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How Round is a Circle? Constructions of Double and Circular Planar Near-Rings

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Abstract. A construction procedure for planar near-rings is provided. Amongst others, this method will show why many planar near-rings come in pairs with "opposite" geometries and it will also lead to the construction of many new infinite circular planar near-rings.

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Since its inception, planar near-rings have played an important role on two fronts: in the algebra for their geometric interpretations and outside algebra for their practical applications. Here we will look at the construction of planar near-rings and circular planar near-rings. The construction method is new and will recover many old results. But, in particular, this approach will emphasize the dependence of the planar near-ring multiplications on an existing multiplication from a richer algebraic structure; it will provide many more examples of planar near-rings and explain why planar near-rings often come in pairs with opposite associated geometries; it will show that the two best known planar near-rings in the Euclidean plane have the same source; it will give many more examples of infinite circular planar near-rings and it will confirm the necessity of the language of near-rings in many geometrical considerations.

In a planar near-ring two types of substructures play an important role - either for their practical applications (eg in coding theory or the design of statistical experiments) or for their associated geometry. The first is the set of right identities and the second is a basic block, one for each right identity. These basic blocks form a partition of that part of a planar near-ring where all the action takes place. The class of all concrete planar near-rings

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which have some interesting geometric interpretations, can be divided in two disjoint classes: there are those planar near-rings in which the set of right identities represents the points on some interesting geometric figure (circle, ellipse, hyperbola, parabola, etc) and the blocks are rather straightforward, being just the points on a straight line or a ray. On the other hand there are those planar near-rings where the associated geometry is completely the opposite – the right identities are the points on a straight line or ray and the basic blocks are the ones with the geometrically interesting shapes. As is often said, "opposites attract" and with surprising frequency two of these planar near-rings, one from each group, can be joined in a happy union to form a so called double planar near-ring.

The genetic code of planar near-rings has been known for many years – it is just the Ferrero pairs. Thus this phenomenon can be described genetically, but for the present purposes, a different approach will be more instructive. For near-rings notions not defined here, the reader may consult Pilz [7] or Clay [3].

Many near-ring multiplications on a group are of the form

$$a * b = a\delta(b)$$

where $\delta(b)$ is some "distortion" of b and the product on the right is some existing product – typically a ring or scalar product. We also know that any near-ring product $*$ on a group $(V,+)$ can be described by a mapping $\phi : V \longrightarrow Hom(V,V)$ where $a * b = \phi_b(a)$; here $\phi(b) = \phi_b : V \longrightarrow V$. For example, let V be a vector space over a field F and let $\delta : V \longrightarrow F$ be any F-*homogeneous* function. This means $\delta(vf) = \delta(v)f$ for all $v \in V$ and $f \in F$. (As is usual in the near-ring community, one has to state your alliance. Here we will side with the right: scalars are written on the right and near-rings will be right near-rings.)

Let $\psi : F \longrightarrow Hom(V,V)$ denote the action of F on V, i.e., $\psi(f) = \psi_f : V \to V$, $\psi_f(v) = vf$. This gives a mapping $\phi := \psi \circ \delta : V \longrightarrow Hom(V,V)$ and consequently a near-ring multiplication $a * b = \phi_b(a) = a\delta(b)$. If $|\delta(V)| \geqslant 3$, then $(V,+,*)$ is always a planar near-ring. It is this commutative diagram

$$\begin{tikzcd} V \arrow[rr, "\phi"] \arrow[dr, "\delta"'] & & Hom(V,V) \\ & F \arrow[ur, "\psi"'] & \end{tikzcd}$$

which is the starting point of our considerations. The basic ingredients of our construction are: A group $(V,+)$, a semigroup $(S,\cdot)$, a subset T of V and a mapping $\delta : T \longrightarrow S$. We suppose S acts on V from the right i.e. we have a mapping $V \times S \longrightarrow V, (v,s) = vs$, which is compatible with respect to all the operations: $(v_1+v_2)s = v_1s+v_2s$ and $v(s_1s_2) = (vs_1)s_2$. By $\psi : S \longrightarrow Hom(V,V)$ we denote this action, i.e. $\psi(s) = \psi_s : V \longrightarrow V$ with $\psi_s(v) = vs$. Suppose $TS \subseteq T$ and δ is an S-*homogeneous* function, i.e. $\delta(ts) = \delta(t)s$ for all $t \in T$, $s \in S$. We then have

$$T \xrightarrow{\delta} S \xrightarrow{\psi} Hom(V,V)$$

and we extend $\psi \circ \delta : T \longrightarrow Hom(V,V)$ in the simplest possible way to a mapping on V by

$$\phi(v) = \begin{cases} (\psi \circ \delta)(v) \text{ if } v \in T \\ 0 \text{ if } v \notin T. \end{cases}$$

Then we define

$$w \odot v := \phi_v(w) = \begin{cases} w\delta(v) \text{ if } v \in T \\ 0 \text{ otherwise.} \end{cases}$$

