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A Symplectic Reduction for Pseudo-Riemannian Manifolds with Compatible almost Product Structures

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Abstract. We consider a manifold M with a pseudo-Riemannian metric g and an almost product structure P such that $g(P(X), P(Y)) = -g(X, Y)$. We suppose that the almost product structure P is parallel with respect to the Levi-Civita connection of g. These induce a natural symplectic structure on M. We consider an isometric action of a Lie group G on M preserving the pseudo-Riemannian metric and the almost product structure P. Then we prove a symplectic reduction theorem for such manifolds. We obtain a reduced manifold with a pseudo-Riemannian metric and a parallel almost product structure.

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1. Introduction

The theory of symplectic manifolds has brought a lot of new results in various branches of mathematics and physics. The subject is of a fast development in the recent years. Among the techniques used in the symplectic geometry there is the so called *symplectic reduction.* If (M, ω) is a symplectic manifold with an action of a Lie group G preserving the symplectic form then there may exist the so called *moment map* going from M to the dual space to the

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Lie algebra of G. Then under certain non-degeneracy conditions one gets that the quotients of the level sets of the moment map by the action of the group carry symplectic structures naturally induced by ω. This construction goes back to the work of Jacobi and Liouville who reduced the number of equations in a Hamiltonian system using essentially the symplectic reduction procedure. In the modern context of the symplectic geometry this was proved by Marsden-Weinstein, cf. [20]. Such situation happens when the symplectic manifold has rich symmetry group.

From the geometric point of view the reduction procedure is applied to construct new examples of manifolds with particularly interesting structures. If the manifold M, apart from being symplectic, carries also another structure compatible, in a certain sense, with ω then via the reduction we frequently get new structures on the quotients. In such a way there were constructed new examples of different structures on manifolds: Kähler, hyper-Kähler, quaternionic-Kähler, hypercomplex and para-quaternionic, cf. [11, 13, 9, 14, 28]. Recently we have also many results about the reduction of manifolds carrying contact, Sasaki, 3-Sasaki structures, cf. [2, 9, 10, 17, 31, 5, 6]. Moreover, via the symplectic reduction, some previously known examples of such manifolds got new description via the reduction procedure. The subject is currently under intensive research by mathematicians and physicists from different scientific backgrounds.

In our paper we consider a pseudo-Riemannian manifold (M, g) equipped with an almost product structure P. We assume that g and P are compatible, i.e. (2.1) holds. Then we put $\omega(X, Y) := g(P(X), Y)$ and get a 2-form on M. We suppose that ω is a symplectic form. We consider also a Lie group G acting on M by isometries and leaving invariant the almost product structure P. Then we apply the reduction theorem and get a new symplectic manifold with a compatible almost product structure. We also consider a particular case when P is parallel with respect to the Levi-Civita connection. In the last section of our paper we apply our reduction to construct a type of Fubini-Study metric on a space obtained from the reduction of the standard pseudo-Riemannian metric on $\mathbb{R}^{2m}$.

2. Preliminaries

2.1. Pseudo-Riemannian manifolds with compatible almost product structures

Let (M, g, P) be a pseudo-Riemannian manifold with a pseudo-Riemannian metric tensor g and an almost product structure P on M, i.e. P is an endomorphism of TM such that $P^2 = Id$. The almost product structures are well-known objects in differential geometry; as main references we suggest [29, 32].

We suppose that g and P are compatible in the sense that for each $X, Y \in T_xM$ and each $x \in M$ we have that

$$g(P(X), P(Y)) = -g(X, Y). \tag{2.1}$$

A structure (M, g, P) is called *almost para-Hermitian manifold*, cf. [23, 18, 3, 4]. There is defined an almost symplectic form ω such that $\omega(X, Y) = g(P(X), Y)$; it is called the *fundamental* 2*-form associated to an almost para-Hermitian structure*, it is always non-degenerated but usually is not closed. Moreover, ω is compatible with P, i.e.

$$\omega(P(X), P(Y)) = -\omega(X, Y) \tag{2.2}$$

for each $X, Y \in T_xM$ and each $x \in M$. On the linear algebra level it is easy to prove that the manifold M has to be of even dimension, let say $2m$, and the pseudo-Riemannian metric tensor g has to be of the type (m, m), i.e. there exist an orthonormal basis $v_1, \dots, v_m, w_1, \dots, w_m$ such that for all $i, j = 1, \dots, m$

$$\delta_{ij} = g(v_i, v_j) = -g(w_i, w_j) \text{ and } g(v_i, w_j) = 0.$$

The operator P has two eigenvalues $+1$ and -1. Let $\mathcal{D}_\pm$ be the eigenvector bundles associated with these eigenvalues. It follows easily from condition (2.1) that $\mathcal{D}_-$ and $\mathcal{D}_+$ are smooth subbundles of the same dimension m. Moreover, both $\mathcal{D}_-$, $\mathcal{D}_+$ are isotropic with respect to g and ω. Explicitly we mean that for each $X_1, Y_1 \in \mathcal{D}_-$ and for each $X_2, Y_2 \in \mathcal{D}_+$ we have that

$$\omega(X_1, Y_1) = \omega(X_2, Y_2) = 0 \tag{2.3}$$

$$g(X_1, Y_1) = g(X_2, Y_2) = 0. \tag{2.4}$$

Equations (2.3) mean that $\mathcal{D}_\pm$ are complementary Lagrangian subbundles of TM.

