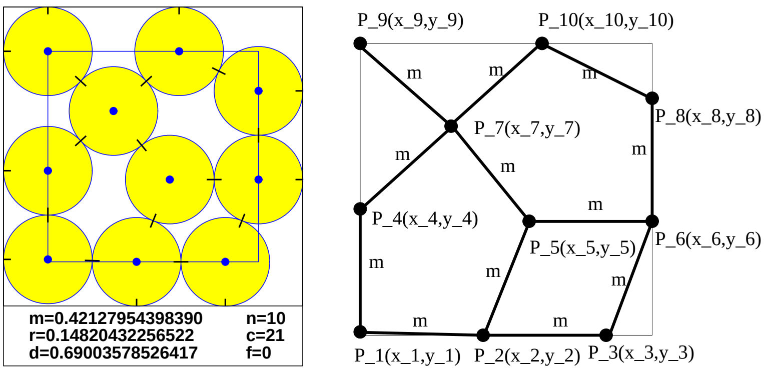


Figure 6. The optimal packing of 10 circles/points in the unit square

The corresponding quadratical system of equations is the following:

$$\begin{aligned}
(x_1-x_2)^2+(y_1-y_2)^2&=m^2 & (x_1-x_4)^2+(y_1-y_4)^2&=m^2\\
(x_2-x_3)^2+(y_2-y_3)^2&=m^2 & (x_2-x_5)^2+(y_2-y_5)^2&=m^2\\
(x_5-x_6)^2+(y_5-y_6)^2&=m^2 & (x_3-x_6)^2+(y_3-y_6)^2&=m^2\\
(x_4-x_7)^2+(y_4-y_7)^2&=m^2 & (x_5-x_7)^2+(y_5-y_7)^2&=m^2\\
(x_7-x_9)^2+(y_7-y_9)^2&=m^2 & (x_7-x_{10})^2+(y_7-y_{10})^2&=m^2\\
(x_8-x_{10})^2+(y_8-y_{10})^2&=m^2 & (x_6-x_8)^2+(y_6-y_8)^2&=m^2
\end{aligned}$$

The points $P_1, P_2, P_3, P_4, P_6, P_8, P_9$, and P_{10} are on the side of the square thus $x_1 = x_4 = x_9 = y_2 = y_3 = 0$ and $x_6 = x_8 = y_9 = y_{10} = 1$. It is easy to see that $y_4 = y_1 + m$, $x_3 = x_2 + m$ and $y_8 = y_6 + m$. $P_2P_3P_5P_6$ is a rhombus thus $x_5 = 1 - m$ and $y_5 = y_6$. In the $P_4P_7P_9$ and $P_9P_7P_{10}$ isosceles triangulars (thus the points P_4, P_7 and P_{10} are on a straight line) these equalities hold: $y_7 = (1 + y_1 + m)/2$ and $x_7 = x_{10}/2$.

Using the previous observations all variables are eliminated with the exception of x_2, x_{10}, y_1, y_5 and m. The system of equations is then reduced to the form:

$$
\begin{aligned}
x_2^2 + y_1^2 &= m^2, \\
x_{10}^2 + (1 - y_1 - m)^2 &= (2m)^2, \\
(1 - x_{10})^2 + (1 - y_5 - m)^2 &= m^2, \\
(1 - x_2 - m)^2 + y_5^2 &= m^2, \\
(2 - 2m - x_{10})^2 + (2y_5 - 1 - y_1 - m)^2 &= (2m)^2.
\end{aligned}
$$

Let us determine the minimal polynomial with Maple 8 based on the Groebner package:
`>with(Groebner):univpoly(m,[polynomials],`$\{x_2, y_1, x_{10}, y_5, m\}$`);`.
The obtained minimal polynomial $P_{10}(m)$ is given in the following subsection.

5.3. A list of the known minimal polynomials $P_n(m)$ $(2 \leq n \leq 100)$

$n = 2$ $\quad m^2 - 2$

$n = 3$ $\quad m^4 - 16m^2 + 16$

$n = 4$ $\quad m - 1$

$n = 5$ $\quad 2m^2 - 1$

$n = 6$ $\quad 36m^2 - 13$

$n = 7$ $\quad m^2 - 8m + 4$

$n = 8$ $\quad m^4 - 4m^2 + 1$

$n = 9$ $\quad 2m - 1$

$n = 10$
$$
\begin{aligned}
&1180129m^{18} - 11436428m^{17} + 98015844m^{16} - 462103584m^{15} \\
&+1145811528m^{14} - 1398966480m^{13} + 227573920m^{12} + 1526909568m^{11} \\
&-1038261808m^{10} - 2960321792m^{9} + 7803109440m^{8} - 9722063488m^{7} \\
&+7918461504m^{6} - 4564076288m^{5} + 1899131648m^{4} - 563649536m^{3} \\
&+114038784m^{2} - 14172160m + 819200
\end{aligned}
$$