Quite often $(V, +, \cdot)$ is a ring and $(S, \cdot)$ is a group. We can then define a second operation $*$ on V by

$$w * v := \begin{cases} w(v\delta(v)^{-1}) \text{ if } v \in T \\ 0 \text{ otherwise.} \end{cases}$$

This operation is well-defined since $\delta(v) \in S$ for $v \in T$ and $(S, \cdot)$ is a group; hence $\delta(v)^{-1} \in S$ and $w(v\delta(v)^{-1}) \in V(TS) \subseteq VT \subseteq V$. In general, these operations need not give rise to a planar near-ring, or even a near-ring. Thus the first four results can correctly be predicted: the requirements on the quadruple (V, S, T, δ) such that $V_1 := (V, +, \odot)$ and $V_2 := (V, +, *)$ are near-rings and planar near-rings.

Proposition 1. *Let $(V, +)$ be a group, $(S, \cdot)$ a semigroup which acts on V from the right, $T \subseteq V$ with $TS \subseteq T$ and $\delta : T \longrightarrow S$ an S-homogeneous function. We suppose for $v \in V$ and $s \in S$, that $vs \in T$ implies $v \in T$. Then $V_1 = (V, +, \odot)$ is a near-ring.*

Proof. To verify the right distributivity is straightforward. The associativity follows from $vs \in T$ if and only if $v \in T$ for any $v \in V, s \in S$. □

This result is as general as it can be, in the sense that any near-ring $(V, +, \cdot)$ can be realized in this way: Take $S = (V, \cdot)$, $T = V$ and δ the identity map. Then $a \odot b = ab$ for all $a, b \in V$. But one is interested in more interesting choices for S and T. We shall see later that all the field, ring and near-field generated planar near-rings are of this type. Now we only give one other example. Let X be a non-empty set, G a group and $\alpha : G \longrightarrow X$ a fixed mapping. Then $V = M(X, G)$ is a group with respect to function addition and $S = M(X)$ is a semigroup with respect to composition. S acts on V from the right via composition. Let $T = V$ and define $\delta : T \longrightarrow S$ by $\delta(t) := \alpha \circ t$. Then δ is an S-homogeneous mapping and the corresponding near-ring $(V, +, \odot)$ with $v \odot w = v\delta(w) = v \circ \alpha \circ w$ is just the sandwich near-ring $M(X, G, \alpha)$.

Let us recall the definition of a planar near-ring. In any near-ring N there is an equivalence operation $=_m$ defined by : $a =_m b$ if and only if $na = nb$ for all $n \in N$. In this case, a and b are said to be *equivalent multipliers.*

A near-ring N is a *planar near-ring* if $|N/=_m| \geq 3$ and for all $a, b, c \in N$ with $a \neq_m b$, the equation $xa = xb + c$ has a unique solution in N. In a planar near-ring, the following subsets play an important role:

- the equivalence class of 0, denoted by $[0]$; in fact, its importance is really because of what is outside of it, namely $N^{\#} := N \setminus [0]$.
- $\mathcal{R}$ which denotes the set of all right identities. It can be shown that $\mathcal{R} = \{1_a | a \in N^{\#}\}$ where 1_a is the unique element of N for which $1_a a = a$ $(a \in N^{\#})$.

- the basic blocks $aN^{\#}$, $a \neq 0$. When $a \in N^{\#}$, a basic block is often denoted by $\mathcal{B}_a = aN^{\#}$. The notion of a basic block is not standard, often it is used to denote other subsets of N, eg $\{a, -a\}N^{\#}$ (a any nonzero element of N) – if we want to do this, it will be stated explicitly.

We will say the semigroup S *acts faithfully* on V if $vs_1 = vs_2$ for all $v \in V$ implies $s_1 = s_2$ ($s_i \in S$) and $vs = 0$ ($v \in V, s \in S$) implies $v = 0$.

Proposition 2. *Let* $(V, +)$ *and* $(S, \cdot)$ *be non-trivial groups such that* S *acts faithfully on* V *from the right. Let* $T \subseteq V^* := V\backslash\{0\}$ *with* $TS \subseteq T$ *and let* $\delta : T \longrightarrow S$ *be an* S*-homogeneous function. We suppose:*

(i) $|\delta(T)| \geq 2$ *and*

(ii) *for all* $v \in V$ *and* $t \in T$ *with* $\delta(t) \neq 1_s$, *there is a unique* $x \in V$ *with* $x = x\delta(t) + v$.