Example 2.1. Let $M = \mathbb{R}^m \times \mathbb{R}^m$ and let $(x_1, \dots, x_m, y_1, \dots, y_m)$ be the global coordinates on M. Then for all $i, j = 1, \dots, m$ we put:

$$P(\tfrac{\partial}{\partial x_i}) = \tfrac{\partial}{\partial y_i},\ P(\tfrac{\partial}{\partial y_i}) = \tfrac{\partial}{\partial x_i},$$
$$\delta_{ij} = g(\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_j}) = -g(\tfrac{\partial}{\partial y_i}, \tfrac{\partial}{\partial y_j}),\ 0 = g(\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial y_j})$$

and get (M, g, P) which is the standard flat model of an almost para-Hermitian manifold. We shall return to this example in the last section of our paper.

An almost para-Hermitian manifold may be defined in an equivalent way.

Observation 2.1. *A triple (M, ω, P) such that ω is an almost symplectic form and P is a compatible almost product structure, in the sense of* (2.2)*, determines an almost para-Hermitian structure on M.*

Proof. In fact, we put $g(X, Y) := \omega(P(X), Y)$ and get a pseudo-Riemannian metric compatible with P. □

Observation 2.2. *A quadruple $(M, g, \mathcal{D}_-, \mathcal{D}_+)$ such that g is a pseudo-Riemannian metric tensor and $\mathcal{D}_\pm$ are complementary isotropic subbundles of TM, determines an almost para-Hermitian structure on M.*

Proof. In fact, we define an endomorphism $P : TM \to TM$ by assuming that $P|_{\mathcal{D}_\pm} := \pm id$. This gives an almost product structure compatible with g. □

Observation 2.3. *A quadruple $(M, \omega, \mathcal{D}_-, \mathcal{D}_+)$ such that ω is an almost symplectic form and $\mathcal{D}_\pm$ are Lagrangian transversal subbundles of TM, determines an almost para-Hermitian structure on M.*

Proof. In fact, we define $P : TM \to TM$ as in Observation 2.2 and then a pseudo-Riemannian metric such that $g(X, Y) := \omega(P(X), Y)$. Then easy calculation proves that (M, g, P) is an almost para-Hermitian manifold. □

Observation 2.4. *A triple (M, g, ω) such that g is a pseudo-Riemannian metric tensor, ω is an almost symplectic form satisfying $\omega^\sharp \circ g^\flat = g^\sharp \circ \omega^\flat$, determines an almost para-Hermitian structure on M. The symbols $\sharp$ and $\flat$ denote the musical isomorphism defined by a bilinear map,* cf. [19].

Proof. We put $P := g^\sharp \circ \omega^\flat$ and easily get that together with g it defines an almost para-Hermitian structure. □

Let g be a pseudo-Riemannian metric tensor and P an almost product structure which are compatible, in the sense that (2.1) holds. In this context the following definition is rather natural.

Definition 2.1. (cf. [4]) *An almost para-Hermitian manifold (M, g, P) is said to be para-Kähler if and only if $\nabla^g P = 0$ where ∇^g is the Levi-Civita connection of g.*

The *integrability* of P means that an associated $\mathcal{G}$-structure is integrable, cf. [27]. A necessary and sufficient condition for the integrability of P is that the associated Nijenhuis tensor $N(P)(X, Y) := [P(X), P(Y)] - P[P(X), Y] - P[X, P(Y)] + [X, Y]$ vanishes identically on M. Another, equivalent condition for the integrablity of P is that the distributions $\mathcal{D}_-, \mathcal{D}_+$ are integrable.

Since we have equivalent descriptions of an almost para-Hermitian manifolds as in Observations 2.1, 2.2, 2.3 and 2.4 then a para-Kähler manifold may be also described in different ways. In the following observation we list such descriptions.

Observation 2.5. *The following conditions are equivalent:*

- *An almost para-Hermitian structure (M, ω, P) is para-Kähler,*
- *$d\omega = 0$ and $N(P) = 0$,*
- *$\nabla^g \Gamma(\mathcal{D}_\pm) \subset \Gamma(\mathcal{D}_\pm)$ where ∇^g is the Levi-Civita connection of g,*
- *$d\omega = 0$ and $\mathcal{D}_\pm$ are integrable subbundles of TM,*
- *$\nabla^g \omega = 0$ where ∇^g is the Levi-Civita connection of g.*

Proofs of the above properties may be found in [3, 4].

Corollary 2.1. *A para-Kähler manifold is uniquely determined by a symplectic form and two transversal Lagrangian foliations.*

Remark 2.1. There is a wide range of classes of manifolds between almost para-Hermitian and para-Kähler manifolds. A rich bibliography of this subject may be found in [3, 4].

2.2. Lorentz numbers

We consider the algebra of Lorentz numbers $\mathbb{L}$ and recall here some properties of $\mathbb{L}$. We would like to stress its geometrical similarity to the field of the complex numbers. Then we shall get an almost para-Hermitian structure on manifolds as a natural analogue of the almost Hermitian structures.

