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A Characterization of Isoparametric Hypersurfaces of Clifford Type

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Abstract. Let M be an isoparametric hypersurface with four distinct principal curvatures in the unit sphere $\mathbb{S} \subseteq \mathbb{R}^{2l}$ with focal manifolds M_+ and M_-. Let $\mathcal{U}$ be a vector space of symmetric $(2l \times 2l)$-matrices such that each matrix in $\mathcal{U}\backslash\{0\}$ is regular, and assume that M_+ is the intersection of $\mathbb{S}$ with the quadrics $\{x \in \mathbb{R}^{2l} \mid \langle x, Ax\rangle = 0\}$, $A \in \mathcal{U}$. Then $\mathcal{U}$ is generated by a Clifford system and M is an isoparametric hypersurface of Clifford type provided that $\dim M_+ \geq \dim M_-$. The proof of this theorem is based on properties of quadratic forms vanishing on M_+ and on a structure theorem for isoparametric triple systems, which we prove in this paper.

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1. Introduction

Let M be an isoparametric hypersurface with four distinct principal curvatures in the unit sphere $\mathbb{S}$ of the Euclidean vector space $\mathbb{R}^{2l}$ such that for a suitable Cartan-Münzner polynomial F we have $M = \mathbb{S} \cap F^{-1}(0)$, $M_+ = \mathbb{S} \cap F^{-1}(1)$, and $M_- = \mathbb{S} \cap F^{-1}(-1)$, cf. [6], Theorem 4. If F is defined by means of a Clifford sphere Σ as in [2], 4.1, then the focal manifold M_+ is the intersection of $\mathbb{S}$ with the quadrics $\{x \in \mathbb{R}^{2l} \mid \langle x, Qx\rangle = 0\}$, $Q \in \Sigma$, see [2], 4.2. In this paper we prove a partial converse to this statement: Let $\mathcal{U}$ be a subspace of the vector space $\mathcal{S}_{2l}$ of symmetric $(2l \times 2l)$-matrices with the property that each matrix in $\mathcal{U}\backslash\{0\}$ is regular, and assume that M_+ is the intersection of $\mathbb{S}$ with the quadrics $\{x \in \mathbb{R}^{2l} \mid \langle x, Ax\rangle = 0\}$, $A \in \mathcal{U}$. Then $\mathcal{U}$ is generated by a Clifford sphere and M is an isoparametric hypersurface of Clifford type provided that $\dim M_+ \geq \dim M_-$. The proof of this result is based on a structure theorem for isoparametric triple systems and on properties of quadratic forms vanishing on M_+, see Theorem 2.1 and Proposition 3.1. Note that subspaces $\mathcal{U}$ with the property that each matrix in $\mathcal{U}\backslash\{0\}$ is regular are not rare at all: The n-dimensional subspaces with this property form an open subset in the Grassmannian of n-dimensional subspaces of $\mathcal{S}_{2l}$. In particular, every subspace of dimension $\dim \mathbb{R}\Sigma$ which is sufficiently close to $\mathbb{R}\Sigma$ in the Grassmann topology has this property.

Isoparametric triple systems were introduced by Dorfmeister and Neher in [1] and were investigated in several subsequent papers. Theorem 2.1 is related to the main result of [1]. In [4], we used isoparametric triple systems in our proof that the incidence structures associated with isoparametric hypersurfaces with four distinct prinicipal curvatures in spheres are Tits buildings of type C_2.

In this paper, we will not give a detailed introduction to the theory of isoparametric hypersurfaces and the corresponding isoparametric triple systems. The reader is referred to [6], [2], [1], [7], or to [3], [4], where we treated isoparametric hypersurfaces from an incidence-geometric point of view. We will, however, present parts of these theories in the following sections as far as the topics of the present paper are concerned.