Then $V_1 = (V, +, \odot)$ *is a planar near-ring.*

Proof. Since S acts faithfully on V, it follows from $(v1_s - v)s = 0$ that $v1_s = v$. Furthermore, $vs \in T$ ($v \in V, s \in S$) implies $v \in T$ (since S is a group). Proposition 1 is thus applicable and we only have to verify the planarity. It can be verified that $a =_m b$ if and only if (1) $a, b \in T$ and $\delta(a) = \delta(b)$ or (2) $a \notin T$ and $b \notin T$. Since $|\delta\,(T)| \geq 2$, there are $t_1, t_2 \in T$ with $\delta(t_1) \neq \delta(t_2)$ and since $0 \notin T, |V/=_m| \geq 3$. Let $a, b, c, \in V$ with $a \neq_m b$. We have to find a unique solution for the equation $x \odot a = x \odot b + c$. For this we consider the three cases:
If $a \in T$ and $b \notin T$, then $x = c\delta(a)^{-1}$.
If $a \notin T$ and $b \in T$, then $x = (-c)\delta(b)^{-1}$.
If $a \in T$ and $b \in T$, then $\delta(a) \neq \delta(b)$. Hence $\delta(b\delta(a)^{-1}) \neq 1_s$ and by assumption the equation $x = x\delta(b\delta(a)^{-1}) + c\delta(a)^{-1}$ has a unique solution. □

For this planar near-ring $V_1 := (V, +, \odot)$, we have $[0] = V\backslash T$ and $V_1^{\#} = T$. Note that in general $V^{\#}$ and V^* need not coincide. For any $t \in T$, $1_t = t\delta(t)^{-1}$ and the set of right identities $\mathcal{R} = \{t \in T \mid \delta(t) = 1_s\}$. The basic block $\mathcal{B}_t$ determined by $t \in T$, is given by

$$\mathcal{B}_t = \{b \in T \mid b\delta(b)^{-1} = 1_t = t\delta(t)^{-1}\}.$$

We now consider various applications of this result.

Example 3. Many of the examples below are known (cf [1], [2] or [3]) and can be accommodated by a general procedure described by Clay (see example 3.3 below). They are reproduced here to emphasize the point that the construction method of Proposition 2 usually produces planar near-rings with interesting sets of right identities.

3.1. All the field, near-field and ring generated planar near-rings are given by the procedure described in Proposition 2. For example, let $(V, +, \cdot)$ be a ring with identity and with group of units $\mathcal{U}(V)$. Then $S = \mathcal{U}(V)$ acts faithfully on V from the right (via the ring multiplication). Let $T \subseteq \mathcal{U}(V)$ with $TS \subseteq T$, $|T| \geq 2$ and such that if $t \in T\backslash\{1\}$, then $1 - t \in \mathcal{U}(V)$. Define $\delta : T \longrightarrow S$ by $\delta(t) = t$. Then $(V, +, \odot)$ is a planar near-ring.

3.2. A well-known and good source of planar near-rings is given by: Let V be a vectorspace over a field F. Any function $f : V \to F$ which satisfies $f(vf(w)) = f(v)f(w)$ for all $v, w \in V$

and $|f(V)| \geq 3$, gives rise to a planar near-ring with multiplication $vw = vf(w)$. This procedure can be described more generally in the context of Proposition 2 and this example will exploit this idea. Let $V = (\mathbb{R}, \mathbb{R}, \ldots, \mathbb{R})$ be the direct sum of $n \geq 1$ copies of the additive group of reals $(\mathbb{R}, +)$. Then V is a vector space over $\mathbb{R}$. In all the subexamples below, we will take S a subgroup of $(\mathbb{R}^*, \cdot)$, $T \subseteq V^*$ and $\delta : T \longrightarrow S$ a function. S acts faithfully on V from the right (via the scalar multiplication). In view of Proposition 2, we only have to verify that $TS \subseteq T$, that δ is S-homogeneous and that $|\delta(T)| \geq 2$ in order to get a planar near-ring $(V, +, \odot)$ for the various choices of S, T and δ. In each of these planar near-rings, $V^{\#} = T$ and the right identities are given by $\mathcal{R} = \{(x_1, x_2, \ldots, x_n) \in T \mid \delta(x_1, x_2, \ldots, x_n) = 1\}$. For $a = (a_1, a_2, \ldots, a_n) \in T$, at least one $a_i \neq 0$, say $a_{i_0} \neq 0$. Then $\mathcal{B}_a = \{(x_1, x_2, \ldots .x_n) \in T \mid x_i = \dfrac{a_i}{a_{i_o}} x_{i_o}$ for all $i = 1, 2, \ldots, n$ and $a_{i_o} x_{i_o} \in S\}$. This means, for example, if $S = (\mathbb{R}^+, \cdot)$ that $\mathcal{B}_a$, is the "ray" in the n-dimensional space $\mathbb{R}^n$, starting at the origin but not including the origin, and through the point $(a_1, a_2, \ldots, a_n)$. In any case, depending on the choice for S, we see that the elements of $\mathcal{B}_a$ can at most be the points on a straight line with the origin excluded and that $\mathcal{B}_a$ is independent of δ. We thus only describe the right identities and, for the obvious geometric advantages, mostly for the cases $n = 2$ or $n = 3$.