The algebra $\mathbb{L}$ may be defined as $\mathbb{L} = \{u + \tau v | u, v \in \mathbb{R}\}$ with the assumption that the *imaginary unit* τ has the property that $\tau^2 = 1$. More precisely, in the set $\mathbb{L}$ we have two internal operations: sum and product. They are defined as follows:

$$\begin{aligned}(u_1 + \tau v_1) + (u_2 + \tau v_2) &:= (u_1 + u_2) + \tau(v_1 + v_2)\\ (u_1 + \tau v_1) \cdot (u_2 + \tau v_2) &:= (u_1 u_2 + v_1 v_2) + \tau(u_1 v_2 + u_2 v_1).\end{aligned}$$

With these operations the set $\mathbb{L}$ is an associative, commutative algebra over $\mathbb{R}$ with unity; this algebra is called *the algebra of Lorentz numbers.* One can find a very beautiful exposition of algebraic properties and geometric application of $\mathbb{L}$ in [25, 12]. The algebra $\mathbb{L}$ admits the zero divisors: they are the numbers of the type $u \pm \tau u$ where $u \in \mathbb{R} \setminus \{0\}$. There is naturally defined conjugation in $\mathbb{L}$, namely $\overline{u + \tau v} := u - \tau v$. Moreover, we put $\mathsf{Re}(u + \tau v) = u$ and $\mathsf{Im}(u + \tau v) = v$. Let $z = u + \tau v$ be an element of $\mathbb{L}$ which is not a zero divisor. Then there exists the inverse of z and we have that $z^{-1} = \overline{z}/(z\overline{z})$. The algebra $\mathbb{L}$ is isomorphic, as an algebra, to $\mathbb{R}\oplus\mathbb{R}$ via the map $\Phi : \mathbb{L} \to \mathbb{R}\oplus\mathbb{R}$ defined by $\Phi(u+\tau v) := (u+v, u-v)$. The inverse of this isomorphism is given by $\Phi^{-1}(x, y) = (1/2)(x + y, x - y)$. The algebra $\mathbb{R} \oplus \mathbb{R}$ is usually denoted by $\mathbb{B}$, cf. [3, 4]. We observe that the multiplication by τ on $\mathbb{L}^m$ defines a product structure in the vector space $\mathbb{R}^{2m} \cong \mathbb{L}^m$. Let $v = (z_1, \dots , z_m),\ w = (w_1, \dots , w_m) \in \mathbb{L}^m$; suppose also that $z_j = a_j + \tau b_j$ and $w_j = c_j + \tau d_j$ for each $j = 1, 2, \dots , m$. Then the following formula

$$\ll v, w \gg := \mathsf{Re} \sum_{j=1}^{m} z_j \overline{w}_j = \sum_{j=1}^{m} (a_j c_j - b_j d_j)$$

gives the standard scalar product of signature (m, m) on $\mathbb{L}^m \cong \mathbb{R}^{2m}$. Moreover, we have that $\ll \tau v, \tau w \gg = - \ll v, w \gg$. This justifies the use of the algebra $\mathbb{L}$ as a geometric model for manifolds of signature (m, m) with a compatible almost product structure. Moreover, the property (2.1) arises naturally in this context.

Remark 2.2. There is a canonical isomorphism $\mathbb{L}^m \cong \mathbb{R}^m \times \mathbb{R}^m$ which is an extension of the isomorphism Φ. This isomorphism transfers the scalar product $\ll\ ,\ \gg$ into the scalar product g and the multiplication by τ into the almost product structure P, cf. Example 2.1. The Lorentz numbers seams to be a good instrument to study pseudo-Riemannian geometry. There is a wide theory of functions over the algebra $\mathbb{L}$ which finds applications in geometry, cf. [25, 3, 4, 26, 15]. For instance, these numbers may be useful to describe the isometry groups of pseudo-Riemannian plane, cf. [12], or to describe minimal immersions of pseudo-Riemannian manifolds and more generally to study the geometry of such manifolds, cf. [3, 4, 7, 15, 16, 28]. The differential calculus over Lorentz numbers is developed in [23, 24, 18, 26, 15]. The study of manifolds modelled on the algebra of Lorentz numbers goes back to the work of Rashevski, cf. [23] and then was developped by many authors, cf. [18, 24, 25]. A wide bibliography about the subject may be found in [3, 4]. In the particular case of $m = 1$ we get the so called *Lorentz surfaces* which are studied intensively in the recent years, cf. [30].

2.3. Moment map for symplectic manifolds

We shall recall the basic definitions here. Let (M, ω) be a symplectic manifold and suppose that there is given an action $\phi : G \times M \to M$ of a Lie group G on M which preserves the symplectic 2-form ω. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ the dual space. Then the group G acts on $\mathfrak{g}$ by the adjoint action $ad : G \to Aut(\mathfrak{g})$ and then it induces the coadjoint action on $\mathfrak{g}^*$ such that for each $a \in G$ and $l \in \mathfrak{g}^*$ we have that $a \cdot l = l \circ ad_g^{-1}$. Each element A of $\mathfrak{g}$ determines a vector field, denoted by $\widetilde{A}$, on M in the following way: if a_t is a 1-parameter subgroup of G generated by A then $(t, x) \to \phi(a_t, x)$ is the flow which defines $\widetilde{A}$. This vector field is usually called the *infinitesimal generator* of the operation on M associated to A.