$n = 11$ $\quad m^8 + 8m^7 - 22m^6 + 20m^5 + 18m^4 - 24m^3 - 24m^2 + 32m - 8$

$n = 12$ $\quad 225m^2 - 34$

$n = 13$
$$
\begin{aligned}
&5322808420171924937409m^{40} + 5867739593380498886173232m^{39} \\
&+1302444884533227120326692 8m^{38} - 1298840956705690999017 0432m^{37} \\
&-6697217539589294973937 2512m^{36} - 2714511572112816542521 75360m^{35} \\
&+1438322342979585076139742976m^{34} - 3354298954676639164979 96800m^{33} \\
&-6543699259726848821592216832m^{32} + 9441371361011345362166468608m^{31} \\
&+10182180602633501397232254976m^{30} - 4224601986454107192266162 1760m^{29} \\
&+37620100408876038921186476032m^{28} + 28699095956807539331396009984m^{27} \\
&-102587608293645346411004952576m^{26} + 103509313296807875445571190784m^{25}
\end{aligned}
$$

$$
\begin{aligned}
&-23909360523055293307841740800m^{24} - 62735581440162634955836358656m^{23}\\
&+8845487155196314204195258368m^{22} - 53012494559549527012040245248m^{21}\\
&+2135173605242212884072628224m^{20} + 26378985900767549703436894208m^{19}\\
&-26497225761631816480192462848m^{18} + 12731474183761933022491836416m^{17}\\
&-3984323399280382686621859840m^{16} - 4422001291286852186186711040m^{15}\\
&+3658751900977247115934695424m^{14} - 1429726216634427968279543808m^{13}\\
&+57770773621828718826618880m^{12} + 27558237068869986131797606 4m^{11}\\
&-17163231072528337551228928 0m^{10} + 46974915155899860050247680m^{9}\\
&+17600674325965992414412 80m^{8} - 7491112055212411797372928m^{7}\\
&+365299850469661428259225 6m^{6} - 1072642406499215430647808m^{5}\\
&+217086289997205686190080m^{4} - 30811405631471617048576m^{3}\\
&+2960075719794736758784m^{2} - 174103532094609162240m\\
&+4756927106410086400
\end{aligned}
$$

$n = 14$	$13m^2 - 16m + 4$
$n = 15$	$2m^4 - 4m^3 - 2m^2 + 4m - 1$
$n = 16$	$3m - 1$
$n = 17$	$m^8 - 4m^7 + 6m^6 - 14m^5 + 22m^4 - 20m^3 + 36m^2 - 26m + 5$
$n = 18$	$144m^2 - 13$
$n = 19$	$242m^{10} - 1430m^9 - 8109m^8 + 58704m^7 - 78452m^6$ $-2918m^5 + 43315m^4 + 39812m^3 - 53516m^2 + 20592m$ -2704
$n = 20$	$128m^2 - 96m + 17$
$n = 23$	$16m^4 - 16m^2 + 1$
$n = 24$	$m^4 - 16m^3 + 20m^2 - 8m + 1$
$n = 25$	$4m - 1$
$n = 27$	$1600m^2 - 89$
$n = 30$	$1202m^2 - 252m + 13$
$n = 34$	$m^4 + 28m^3 - 10m^2 - 4m + 1$
$n = 35$	$46m^4 - 84m^3 + 50m^2 - 12m + 1$
$n = 36$	$5m - 1$
$n = 39$	$1732m^2 - 68m - 17$
$n = 42$	$864m^2 - 360m + 37$
$n = 52$	$7056m^2 - 193$
$n = 56$	$1715m^2 - 588m + 50$
$n = 99$	$28900m^2 - 389$

5.4. An experimental way to guess minimal polynomials using Maple 8

Recently M. Cs. Markót and T. Csendes [9, 10] have developed a reliable numerical computer aided method to find the optimal solution of the circle packing problem. This approach is based on interval arithmetic computations and gives high accuracy numerical results. They studied the $n = 28, 29$, and 30 cases. If the precision of the computation is good enough, sometimes the minimal polynomial can be guessed using e.g. Maple 8. Applying the

```
>Digits:=a;
>with(PolynomialTools):MinimalPolynomial(m,b);
```

commands, where a is the accuracy of approximation of m, and b is the degree of the approximating minimal polynomial. Table 7 summarizes the accuracy necessary to find the exact minimal polynomial $P_n(m)$.

n	degree	accuracy	n	degree	accuracy
2	2	3	18	2	10
3	4	10	19	10	58
4	1	3	20	2	10
5	2	4	23	4	10
6	2	9	24	4	10
7	2	6	25	1	4
8	4	5	27	2	15
9	1	3	30	2	13
10	18	193	34	4	10
11	8	20	35	4	13
12	2	11	36	1	4
13	40	1217	39	2	13
14	2	7	42	2	13
15	4	7	52	2	14
16	1	4	56	2	14
17	8	19	99	2	17

Table 7. The necessary accuracy in digits to determine the exact minimal polynomial $P_n(m)$

6. Summary

In this work we investigated the relations between the parameters of four equivalent allocation problems. We proved sharp constant bounds on the density of packings. Some new concepts (optimal substructure, containment graph and generalized minimal polynomial) have been introduced. Based on optimal substructures, we have calculated some new minimal polynomials.

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