2. Structure theorem for isoparametric triple systems

Let V denote a Euclidean vector space with unit sphere $\mathbb{S}$. An *isoparametric hypersurface* in $\mathbb{S}$ is a compact, connected smooth hypersurface of $\mathbb{S}$ with constant principal curvatures. Each isoparametric hypersurface gives rise to an *isoparametric family* of parallel hypersurfaces. The sphere $\mathbb{S}$ is foliated by these parallel hypersurfaces and the two focal manifolds, see [6], Theorem 4. All these manifolds may be described by means of a Cartan-Münzner polynomial, see [6], Theorem 2. More precisely, there exists a homogeneous polynomial function $F : V \to \mathbb{R}$ such that the family of isoparametric hypersurfaces is given by $\mathbb{S}\cap F^{-1}(\rho)$, $-1 < \rho < 1$, and the two focal manifolds are $M_+ = \mathbb{S}\cap F^{-1}(1)$ and $M_- = \mathbb{S}\cap F^{-1}(-1)$. We set $M = \mathbb{S}\cap F^{-1}(0)$. Every isoparametric family has the geometric property that a great circle S which intersects M orthogonally at one point intersects M and the two focal manifolds orthogonally at each intersection point. Moreover, the points of $S\cap M_+$ and $S\cap M_-$ follow on S alternatingly at spherical distance π/g, where g denotes the number of distinct principal curvatures of M, see [6], Section 6, cf. also [5], Proposition 3.2. We say that such a great circle S is normal to M.

For $g = 4$, there is a triple product $\{\cdot,\cdot,\cdot\}$ on V associated with the Cartan-Münzner polynomial F. In this way, $(V, \langle\cdot,\cdot\rangle, \{\cdot,\cdot,\cdot\})$ becomes an isoparametric triple system, see [1]. The focal manifolds are given by $M_+ = \{x \in \mathbb{S} \mid \{x,x,x\} = 3x\}$ and $M_- = \{y \in \mathbb{S} \mid \{y,y,y\} = 6y\}$. For $x, y \in V$ and $\lambda \in \mathbb{R}$ we put $T(x,y) : V \to V : z \mapsto \{x,y,z\}$, $T(x) = T(x,x)$, and $V_\lambda(x) = \{z \in V \mid T(x)(z) = \lambda z, \langle x,z\rangle = 0\}$. If $x \in M_+$ and $y \in M_-$ then we have $V = \mathrm{span}\{x\} \oplus V_3(x) \oplus V_1(x) = \mathrm{span}\{y\} \oplus V_0(y) \oplus V_2(y)$ (Peirce decompositions). The dimensions of the Peirce spaces $V_3(x)$, $V_1(x)$, $V_0(y)$, and $V_2(y)$ are given by $m_1 + 1$, $m_1 + 2m_2$, $m_2 + 1$, and $2m_1 + m_2$, respectively, where m_1 and m_2 denote the multiplicities of the principal curvatures of M, see [1], Theorem 2.2. Furthermore, the following identities hold ($u_0, v_0, w_0 \in V_0(y)$, $u_3, v_3, w_3 \in V_3(x)$):

(1) $\{u_0, y, v_0\} = 0,$
(2) $\{u_0, v_0, w_0\} = 2(\langle u_0, v_0\rangle w_0 + \langle v_0, w_0\rangle u_0 + \langle w_0, u_0\rangle v_0),$
(3) $\{u_3, x, v_3\} = 3\langle u_3, v_3\rangle x,$
(4) $\{u_3, v_3, w_3\} = \langle u_3, v_3\rangle w_3 + \langle v_3, w_3\rangle u_3 + \langle w_3, u_3\rangle v_3.$

These identities correspond to equations 2.3, 2.6, 2.10, and 2.13 in [1]. In particular, we have $\mathbb{S} \cap V_3(x) \subseteq M_+$ and $\mathbb{S} \cap V_0(y) \subseteq M_-$ by identities (2) and (4). The points of M_+ (of M_-) with spherical distance $\pi/4$ from y (from x) are precisely the points in $\mathbb{S} \cap ((1/\sqrt{2})y + V_0(y))$ (in $\mathbb{S} \cap ((1/\sqrt{2})x + V_3(x))$, respectively), cf. [4], Section 2 and 3.1. Hence, on every great circle through $x \in M_+$ and a point $z_3 \in \mathbb{S} \cap V_3(x) \subseteq M_+$ there is a point $(1/\sqrt{2})(x + z_3) \in M_-$ and, in fact, such a great circle is normal to M, cf. [6], Section 6, and [4], Corollary 3.3. Analogously, every great circle through $y \in M_-$ and a point of $\mathbb{S} \cap V_0(y)$ is normal to M. We are now ready to prove the following structure theorem:

2.1 Theorem. *Let $(V, \langle \cdot, \cdot \rangle, \{\cdot, \cdot, \cdot\})$ be an isoparametric triple system. Let S be a great circle of $\mathbb{S}$ normal to M which intersects M_+ at the four points $\pm p, \pm q$ and M_- at the four points $\pm r, \pm s$. Then V decomposes as an orthogonal sum*

$$V = \operatorname{span}\{p, q, r, s\} \oplus V_3'(p) \oplus V_3'(q) \oplus V_0'(r) \oplus V_0'(s),$$

where $V_3'(p)$, $V_3'(q)$, $V_0'(r)$, $V_0'(s)$ are defined by $V_3(p) = V_3'(p) \oplus \operatorname{span}\{q\}$, $V_3(q) = V_3'(q) \oplus \operatorname{span}\{p\}$, $V_0(r) = V_0'(r) \oplus \operatorname{span}\{s\}$, and $V_0(s) = V_0'(s) \oplus \operatorname{span}\{r\}$.

Proof. The points of M_+ and M_- follow on S alternatingly at spherical distance $\pi/4$. Hence we have $\langle p, q\rangle = \langle r, s\rangle = 0$, and without loss of generality we may assume that $r = (1/\sqrt{2})(p + q)$ and $q = (1/\sqrt{2})(r + s)$.

First we want to justify the definition of $V_3'(p)$ by $V_3(p) = V_3'(p) \oplus \operatorname{span}\{q\}$. For this purpose we have to show that $q \in V_3(p)$. We have $\langle r, p\rangle = 1/\sqrt{2}$ and hence $r \in \mathbb{S} \cap \big((1/\sqrt{2})p + V_3(p)\big)$, as mentioned above. This implies that $q \in V_3(p)$ because of $r = (1/\sqrt{2})(p + q)$. Analogously we see that the definitions of $V_3'(q)$, $V_0'(r)$, and $V_0'(s)$ make sense.

Next we consider the action of the operators $T(p, q)$ and $T(r, s)$ on these subspaces of V. For $x \in V_0'(s)$ we have $T(r, s)(x) = \{r, s, x\} = 0$ by identity (1). In the same way we see that $T(r, s)$ maps $V_0'(r)$ to $\{0\}$. Now let $y \in V_3'(q)$. As remarked above, we have $r = (1/\sqrt{2})q + r_3$ and $s = (1/\sqrt{2})q + s_3$ with $r_3, s_3 \in V_3(q)$. Note that $\langle r_3, y\rangle = \langle s_3, y\rangle = 0$ and $\langle r_3, s_3\rangle = -1/2$. Then the identities (3) and (4) imply that

$$T(r, s)(y) = \{(1/\sqrt{2})q + r_3, y, (1/\sqrt{2})q + s_3\} = (3/2)y - (1/2)y = y,$$

i.e. $T(r, s)$ acts on $V_3'(q)$ as the identity. Analogously, $T(r, s)$ acts on $V_3'(p)$ as $-\mathrm{id}$. Hence we have proved that $T(r, s)|_{V_0'(r)} = T(r, s)|_{V_0'(s)} = 0$, $T(r, s)|_{V_3'(q)} = \mathrm{id}$, and $T(r, s)|_{V_3'(p)} = -\mathrm{id}$.

Using the above identities, it can be shown in the same way that $T(p, q)|_{V_3'(p)} = T(p, q)|_{V_3'(q)} = 0$, $T(p, q)|_{V_0'(r)} = -\mathrm{id}$, and $T(p, q)|_{V_0'(s)} = \mathrm{id}$. Since the operators $T(p, q)$ and $T(r, s)$ are self-adjoint, we get $\operatorname{span}\{p, q, r, s\} \oplus V_3'(p) \oplus V_3'(q) \oplus V_0'(r) \oplus V_0'(s) \leq V$. The claim follows because of $\dim V = 2 + 2m_1 + 2m_2$. □

3. Quadratic forms vanishing on a focal manifold

Every quadratic form on the Euclidean vector space $\mathbb{R}^{2l}$ may be described by a uniquely determined symmetric matrix. The subspace $\mathcal{A}(M_+) = \{A \in \mathcal{S}_{2l} \mid \langle x, Ax\rangle = 0$ for every $x \in M_+\}$ of $\mathcal{S}_{2l}$ corresponds to the vector space of quadratic forms vanishing on M_+. This subspace may be of interest in the classification of isoparametric hypersurfaces with four distinct principal curvatures in spheres. The following proposition explains the structure of $\mathcal{A}(M_+)$.