There is a natural pairing $< \, , \, >: \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ such that for each $l \in \mathfrak{g}^*$ and $A \in \mathfrak{g}$ we have that $< l, A >= l(A)$.

Definition 2.2. (cf. [1]) *A map $\mu : M \to \mathfrak{g}^*$ is said to be a moment map related to the action ϕ if and only if μ is equivariant with respect to the action of G and for each $X \in T_xM$ $(x \in M)$ and each $A \in \mathfrak{g}$*

$$< d\mu(X), A >= \omega(\widetilde{A}_x, X). \tag{2.5}$$

The name of this map comes from the classical mechanics, cf. [1, 11]. We would like only to underline some well-known facts. If there exist two moment maps then they differs by a constant element of $\mathfrak{g}^*$, clearly when M is connected. A moment map may not exist for a given action of a Lie group. If $H^1(M) = 0$ then there always exists a map satisfying condition (2.5). However a moment map always exists if G is semi-simple. For detailed treatment of this subject look to [1, 11, 20, 19].

3. Reduction theorem

We make the following assumptions throughout all of this section:

(i) (M, g, P) is an almost para-Hermitian manifold and ω is the 2-form on M determined by the almost para-Hermitian structure; $d\omega = 0$.

(ii) G is a Lie group and $\phi : G \times M \to M$ is an action on the left preserving g and P. We assume that there exists a moment map $\mu : M \to \mathfrak{g}^*$ for the action ϕ.

(iii) l is an element of $\mathfrak{g}^*$ which is invariant by the coadjoint action of G and such that $\mu^{-1}(l)$ is not empty.

(iv) G acts free and properly on $\mu^{-1}(l)$, here properly means that if (x_i) and $(\phi(a_i, x_i))$ are convergent in $\mu^{-1}(l)$ then (a_i) has a convergent subsequence in G.

Theorem 3.1. *If the pseudo-Riemannian metric restricted to any orbit of the action G in $\mu^{-1}(l)$ is non-degenerated then $\frac{\mu^{-1}(l)}{G}$ is an almost para-Hermitian manifold of dimension* $\dim M - 2\dim G$ *with the closed fundamental* 2*-form associated to the almost para-Hermitian structure. If (M, g, P) is para-Kähler then $\frac{\mu^{-1}(l)}{G}$ is also para-Kähler.*

Proof. We follow the original proof of the Kähler reduction theorem with a particular attention to the signature of the induced pseudo-Riemannian metrics on the submanifolds and the quotient manifolds. We suppose that (M, g, P) is an almost para-Hermitian manifold and the fundamental 2-form ω is closed. For convenience, we put $M_0 = \mu^{-1}(l)$ and let $x \in M_0$. We observe that for each $X \in T_xM$ we have that: $d\mu(X) = 0 \Leftrightarrow$ for each $A \in \mathfrak{g}$ $\omega(\widetilde{A}, X) = 0$. We put $p := \dim G$. Since the form ω is of maximal rank and since $\dim G = \dim\{\widetilde{A}_x : A \in \mathfrak{g}\}$ for G acts free then the dimension of $\ker d_x\mu$ is constant and equal to $\dim M - p$. Hence $d_x\mu : T_xM \to \mathfrak{g}^*$ is surjective and hence l is a regular value of μ. For each $x \in M_0$ we put $V_x = \{\widetilde{A}_x \ : \ A \in \mathfrak{g}\}$. Then it is clear that

$$V := \bigcup_{x\in M_0} V_x \to M_0$$

is a vector bundle of rank p. Since l is fixed by the coadjonit action of G then it follows that V is a subbundle of TM_0. Then we define new subbundles W and H of TM such that for each $x \in M_0$ we have that $W_x = P(V_x)$ and $H_x = (V_x + W_x)^\perp$. We have the following properties of these bundles:

1. The restriction of the pseudo-Riemannian metric g to V gives a non-degenerated scalar product in each fibre of V,
2. $\dim V_x = \dim W_x = p$ and the pseudo-Riemannian metric g restricted to W_x is non-degenerated for each $x \in M_0$,
3. $T_xM_0 = W_x^\perp$ for each $x \in M_0$,
4. $P(H) \subset H$,
5. the pseudo-Riemannian metric g restricted to the fibres of H is non-degenerated, for each $x \in M_0$

$$T_xM = W_x \oplus \overbrace{V_x \oplus H_x}^{T_xM_0},$$

moreover, the decomposition above is orthogonal and the scalar product g restricted to W_x, V_x and H_x is non-degenerated for each $x \in M_0$.