3.2.1. Let p be a positive odd integer and choose $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ fixed such that $T := \{a = (a_1, a_2, \ldots, a_n) \in V \mid \sum_{i=1}^{n} \alpha_i a_i^p \neq 0\}$ has at least two distinct elements a and b for which $\sum_{i=1}^{n} \alpha_i a_i^p \neq \sum_{i=1}^{n} \alpha_i b_i^p$. Let $S = (\mathbb{R}^*, \cdot)$ and define $\delta : T \longrightarrow S$ by $\delta(a_1, a_2, \ldots, a_n) = \left(\sum_{i=1}^{n} \alpha_i a_i^p\right)^{\frac{1}{p}}$. Then $(V, +, \odot)$ is a planar near-ring with right identities $\mathcal{R} = \{(x_1, x_2, \ldots, x_n) \in T \mid \sum_{i=1}^{n} \alpha_i x_i^p = 1\}$. There are, of course, many interesting geometric figures associated with this set for various p, n and α_i. For example, if $n = 2$, $p = 3$, $\alpha_1 = \alpha_2 = 1$ we get $x^3 + y^3 = 1$; for $n = 2$, $p = 1$ and $\alpha_2 = \alpha_1 = 1$ we have $y = 1 - x$, etc.

3.2.2. Let p be a positive even integer, $S = (\mathbb{R}^+, \cdot)$ and choose $\alpha_1, \alpha_2, \ldots, a_n \in \mathbb{R}$ fixed such that $T := \{a = (a_1, a_2, \ldots, a_n) \mid \sum_{i=1}^{n} \alpha_i a_i^p > 0\}$ has at least two elements for which $\sum_{i=1}^{n} \alpha_i a_i^p \neq \sum_{i=1}^{n} \alpha_i b_i^p$. Define $\delta : T \longrightarrow S$ by

$$\delta(a_1, a_2, \ldots, a_n) = \left(\sum_{i=1}^{n} \alpha_i a_i^p\right)^{\frac{1}{p}}.$$

Then $(V, +, \odot)$ is a planar near-ring. For $n = p = 2$, the set of right identities $\mathcal{R}$ consists of all the points on a circle if $\alpha_1 = \alpha_2 > 0$; all the points on an ellipse if $\alpha_1 > 0, \alpha_2 > 0$ and all the points on a hyperbola if $\alpha_1 \alpha_2 < 0$. For $n = 3$ and $p = 2$, $\mathcal{R}$ gives all the points on a sphere if $\alpha_1 = \alpha_2 = \alpha_3 > 0$ and all the points on a hyperboloid if $\alpha_1 = \alpha_2 = 1, \alpha_3 = -1$.

3.2.3. Let p be any positive real number, $S = (\mathbb{R}^+, \cdot)$ and choose $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ fixed such that $T := \{(a_1, a_2, \ldots, a_n) \mid \sum_{i=1}^{n} \alpha_i a_i^p \neq 0\}$ has at least two elements for which $\sum_{i=1}^{n} \alpha_i a_i^p \neq$

$\sum_{i=1}^{n} \alpha_i b_i^p$. Define $\delta : T \longrightarrow S$ by

$$\delta(a_1, a_2, \dots, a_n) = \left(\left| \sum_{i=1}^{n} \alpha_i a_i^p \right| \right)^{\frac{1}{p}}.$$

Then $(V, +, \odot)$ is a planar near-ring. For $n = p = 2$ and $\alpha_1 = 1$, $\alpha_2 = -1$ the right identities are the points on $|x^2 - y^2| = 1$ and for $n = 2$, $p = \frac{2}{3}$ and $\alpha_1 = \alpha_2 = 1$, the right identities are given by $x^{2/3} + y^{2/3} = 1$.

3.2.4. Let $S = (\mathbb{R}^+, \cdot)$ and let $T = \{(a_1, a_2, \dots, a_n) \mid a_1 a_2 \cdots a_n \neq 0\}$. Define $\delta : T \longrightarrow S$ by

$$\delta(a_1, a_2, \dots, a_n) = \sqrt[n]{|a_1 a_2 \cdots a_n|}.$$

We get a planar near-ring and for $n = 2$, the right identities are the points $|xy| = 1$.

3.2.5. Let $S = (\mathbb{R}^+, \cdot)$ and choose $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$ fixed such that $T := \{(a_1, a_2, \dots, a_n) \mid \sum_{i=1}^{n} \alpha_i |a_i| \neq 0\}$ has at least two elements such that $\delta : T \longrightarrow S$ defined by

$$\delta(a_1, a_2, \dots, a_n) = \sum_{I=1}^{n} \alpha_i |a_i|$$

has $|\delta(T)| \geq 2$.

3.2.6. Let $T = V^*, S = (\mathbb{R}^*, \cdot)$ and define δ by

$$\delta(a_1, a_2, \dots, a_n) = \begin{cases} a_i \text{ if } a_1 = a_2 = \cdots = a_{i-1} = 0, i < n \\ a_n \text{ otherwise.} \end{cases}$$

For $n = 3$, the right identities of $(V, +, \odot)$ are $\mathcal{R} = \{(x, y, z) \mid x = 1 \text{ or } (x = 0, y = 1) \text{ or } (x = y = 0, z = 1)\}$.