3.1 Proposition. *Let M_+ and M_- be the focal manifolds of an isoparametric hypersurface with four distinct principal curvatures in the unit sphere $\mathbb{S} \subseteq \mathbb{R}^{2l}$ and let $\mathcal{A}(M_+) = \{A \in \mathcal{S}_{2l} \mid \langle x, Ax\rangle = 0$ for every $x \in M_+\}$. Then the following two statements hold:*

(i) *For every $A, B \in \mathcal{A}(M_+)$ we have $ABA \in \mathcal{A}(M_+)$.*
(ii) *For every $A \in \mathcal{A}(M_+)$ we have $A = \sum_{i=1}^{r} \lambda_i (Q_{\lambda_i} - Q_{-\lambda_i})$ with $Q_{\lambda_i} - Q_{-\lambda_i} \in \mathcal{A}(M_+)$, where the $\pm\lambda_i$ denote the non-zero eigenvalues of A and the $Q_{\pm\lambda_i}$ denote the orthogonal projections onto the eigenspaces of these eigenvalues. Moreover, the eigenvalues λ_i and λ_{-i} have the same multiplicity $\mu_i \geq m_2 + 1$, and their eigenspaces are contained in $\mathbb{R}M_-$.*

Proof. Let $A, B \in \mathcal{A}(M_+)$ and $x \in M_+$. The quadratic form $b_1 : \mathbb{R}^{2l} \to \mathbb{R}^{2l} : z \mapsto \langle z, Az\rangle$ vanishes on M_+. The differential $\mathrm{D}(b_1)_x$ vanishes on the tangent space $\mathrm{T}_x M_+$, i.e. we have $\langle Ax, u\rangle = 0$ for every $u \in T_x M_+$. By [4], 3.3, we have $T_x M_+ = V_1(x)$. We conclude that $Ax \in V_3(x)$. Hence we get $(1/\|Ax\|)Ax \in M_+$ provided that $Ax \neq 0$. In any case we have $\langle Ax, B(Ax)\rangle = 0$. Since $x \in M_+$ was chosen arbitrarily, we get $\langle y, (ABA)y\rangle = 0$ for every $y \in M_+$. This proves (i).

Let $A^{(1)} \in \mathcal{A}(M_+)\backslash\{0\}$. For each $\lambda \in \mathbb{R}$ we denote by Q_λ the orthogonal projection onto the subspace $E_\lambda = \{x \in \mathbb{R}^{2l} \mid A^{(1)}x = \lambda x\}$. The distinct absolute values of non-zero eigenvalues of $A^{(1)}$ are denoted by $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\lambda_1 > \lambda_2 > \ldots > \lambda_r$. Then we have $A^{(1)} = \sum_{i=1}^{r} \lambda_i (Q_{\lambda_i} - Q_{-\lambda_i})$. By (i), the sequence $((A^{(1)}/\lambda_1)^{2n+1})_{n\in\mathbb{N}}$ is contained in $\mathcal{A}(M_+)$. Hence we get $\lim_{n\to\infty}(A^{(1)}/\lambda_1)^{2n+1} = Q_{\lambda_1} - Q_{-\lambda_1} \in \mathcal{A}(M_+)$. We set $A^{(2)} = A^{(1)} - \lambda_1(Q_{\lambda_1} - Q_{-\lambda_1}) \in \mathcal{A}(M_+)$. Then we get $\lim_{n\to\infty}(A^{(2)}/\lambda_2)^{2n+1} = Q_{\lambda_2} - Q_{-\lambda_2} \in \mathcal{A}(M_+)$. By proceeding in this way, we obtain $Q_{\lambda_i} - Q_{-\lambda_i} \in \mathcal{A}(M_+)$ for $i = 1, \ldots, r$. Choose $i \in \{1, \ldots, r\}$ arbitrarily. In order to complete the proof of (ii) it suffices to show that λ_i and $-\lambda_i$ are both eigenvalues of $A^{(1)}$ with the same multiplicity $\mu_i \geq m_2 + 1$ and that their eigenspaces are contained in $\mathbb{R}M_-$. Let x_i denote an eigenvector of $A^{(1)}$ to an eigenvalue with absolute value λ_i with $\|x_i\| = 1$. If this eigenvalue is positive, we put $\varepsilon = 1$, otherwise $\varepsilon = -1$. Note that the quadratic form $b_2 : \mathbb{R}^{2l} \to \mathbb{R} : y \mapsto \langle y, (Q_{\lambda_i} - Q_{-\lambda_i})y\rangle$ takes its maximum 1 (minimum -1) on $\mathbb{S}$ at x_i for $\varepsilon = 1$ ($\varepsilon = -1$). Let S denote a great circle of $\mathbb{S}$ through x_i normal to M. By Theorem 2.1, we have for $V = \mathbb{R}^{2l}$