Property 1 follows just from the assumption of our theorem about the non-degeneracy of the restriction of the pseudo-Riemannian metric g to the orbits in M_0. Property 2 follows from the fact that the almost product structure P is an isomorphism and preserves the pseudo-Riemannian metric with the opposite sign. Let $X \in T_xM$; then $X \in T_xM_0 \Leftrightarrow$

$$0 = d\mu(X) = \omega(\widetilde{A}, X) = g(P(\widetilde{A}), X)$$

for each $A \in \mathfrak{g} \Leftrightarrow X \in W^\perp$; hence property 3 follows. Since W_x has a non-degenerated pseudo-Riemannian scalar product then it follows that we have an orthogonal decomposition $T_xM = W_x \oplus T_xM_0$. Let $A \in \mathfrak{g}$ and $X \in H_x$ then

$$g(\widetilde{A}, P(X)) = -g(P(\widetilde{A}), P^2(X)) = -g(P(\widetilde{A}), X) = 0$$

because $X \in W_x^\perp$ and hence $P(H_x) \subset V_x^\perp$; on the other hand

$$g(P(\widetilde{A}), P(X)) = -g(\widetilde{A}, X) = 0$$

because $X \in V_x^\perp$ and hence $P(H_x) \subset W_x^\perp$. Whole together we get that

$$P(H_x) \subset V_x^\perp \cap W_x^\perp = (V_x + W_x)^\perp = H_x$$

for each $x \in M_0$. Hence property 4 follows. Since g restricted to W_x is non-degenerated then also g restricted to $T_xM_0 = W_x^\perp$ is non-degenerated. Since $V_x \subset T_xM_0$ and g restricted to V_x is non-degenerated then we have an orthogonal decomposition $T_xM_0 = V_x \oplus H_x$ and the restriction of g to H is non-degenerated too. Hence property 5 follows.

It is clear that the bundle $H \to M_0$ carries the induced pseudo-Riemannian scalar product g^H and an almost product structure P^H on each fibre. Moreover, the condition that $g^H(P^H(X), P^H(Y)) = -g^H(X,Y)$ holds for each $X, Y \in H_x$. The Levi-Civita connection ∇^g on M may be pulled back to the bundle $TM|_{M_0} \to M_0$. Since $H \to M_0$ is a vector subbundle, with a non-degenerated pseudo-Riemannian scalar product, of $TM|_{M_0} \to M_0$ then the pull-back connection of ∇^g on $TM|_{M_0} \to M_0$ determines, via the orthogonal projection, the connection ∇^H on $H \to M_0$. Explicitly, we have that $\nabla^H_X Y = \pi^H(\nabla^g_X Y)$ where $X \in \Gamma(TM_0)$, $Y \in \Gamma(H)$ and $\pi^H : TM|_{M_0} \to H$ is the orthogonal projection on each fibre. In the present paper we use the symbol Γ to denote smooth sections of the respective bundles. Then standard calculations show that g^H is parallel with respect to the connection ∇^H. The symplectic form ω restricts to the form ω^H on the fibres of $H \to M_0$. This form is determined by g^H and P^H in the sense that for each $X, Y \in \Gamma(H)$ we have that $\omega^H(X,Y) = g^H(P^H(X), Y)$.

Let $a \in G$ then we denote by $\phi_a : M \to M$ the map such that $\phi_a(x) = \phi(a,x)$. Then we have that $d\phi_a(\widetilde{A}_x) = \widetilde{ad_a(A)}_{\phi_a(x)}$ and then it follows that $d\phi_a(V_x) = V_{\phi(a,x)}$ for all $x \in M_0$. Since g and P are invariant with respect to the action ϕ then it follows that $d\phi_a(W_x) = W_{\phi(a,x)}$ and $d\phi_a(H_x) = H_{\phi(a,x)}$ for all $x \in M_0$. Hence for each $a \in G$ we have the following commuting diagram

$$\begin{array}{ccc} H & \xrightarrow{d\phi_a} & H \\ \downarrow & & \downarrow \\ M_0 & \xrightarrow{\phi_s} & M_0 \end{array} \tag{3.1}$$

Hence each element a of G defines an automorphism (3.1) of the bundle $H \to M_0$. Moreover, from the definition of ∇^H, P^H, g^H and ω^H it is straightforward to prove that they are invariant with respect to such automorphisms, i.e.

$$(\phi_a)_* P^H = P^H, \; (\phi_a)_* g^H = g^H, (\phi_a)_* \omega^H = \omega^H (\phi_a)_* \nabla^H = \nabla^H \tag{3.2}$$

for each $a \in G$. The group G acts freely and properly on M_0. Hence $\overline{M} := \frac{M_0}{G}$ is a smooth manifold and there is given there canonical submersion $p : M_0 \to \overline{M}$. Then we observe that the following diagram of vector bundles

$$\begin{array}{ccc} H & \xrightarrow{dp} & T\overline{M} \\ \downarrow & & \downarrow \\ M_0 & \xrightarrow{p} & \overline{M} \end{array}$$

commutes and the map dp is an isomorphism when restricted to the fibres. Since g^H, P^H, ω^H and ∇^H are invariant with respect to the action of G, cf. (3.2), then it follows that they may be projected to respective structures $\overline{g}$, $\overline{P}$, $\overline{\omega}$ and $\overline{\nabla}$ on $\overline{M}$. From the construction it follows that for each $\overline{X}, \overline{Y} \in \Gamma(T\overline{M})$ we have the compatibility conditions:

$$\overline{g}(\overline{P}(\overline{X}), \overline{P}(\overline{Y})) = -\overline{g}(\overline{X}, \overline{Y}) \quad \text{and} \quad \overline{g}(\overline{P}(\overline{X}), \overline{Y}) = \overline{\omega}(\overline{X}, \overline{Y}). \tag{3.3}$$