3.2.7. Let $T = V^*, S = (\mathbb{R}^+, \cdot)$ and define $\delta : T \longrightarrow S$ by

(i) $\delta(a_1, a_2, \dots, a_n) = \max\{|a_1|, |a_2|, \dots, |a_n|\}$ or

(ii) $\delta(a_1, a_2, \dots, a_n) = \min\{|a_1|, |a_2|, \dots, |a_n|\}$.

3.2.8. Let $\beta : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form, $T := V^*$ and $S := (\mathbb{R}^+, \cdot)$. Define $\delta : T \to S$ by $\delta(x) = \sqrt{|\beta(x)|}$. Then $|\delta(T)| \geq 2$ ensures that a planar near-ring is obtained.

3.3. (Anshel and Clay [1], Clay [3]). Let V be a vector space over $\mathbb{R}$. Let $T = V^*$ and let $S = (\mathbb{R}^+, \cdot)$.

(i) If V is a normed space with norm $|| \cdot ||$, let $\delta(t) = ||t||$. Then $(V, +, \odot)$ is a planar near-ring.

(ii) Suppose there is a function $\phi : V \longrightarrow \mathbb{R}$ with the property that there is a fixed $\alpha \in \mathbb{R}^*$ such that for all $r \in \mathbb{R}, r \geq 0$ and for all $v \in V$, we have $\phi(rv) = r^\alpha \phi(v)$. Let $\delta(t) := |\phi(t)|^{\frac{1}{\alpha}}$. Once again a planar near-ring is obtained.

As mentioned earlier, many of the examples in 3.2 are covered by these two examples. But 3.4 below shows that one can consider more general cases:

3.4. Let X be a non-empty set and let $V = \mathbb{R}^X$ with pointwise addition. Let $S = (\mathbb{R}^+, \cdot)$ and $T = \{f \in V \mid f \neq 0 \text{ and } f \text{ is bounded}\}$. Define $\delta : T \longrightarrow S$ by $\delta(f) = \sup\{\ |f(x)| \mid x \in X\}$. Once again a planar near-ring $(V, +, \odot)$ is obtained.

One should not be coerced into thinking that this procedure always leads to uninteresting blocks:

3.5. Let F be a field with $|F| \geq 3$ and let V be the direct sum of n copies of $(F, +)$, $n \geq 2$. Then V is a vector space over F. For each $i = 1, 2, \ldots, n$ let $\alpha_i : (F^*, \cdot) \longrightarrow (F^*, \cdot)$ be a group automorphism with $\alpha_1 = 1$. Extend each α_i to F by setting $\alpha_i(0) = 0$. Let $T = \{(a_1, a_2, \ldots, a_n) | \ a_i \in F^* \text{ for all } i\}$ and let $S = \{(\alpha_1(b),\ \alpha_2(b), \ldots, \alpha_n(b) | \ b \in F^*\}$. Then S is a group with respect to componentwise multiplication. Define $V \times S \longrightarrow V$ by componentwise multiplication and let $\delta : T \longrightarrow S$ be given by

$$\delta(a_1, a_2, \ldots, a_n) = (\alpha_1(a_1), \ldots, \alpha_n(a_1)).$$

For $n = 2$, we get the planar near-ring as described by Clay in [3] which is based on one given by van der Walt with $F = \mathbb{R}$ and $\alpha_2(x) = x^3$. In this case $\mathcal{R} = \{(1, y) | \ y \neq 0\}$ and for $(a_1, a_2) \in T$ a basic block is given by $(a_1, a_2) \odot V^{\#} = \{(x, y) | \ y = \frac{a_2}{a_1^3} x^3, x \neq 0\}$. For $n = 3$, one can get a variety of interesting space curves in $\mathbb{R}^3$ as basic blocks by choosing the $\alpha_i's$, for example, from the following automorphisms

$$\alpha(x) = x^3,\ \alpha(x) = \tfrac{1}{x},\ \alpha(x) = \begin{cases} x^2 \text{ if } x > 0 \\ -x^2 \text{ if } x < 0 \end{cases},\ \alpha(x) = \begin{cases} \sqrt{x} \text{ if } x > 0 \\ \sqrt{-x} \text{ if } x < 0 \end{cases} \text{ and } \alpha(x) = x.$$

To complete this set of examples we may mention that the examples of planar near-rings given by Clay [3] in Theorems 4.19 and 4.22 can also be described in terms of our Proposition 2. We now look at the second operation.