$$V = \mathrm{span}\{p, q, r, s\} \oplus V_3'(p) \oplus V_3'(q) \oplus V_0'(r) \oplus V_0'(s),$$

where $\pm p, \pm q$ ($\pm r, \pm s$) are the four intersection points of M_+ (M_-) with S. The map $b_2|_{\mathrm{span}\{p,q,r,s\}}$ is a quadratic form on the two-dimensional vector space $\mathrm{span}\{p, q, r, s\}$. It vanishes at $\pm p, \pm q$ and hence takes its maximum 1 on S at $\pm r$ and its minimum -1 at $\pm s$, or vice versa. Hence we get $x_i \in S \cap M_-$. Without loss of generality we may assume that $x_i = r$. Then we have $b_2(r) = \varepsilon$ and $b_2(s) = -\varepsilon$. This implies that $(Q_{\lambda_i} - Q_{-\lambda_i})(r) = \varepsilon r$ and $(Q_{\lambda_i} - Q_{-\lambda_i})(s) = -\varepsilon s$. Now choose $s' \in V_0(r) \cap \mathbb{S}$ arbitrarily. Then the great circle of $\mathbb{S}$ through r and s' is normal to M and we see as before that $(Q_{\lambda_i} - Q_{-\lambda_i})(s') = -\varepsilon s'$. Hence, we obtain $V_0(r) \subseteq E_{(-\varepsilon)\lambda}$. Analogously, we get $V_0(s) \subseteq E_{\varepsilon\lambda}$. We choose orthonormal bases of $V_3'(p)$, $V_3'(q)$, $V_0(r)$, and $V_0(s)$. In this way we get an orthonormal base of V, and we describe the quadratic form b_2 by a matrix with respect to this base. The trace of this matrix is equal to 0 since $b_2(x) = b_2(y) = 0$ (because of $V_0(r), V_0(s) \subseteq \mathbb{R}M_+$), $b_2(z) = -\varepsilon\langle z, z\rangle$, and $b_2(w) = \varepsilon\langle w, w\rangle$ for $x \in V_3'(p)$, $y \in V_3'(q)$, $z \in V_0(r)$, $w \in V_0(s)$, where $\dim V_0(r) = \dim V_0(s) = m_2 + 1$. We conclude that also the trace of the matrix $Q_{\lambda_i} - Q_{-\lambda_i}$ is equal to 0. Hence, λ_i and $-\lambda_i$ are eigenvalues of $A^{(1)}$ with the same multiplicity $\mu_i \geq m_2 + 1$. This completes the proof. □

4. Main theorem

In this section we apply the results of Proposition 3.1 in order to prove a characterization of isoparametric hypersurfaces of Clifford type under the assumption that $\dim M_+ \geq \dim M_-$. This assumption is equivalent to $m_1 \leq m_2$ because of $\dim M_+ = m_1 + 2m_2$ and $\dim M_- = m_2 + 2m_1$, see [6], proof of Theorem 4, cf. [4], 3.1. Note that this condition is satisfied for all isoparametric hypersurfaces of Clifford type except for a small number of exceptions, see [2], 4.3 and 7.