Moreover, easy calculation gives that $\overline{\nabla}\overline{g} = 0$. Let X, Y be sections of the bundle $H \to M_0$ which projects on the given vector fields $\overline{X}, \overline{Y}$ on $\overline{M}$. Then we have that

$$\begin{aligned} \overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X} &= dp(\nabla^H_X Y - \nabla^H_X Y) \\ &= dp(\pi^H(\nabla^g_X Y - \nabla^g_X Y)) \\ &= dp([X, Y]) \\ &= [\overline{X}, \overline{Y}]. \end{aligned}$$

Hence $\overline{\nabla}$ is the Levi-Civita connection of $\overline{g}$. In the similar way, may be proved that $\overline{\omega}$ is closed. Hence $(\overline{M}, \overline{g}, \overline{P})$ is an almost para-Hermitian manifold with $d\overline{\omega} = 0$.

If (M, g, P) is para-Kähler then $\nabla^g P = 0$. This implies that $\nabla^H P^H = 0$ and then since ∇^H projects onto the Levi-Civita connection of $\overline{g}$ then it is easy to prove, lifting vector fields from $T\overline{M}$ to horizontal vector fields on $TM|_{M_0} \to M_0$, that $\overline{\nabla}\ \overline{P} = 0$ and then $(\overline{M}, \overline{g}, \overline{P})$ is a para-Kähler manifold. □

Remark 3.1. Since the structures on $\overline{M}$ are given by the pseudo-Riemannian submersion $p : M_0 \to \overline{M}$ then we get formulas for the curvature of $\overline{M}$ via O'Neill formulas, cf. [21, 22]. In fact, for each local $\overline{X}, \overline{Y} \in \Gamma(T\overline{M})$ spanning a non-degenerated 2-plane in $T\overline{M}$ we have that

$$\overline{K}(\mathrm{span}\{\overline{X}, \overline{Y}\}) = K(\mathrm{span}\{X, Y\}) + \frac{3}{4}\frac{g([X, Y]^v, [X, Y]^v)}{\overline{Q}(\overline{X}, \overline{Y})} \tag{3.4}$$

where in equation (3.4): X, Y denote the horizontal liftings of $\overline{X}, \overline{Y}$ to TM_0, $[X, Y]^v$ is the vertical component of $[X, Y]$ in $\Gamma(TM_0)$,

$$\overline{Q}(\overline{X}, \overline{Y}) := \overline{g}(\overline{X}, \overline{X})\overline{g}(\overline{Y}, \overline{Y}) - (\overline{g}(\overline{X}, \overline{Y}))^2,$$

and $\overline{K}(\mathrm{span}\{\overline{X}, \overline{Y}\})$, $K(\mathrm{span}\{X, Y\})$, are respective sectional curvatures on $\overline{M}$ and of M_0.

Remark 3.2. The action ϕ of the Lie group G on M on the left may be easily substituted by an action on the right and the reduction theorem is still valid under analogous assumptions as in Theorem 3.1.

4. Examples

Example 4.1. We continue to consider Example 2.1; $M = \mathbb{R}^m \times \mathbb{R}^m$ for $m > 0$. The manifold M has natural global coordinates $(x_1, \dots, x_m, y_1, \dots, y_m)$. There is given a pseudo-Riemannian metric g on M such that with respect to the canonical coordinates it is of the form

$$g = \sum_{j=1}^{m} (dx_j^2 - dy_j^2)$$

and there is given an almost product structure P such that

$$P = \sum_{j=1}^{m} (\frac{\partial}{\partial y_j} \otimes dx_j + \frac{\partial}{\partial x_j} \otimes dy_j).$$

In other words it means that $P(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}$ and $P(\frac{\partial}{\partial y_i}) = \frac{\partial}{\partial x_i}$ for each $i = 1, 2, \dots, n$. The Levi-Civita connection of M is given by

$$\nabla \frac{\partial}{\partial x_j} = \nabla \frac{\partial}{\partial y_j} = 0, \; \nabla dx_j = \nabla dy_j = 0$$

for each $i, j = 1, 2, \dots, m$. Hence it is clear that $\nabla g = 0$ and $\nabla P = 0$. It means that (M, g, P) is a para-Kähler manifold of dimension $2m$. The symplectic form ω associated with g and P is given by

$$\omega = 2 \sum_{j=1}^{m} dx_j \wedge dy_j.$$

Then we consider the Lie group $G = (\mathbb{R}, +)$ and the action $\phi : G \times M \to M$ such that for each $t \in \mathbb{R}$ and for each $x = (x_1 \dots, x_m)$, $y = (y_1 \dots, y_m)$

$$\phi(t, (x, y)) = \big(x \cosh(t) + y \sinh(t), x \sinh(t) + y \cosh(t)\big). \tag{4.1}$$

Then it is an easy exercise to prove that the action ϕ is isometric on (M, g) and it preserves the almost product structure P. Moreover, for a given $A \in \mathbb{R} \cong Lie(G)$ we have that

$$\widetilde{A} = A \sum_{j=1}^{m} (y_j \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial y_j}).$$