Proposition 4. *Let $(V, +, \cdot)$ be a ring with identity and $(S, \cdot)$ a commutative group which acts faithfully on V from the right. We suppose $(v_1 s_1)(v_2 s_2) = (v_1 v_2)(s_1 s_2)$ for all $v_i \in V$ and $s_i \in S$. Let T be a subgroup of $\mathcal{U}(V)$, the group of units of V, and suppose $TS \subseteq T$. Let $\delta : T \longrightarrow S$ be an S-homogenous group homomorphism. Then $V_2 = (V, +, *)$ is a near-ring where* $a * b = \begin{cases} a(b\delta(b)^{-1}) \text{ if } b \in T \\ 0 \text{ if } b \notin T. \end{cases}$

Proof. The right distributivity is obvious, so we only verify the associativity. Firstly note that for any $a, b \in V$ and $s \in S$, $a(bs) = (a1_s)(bs) = (ab)(1_s s) = (ab)s$ and likewise $(as)b = (ab)s$. This means there is no need for brackets in the definition of $a * b$, i.e. we will only write $ab\delta(b)^{-1}$ when applicable. Let $a, b, c, \in V$. Then

$$(a * b) * c = \begin{cases} (ab\delta(b)^{-1})(c\delta(c)^{-1}) \text{ if } c \in T \text{ and } b \in T \\ 0 \text{ otherwise} \end{cases}$$

and

$$a * (b * c) = \begin{cases} a(bc\delta(c)^{-1})\delta(bc\delta(c)^{-1})^{-1} \text{ if } c \in T \text{ and } bc\delta(c)^{-1} \in T \\ 0 \text{ otherwise.} \end{cases}$$

Since T is a subgroup of $\mathcal{U}(V)$ and S acts faithfully on V we have for any $c \in T : b \in T$ if and only if $bc\delta(c)^{-1} \in T$. Furthermore

$$\begin{aligned} a(bc\delta(c)^{-1}\,\delta(bc\delta(c)^{-1})^{-1} &= \\ &= (abc\delta(c)^{-1})\,(\delta(b)\,\delta(c)\,\delta(c)^{-1})^{-1} \\ &= abc(\delta(c)^{-1}\delta(b)^{-1}) \\ &= abc\,(\delta(b)^{-1}\delta(c)^{-1}) \\ &= a(b\delta(b)^{-1})\,(c\delta(c)^{-1}) \end{aligned}$$

which proves the associativity. $\square$

Proposition 5. *Let $(V, +, \cdot)$ be a ring with identity and $(S, \cdot)$ a commutative group which acts faithfully on V from the right. We suppose $(v_1 s_1)(v_2 s_2) = (v_1 v_2)(s_1 s_2)$ for all $v_i \in V$ and $s_i \in S$. Let T be a subgroup of $\mathcal{U}(V)$, the group of units of V, and suppose $TS \subseteq T$. Let $\delta : T \longrightarrow S$ be an S-homogenous group homomorphism. In addition, suppose*

(i) *there are $t_1, t_2 \in T$ with $t_1\delta(t_2) \neq t_2\delta(t_1)$ and*

(ii) *for $a, b \in T$, $a\delta(b) \neq b\delta(a)$ implies $a\delta(b) - b\delta(a) \in \mathcal{U}(V)$.*

Then $V_2 = (V, +, *)$ is a planar near-ring with $V^{\#} = T$, $\mathcal{R} = \{a \in T \mid a = 1_V\delta(a)\}$ and for $a \in T$, a basic block $\mathcal{B}_a$ is given by $\mathcal{B}_a = \{b \in T \mid \delta(b) = \delta(a)\}$.

Proof. We firstly note that for a, $b \in V$, $a =_m b$ if and only if both a and b are not in T or, if both a and b are in T, then $a\delta(a)^{-1} = b\delta(b)^{-1}$. Thus $V^{\#} = T$. By (i) we know $|V/_{=_m}| \geq 3$. Let $a, b, v \in V$ with $a \neq_m b$ and consider the equation $x * a = x * b + v$.
If a and b are in T, then $a\delta(a)^{-1} \neq b\delta(b)^{-1}$. Thus $a\delta(b) \neq b\delta(a)$ and from (ii) the equation has a unique solution $x = v(a\delta(a)^{-1} - b\delta(b)^{-1})^{-1}$ (note that $a\delta(b) - b\delta(a) \in \mathcal{U}(V)$ implies $a\delta(a)^{-1} - b\delta(b)^{-1} \in \mathcal{U}(V)$). If only one of a or b is in T, say $a \in T$ and $b \notin T$, then $x = va^{-1}\delta(a)$ is the unique solution of the equation. Hence V_2 is a planar near-ring. For $a \in T$, $1_a = 1_V\delta(a)$ and $\mathcal{R} = \{a \in T \mid a = 1_V\delta(a)\}$. Lastly, $\mathcal{B}_a = \{b \in T \mid \delta(b) = \delta(a)\}$ since $1_V s = s$ for all $s \in S$. $\square$

Combining Propositions 2 and 5 we get:

Proposition 6. *Let $(V, +, \cdot)$ be a ring with identity and $(S, \cdot)$ a commutative group which acts faithfully on V from the right and satisfies $(v_1 s_1)(v_2 s_2) = (v_1 v_2)(s_1 s_2)$ for all $v_i \in V$ and $s_i \in S$. Let T be a subgroup of $\mathcal{U}(V)$ with $TS \subseteq T$. Let $\delta : T \longrightarrow S$ be an S-homogeneous group homomorphism. We suppose:*