4.1 Theorem. *Let M be an isoparametric hypersurface in $\mathbb{S} \subseteq \mathbb{R}^{2l}$ with four distinct principal curvatures and assume that the two focal manifolds M_+ and M_- satisfy $\dim M_+ \geq \dim M_-$. Let $\mathcal{U}$ denote a subspace of $\mathcal{S}_{2l}$ such that $\mathcal{U}\backslash\{0\}$ consists of regular matrices. Then the following two statements are equivalent.*

(i) $M_+ = \{x \in \mathbb{S} \mid \langle x, Ax\rangle = 0$ *for every* $A \in \mathcal{U}\}$.
(ii) *The subspace $\mathcal{U}$ is generated by a Clifford system, and M is an isoparametric hypersurface of Clifford type associated with this Clifford system.*

Proof. We have already mentioned that (ii) implies (i), see [2], 4.2 (ii). In order to prove (i) ⇒ (ii), choose $A \in \mathcal{U}\backslash\{0\}$ arbitrarily. Assume that A has at least two eigenvalues with different absolute values. Then, by Proposition 3.1, the matrix A has at least four distinct eigenvalues, each of which has at least multiplicity $m_2 + 1$. We conclude that $4(m_2 + 1) \leq 2l = 2(m_1 + m_2 + 1)$. This is equivalent to $m_2 + 1 \leq m_1$ and contradicts $m_1 \leq m_2$. Hence, all eigenvalues of A have the same absolute value and we obtain $A^2 = \langle A, A\rangle\mathrm{id}$ with respect to the scalar product $\langle B, C\rangle = (1/2l)\operatorname{trace}(BC)$ on $\mathcal{S}_{2l}$. Since M_+ is a submanifold of codimension at least 2 in $\mathbb{S}$, the subspace $\mathcal{U}$ is at least two-dimensional. We denote by $\{P_0, P_1, \ldots, P_m\}$ an orthonormal base of $\mathcal{U}$. Then we have $P_i^2 = \mathrm{id}$ for $i = 0, \ldots, m$. Furthermore, for $i, j \in \{0, \ldots, m\}$ with $i \neq j$ we have $\|(1/\sqrt{2})(P_i + P_j)\| = 1$, hence $(1/2)(P_i - P_j)^2 = \mathrm{id}$, which implies that $P_iP_j + P_jP_i = 0$. Therefore $(P_0, P_1, \ldots, P_m)$ is a Clifford system. By (i), we have $M_+ = \{x \in \mathbb{S} \mid \langle x, P_ix\rangle = 0 \text{ for } i = 0, \ldots, m\}$. Since families of isoparametric hypersurfaces are uniquely determined by one of their focal manifolds (cf. [5], 3.2, or [6], Section 6), we conclude that M is an isoparametric hypersurface of Clifford type associated with the Cartan-Münzner polynomial $F : \mathbb{R}^{2l} \to \mathbb{R} : x \mapsto \langle x, x\rangle^2 - 2\sum_{i=0}^{m}\langle P_i x, x\rangle^2$, cf. [2], 4.1, 4.2 (ii). This completes the proof. □

References

[1] Dorfmeister, J.; Neher, E.: *An algebraic approach to isoparametric hypersurfaces in spheres I*. Tôhoko Math. J. **35** (1983), 187–224. Zbl 0507.53038

[2] Ferus, D.; Karcher, H.; Münzner, H.-F.: *Cliffordalgebren und neue isoparametrische Hyperflächen*. Math. Z. **177** (1981), 479–502. Zbl 0443.53037

[3] Immervoll, St.: *Smooth generalized quadrangles and isoparametric hypersurfaces of Clifford type*. Forum Math. **14** (2002), 877–899. Zbl 1015.51003

[4] Immervoll, St.: *Isoparametric hypersurfaces and smooth generalized quadrangles*. J. Reine Angew. Math. **554** (2003), 1–17. Zbl pre01868664

[5] Knarr, N.; Kramer, L.: *Projective planes and isoparametric hypersurfaces*. Geom. Dedicata **58** (1995), 192–202. Zbl 0839.53044

[6] Münzner, H. F.: *Isoparametrische Hyperflächen in Sphären I*. Math. Ann. **251** (1980), 57–71. Zbl 0417.53030

[7] Thorbergsson, G.: *A survey on isoparametric hypersurfaces and their generalizations*. In: Dillen, F. J. E., Verstraelen, L. C. A.: Handbook of differential geometry, 963–995. Elsevier Science B. V., 2000 Zbl 0979.53002

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