It is clear that $\mathfrak{g}^* \cong \mathbb{R}$. Then we consider a map $\mu : M \to \mathbb{R} \cong \mathfrak{g}^*$ such that

$$\mu(x_1, \dots, x_m, y_1, \dots, y_m) = \frac{1}{2} \sum_{j=1}^{m} (x_j^2 - y_j^2). \tag{4.2}$$

It follows that $d\mu = \sum_{j=1}^{m} (x_j dx_j - y_j dy_j)$ and easy calculations show that condition (2.5) is verified. Since the Lie group G is commutative than $\frac{1}{2} \in \mathbb{R} \cong \mathfrak{g}^*$ is invariant by the coadjoint action of G. Moreover, μ is G-invariant; hence it is a moment map for the action ϕ. We consider

$$M_0 = \mu^{-1}\big(\frac{1}{2}\big) = \big\{(x_1, \dots, x_m, y_1, \dots, y_m) \in M : \sum_{j=1}^{m}(x_j^2 - y_j^2) = 1\big\}.$$

We observe that M_0 is nothing but $\mathbb{S}^{m-1,m}$, i.e. M_0 is a pseudo-Riemannian space form of signature $(m-1, m)$ and of constant sectional curvature equal to 1, called also *pseudo-sphere*. We observe that for each $A \in \mathfrak{g}$ we have that

$$\begin{aligned} g(\widetilde{A}, \widetilde{A}) &= A^2 g\Big(\sum_{j=1}^{m}(y_j \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial y_j}), \sum_{j=1}^{m}(y_j \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial y_j})\Big) \\ &= -A^2. \end{aligned} \tag{4.3}$$

Hence the restriction of the pseudo-Riemannian metric g on M to the orbits of the action of G on M_0 is non-degenerated; actually g restricted to the orbits have the signature -1. It also follows from equation (4.3) that the action of G on M_0 is locally free because the vector field $\widetilde{A}$ vanishes only when $A = 0$. Moreover, the action of G is proper because cosh and sinh are proper maps, in the topological sense. Then we obtain a para-Kähler structure on the manifold $\overline{M} = \frac{\mathbb{S}^{m-1,m}}{G}$; this manifold is of the signature $(m-1, m-1)$. We denote by $\overline{g}$ and $\overline{P}$ the induced structures on $\overline{M}$. This space corresponds naturally to the complex projective space with the Fubini-Study metric which may be obtained from the reduction construction too.

We shall consider the curvature of the manifold $(\overline{M}, \overline{g}, \overline{P})$ constructed in Example 4.1. Let N denote the normal vector field to M_0; N is given by the formula

$$N = \sum_{j=1}^{m}(x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}).$$

Observation 4.1. *We have that* $P(N) = \widetilde{1}$ *where* $1 \in \mathbb{R} \cong \mathfrak{g}$.

Proof. It follows easily from the definition of the action ϕ, cf. (4.1) by differentiating with respect to t and posing $t = 0$. □

Let $\overline{X}, \overline{Y}$ be a local vector fields on $\overline{M}$ and let X, Y be their horizontal liftings to TM_0. We suppose that $\overline{X}, \overline{Y}$ generate at each point of their domain a 2-dimensional non-degenerated vector subspace of $T\overline{M}$. Hence X, Y have these property too in TM_0. Since M_0 is of constant sectional curvature equal to 1 then from (3.4) it follows that

$$\overline{K}(\mathrm{span}\{\overline{X}, \overline{Y}\}) = 1 + \frac{3}{4}\frac{g([X,Y]^v, [X,Y]^v)}{\overline{Q}(\overline{X}, \overline{Y})}. \tag{4.4}$$

We shall calculate more convenient expression for $[X, Y]^v$. Since

$$\nabla_X^{\mathbb{R}^{2m}} N = X, \quad \nabla_Y^{\mathbb{R}^{2m}} N = Y \text{ and } \nabla^{\mathbb{R}^{2m}} P = 0$$

then these imply that

$$\nabla_X^{\mathbb{R}^{2m}}\big(P(N)\big) = P(X) \quad\text{and}\quad \nabla_Y^{\mathbb{R}^{2m}}\big(P(N)\big) = P(Y).$$

Then it follows that

$$\begin{aligned} g([X,Y],P(N)) &= g(\nabla_X^{\mathbb{R}^{2m}}Y - \nabla_Y^{\mathbb{R}^{2m}}X, P(N)) \\ &= -g(Y,\nabla_X^{\mathbb{R}^{2m}}(P(N))) + g(X,\nabla_Y^{\mathbb{R}^{2m}}(P(N))) \\ &= -g(Y,P(X)) + g(X,P(Y)) \\ &= 2g(X,P(Y)). \end{aligned} \tag{4.5}$$

Since $g(P(N),P(N)) = -1$ and since (4.5) holds we have that

$$[X,Y]^v = -2g(X,P(Y))\cdot P(N) = -2\overline{g}(\overline{X},\overline{P}(\overline{Y}))\cdot P(N). \tag{4.6}$$

Hence from (4.4) and (4.6) we get that

$$\overline{K}(\mathrm{span}\{\overline{X},\overline{Y}\}) = 1 - 3\frac{\overline{g}(\overline{X},\overline{P}(\overline{Y}))^2}{\overline{Q}(\overline{X},\overline{Y})}. \tag{4.7}$$

The above formula has some nice corollaries.