(i) *There is a $t_\circ \in T$ such that $1_V\delta(t_\circ) \neq t_\circ$ and $\delta(t_\circ) \neq 1_S$.*

(ii) *For $t \in T$, $\delta(t) \neq 1_S$ implies $1_V - 1_V\delta(t) \in \mathcal{U}(V)$.*

(iii) *For $t \in T$, $1_V\delta(t) \neq t$ implies $t - 1_V\delta(t) \in \mathcal{U}(V)$.*

Then $(V, +, \odot, *)$ is a double planar near-ring, i.e. both $V_1 = (V, +, \odot)$ and $V_2 = (V, +, *)$ are planar near-rings and $\odot$ and $*$ distribute from the right over each other.

Proof. We start by showing that the requirements of Propositions 2 and 5 are satisfied. Since $\delta(1_V) = 1_S$ and $\delta(t_\circ) \neq 1_S$, we know $|\delta(T)| \geq 2$. Let $t \in T$ with $\delta(t) \neq 1_S$ and let $v \in V$. By (ii) above, the equation $x = x\delta(t) + v$ has a unique solution $x = v\,(1_V - 1_V\delta(t))^{-1}$. Hence V_1 is a planar near-ring.

For $t_1 := t_\circ$ and $t_2 := 1_V$ condition (i) of Proposition 5 is satisfied. Let $a, b \in T$ with $a\delta(b) \neq b\delta(a)$. Let $t := b^{-1}a$. Then $1_V\delta(t) \neq t$, for $1_V\delta(t) = t$ would imply $1_V\delta(b^{-1}a) = b^{-1}a$, i.e. $1_V\delta(b^{-1})\delta(a) = b^{-1}a$, i.e. $1_V\delta(a)\delta(b)^{-1} = b^{-1}a$, i.e. $b\delta(a) = a\delta(b)$; a contradiction. By assumption (iii) we have $t - 1_V\delta(t) \in \mathcal{U}(V)$. Hence $b^{-1}a - 1_V\delta(b^{-1}a) = u$ for some $u \in U(V)$. Thus $a\delta(b) - b\delta(a) = bu\delta(b) \in \mathcal{U}(V)$ and we conclude that V_2 is a planar near-ring.
Finally we show that the two multiplications distribute over each other. Let $a, b, c \in V$. Then
$(a \odot b) * c = \begin{cases} a\delta(b)c\delta(c)^{-1} \text{ if } c, b \in T \\ 0 \text{ otherwise} \end{cases}$
and
$$(a * c) \odot (b * c) = \begin{cases} ac\delta(c)^{-1}\delta(bc\delta(c)^{-1}) \text{ if } c \in T \text{ and } bc\delta(c)^{-1} \in T \\ 0 \text{ otherwise} \end{cases}$$
$$= \begin{cases} ac\delta(c)^{-1}\delta(b) \text{ if } c, b \in T \\ 0 \text{ otherwise.} \end{cases}$$
Also
$(a * b) \odot c = \begin{cases} ab\delta(b)^{-1}\delta(c) \text{ if } c, b \in T \\ 0 \text{ otherwise} \end{cases}$
and
$(a \odot c) * (b \odot c) = \begin{cases} ab\delta(c)\delta(b)^{-1} \text{ if } c, b \in T \\ 0 \text{ otherwise.} \end{cases}$ □

It should be noted that if $(V, +, \cdot)$ is a field (or skew field), then conditions (ii) and (iii) above are trivially satisfied. Clay [4] has used double planar near-rings with great effect to describe geometry over fields.

Example 7. We will consider various applications of the above result.

7.1. Let $(V, +, \cdot)$ be a subring with identity of the ring $M_n(\mathbb{C})$ of $n \times n$ matrices over the complex field $\mathbb{C}$. Let T be a subgroup of $\mathcal{U}(V)$ and let $S = (\mathbb{R}^+, \cdot)$ such that $TS \subseteq T$ where $V \times S \longrightarrow V$ is given by

$$((a_{ij})_{n\times n}, s) \mapsto (a_{ij}s)_{n\times n}.$$

S acts faithfully on V and $((a_{ij})s_1)((b_{ij})s_2) = (a_{ij})(b_{ij})s_1s_2$.
Define $\delta : T \longrightarrow S$ by $\delta((a_{ij})) = \sqrt[n]{|\det(a_{ij})|}$. Then δ is an S-homogeneous group homomorphism. By varying S and T, we get two well-known examples for the case $n = 2$.

7.1.1. Let $n = 2$, $V = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$ and $T = \mathcal{U}(V)$. As is well-known, V is isomorphic to the complex field via

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \longmapsto a + ib.$$

Here $\delta\left(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\right) = \sqrt{a^2 + b^2}$. The requirements of Proposition 6 are satisfied and the near-ring $(V, +, \odot, *)$ is the well-known example of a double planar near-ring. Moreover, the near-ring $V_2 = (V, +, *)$ serves as motivation for much that is done for circular planar near-rings. We will encounter this example again below, but then from a different source.