Corollary 4.1. *If $m = 2$ then $\overline{M}$ is of dimension 2; let $\overline{X},\overline{Y} \in T\overline{M}$ be a local orthonormal frame in $T\overline{M}$. Then because of the dimension reasons $\overline{P}(\overline{X}) = \pm\overline{Y}$ and $\overline{Q}(\overline{X},\overline{Y}) = -1$. Hence $\overline{K} = \text{constant} = 4$.*

Corollary 4.2. *If $\overline{X},\overline{Y}$ are of the same causality, i.e. $\overline{Q}(\overline{X},\overline{Y}) > 0$, then $\overline{K}(\mathrm{span}\{\overline{X},\overline{Y}\}) \le 1$.*

Corollary 4.3. *If $\overline{X},\overline{Y}$ are of the opposite causality, i.e. $\overline{Q}(\overline{X},\overline{Y}) < 0$, then $\overline{K}(\mathrm{span}\{\overline{X},\overline{Y}\}) \ge 1$.*

Corollary 4.4. *If $\overline{X}$ is causal then $\overline{K}(span\{\overline{X},\overline{P}(\overline{X})\}) = constant = 4$.*

Theorem 4.1. *Suppose that $m \ge 3$, then*

(i) *the sectional curvature in the direction defined by two vectors with the same causality attains each value in the interval $(-\infty, 1]$;*

(ii) *the sectional curvature in the direction defined by two vectors with the opposite causality attains each value in the interval $[1, +\infty)$.*

Proof. Since $\dim\overline{M} \ge 3$ then there exist locally vector fields $\overline{X},\overline{Y}$ such that

$$\overline{g}(\overline{X},\overline{X}) = \overline{g}(\overline{Y},\overline{Y}) = 1 \text{ and } \overline{g}(\overline{X},\overline{Y}) = \overline{g}(\overline{X},\overline{P}(Y)) = 0.$$

Then for each $t \in \mathbb{R}$ the vector fields $\overline{X}, \cosh(t)\overline{Y} + \sinh(t)\overline{P}(\overline{X})$ are orthonormal and have the same causality. Moreover,

$$\overline{K}(\mathrm{span}\{\overline{X}, \cosh(t)\overline{Y} + \sinh(t)\overline{P}(\overline{X})\}) = 1 - 3(\cosh(t))^2$$

and hence (i) follows. In the same spirit we have that

$$\overline{K}(\mathrm{span}\{\overline{X}, \cos(t)\overline{P}(\overline{Y}) + \sin(t)\overline{P}(\overline{X})\}) = 1 + 3(\sin(t))^2 \tag{4.8}$$

$$\overline{K}(\mathrm{span}\{\overline{X}, \sinh(t)\overline{Y} + \cosh(t)\overline{P}(\overline{X})\}) = 1 + 3(\cosh(t))^2. \tag{4.9}$$

Hence (ii) follows from (4.8) and (4.9). □

Remark 4.1. If we consider $l = -\frac{1}{2}$ and the same action of G on $\mu^{-1}(-\frac{1}{2})$ then applying the reduction we obtain the manifold $\frac{\mu^{-1}(-\frac{1}{2})}{G}$. However the manifold $\frac{\mu^{-1}(-\frac{1}{2})}{G}$ with its 'reduced' pseudo-Riemannian metric structure is isometric to the manifold $\frac{\mu^{-1}(\frac{1}{2})}{G}$ where the pseudo-Riemannian metric on the latter manifold is equal to the minus of the metric obtained from the reduction. The isomorphism is given by

$$(x_1, \dots, x_m, y_1, \dots, y_m) \to (y_1, \dots, y_m, x_1, \dots, x_m).$$

Remark 4.2. The use of Lorentz numbers $\mathbb{L}$ gives more elegant description of the moment map given by equation (4.2). In fact, with respect to the isomorphism $\mathbb{L}^m \cong \mathbb{R}^m \times \mathbb{R}^m$ we have that

$$\mu(z_1 \dots, z_m) = \frac{1}{2} \sum_{j=1}^{m} z_j \overline{z}_j.$$

Moreover, the action $\phi : G \times M \to M$ given by formula (4.1) may be rewritten as follows: for each $t \in \mathbb{R}$ and $(z_1, \dots, z_m) \in \mathbb{L}^m \cong M$ we have that

$$\mu(t, (z_1, \dots, z_m)) = \left(e^{t\tau} z_1, \dots, e^{t\tau} z_m\right)$$

where in the formula above we use the exponential function defined on the algebra $\mathbb{L}$, cf. [15].

Remark 4.3. A construction of a para-Kähler projective space was done by some authors in the past. B. Rozenfeld and P. Libermann constructed such a structure using a kind of homogeneous coordinates for manifolds modelled on $\mathbb{P}_n(\mathbb{L})$, cf. [24, 18, 25]. Roughly speaking they used a kind of homogeneous coordinates and used complex-like formulas to get locally an almost para-Hermitian structure. However our construction is similar to the construction done by P. Gadea and Montesinos, cf. [8], where the authors use the algebra $\mathbb{B}$ to construct a para-Kähler Fubini-Study metric.

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