7.1.2. For $n = 2$, let

$$V = \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \mid z, w \in \mathbb{C} \right\}$$

(here $\overline{z} = a - ib$ denotes the conjugate of $z = a + ib$). Then $(V, +, \cdot)$ is isomorphic to the ring of quaternions $\{a + ib + jc + kd \mid a, b, c, d \in \mathbb{R}\}$ via

$$\begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} \mapsto a + ib + jc + kd.$$

Let $T = \mathcal{U}(V)$. Then $\delta \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} = \sqrt{a^2+b^2+c^2+d^2}$ also determines a double planar near-ring (cf [4]).

7.2. Let $\mathbb{R}$ be the field of reals and let $f(x) = x^2 + qx + r$ be a monic polynomial of degree two over $\mathbb{R}$. Then $V = \frac{\mathbb{R}[x]}{<f(x)>} \cong \{a + bt \mid a, b \in \mathbb{R}, t^2 + qt + r = 0\}$ is a commutative ring with identity which we identify with $\mathbb{R} \times \mathbb{R} = \{(a, b) \mid a, b \in \mathbb{R}\}$. The addition is componentwise and the multiplication is given by $(a, b)(c, d) = (ac - bdr, ad + bc - bdq)$. For $(a, b) \in V$, we define the conjugate of (a, b), written as $\overline{(a, b)}$, by $\overline{(a, b)} = (a - bq, -b)$. Let $S = (\mathbb{R}^+, \cdot)$ which we identify with the subset $\{(a, 0) \mid a > 0\}$ of V. With respect to the multiplication in V, we then have that S acts faithfully on V from the right. Let $T := \{(a, b) \in V \mid (a, b)\overline{(a, b)} > 0\}$. Then $T = \{(a, b) \mid a^2 - abq + b^2 r > 0\}$ is a subgroup of $\mathcal{U}(V) = \{(a, b) \mid (a, b)\overline{(a, b)} \neq 0\}$. In general T need not coincide with $\mathcal{U}(V)$. However, $T = \mathcal{U}(V)$ if and only if $\triangle < 0$. In this case, $f(x)$ is irreducible over $\mathbb{R}$ and V is a field with $\mathcal{U}(V) = T = V^*$. Note that if $(a, b) \in \mathcal{U}(V)$, then

$$(a, b)^{-1} = \frac{\overline{(a, b)}}{(a, b)\,\overline{(a, b)}}.$$

Define $\delta : T \longrightarrow S$ by $\delta(a, b) := \sqrt{(a, b)\overline{(a, b)}} = \sqrt{a^2 - abq + b^2 r}$.
If $\Delta := q^2 - 4r \neq 0$, the conditions of Proposition 6 are satisfied and we get a double planar near-ring $(V, +, \odot, *)$.

Our first specific case is the classical one (already encountered in 7.1.1 above):

7.2.1. Let $q = 0, r = 1$. Then $f(x) = x^2 + 1$ and $(V, +, \cdot)$ is just the complex field. Here $V_1 = (V, +, \odot)$ has $a \odot b = a\,|b|$ with $\mathcal{R}_1 = \{(x, y) \mid x^2 + y^2 = 1\}$ and for $a \neq 0$, $(\mathcal{B}_1)_a$ is the ray from 0 through a with 0 excluded. Also, $V_2 = (V, +, *)$ is a planar near-ring where $a * b = \begin{cases} ab\ |b|^{-1} \text{ if } b \neq 0 \\ 0 \text{ otherwise.} \end{cases}$

We have $\mathcal{R}_2 = \{(x, y) \mid y = 0,\ x > 0\}$ and if $a = (a_1, a_2) \neq 0$, then $(\mathcal{B}_2)_a = \{(x, y) \mid x^2 + y^2 = u^2\}$ where $u^2 = a_1^2 + a_2^2$. A block $(a_1, a_2) * V^{\#} + (b_1, b_2)$ is then just a translation of $(a_1, a_2) * V^{\#}$, i.e. the circle with center (b_1, b_2) and radius u where $u^2 = a_1^2 + a_2^2$. This near-ring is the well-known circular planar near-ring and serves as motivation for much that is done in this area.

We recall the definition of a circular planar near-ring. For a planar near-ring $(N,+,\cdot)$, let $\mathcal{B}^* := \{aN^{\#}+b \mid a,b \in N, a \neq 0\}$. An element $aN^{\#}+b$ of $\mathcal{B}^*$ is called a *block* and when $b=0$ we have a *basic block* $aN^{\#}$. A planar near-ring $(N,+,\cdot)$ is called *circular* if it satisfies:

(C1) Every three distinct points x, y and z in N belong to at most one $B \in \mathcal{B}^*$.

(C2) Every two distinct points x and y belong to at least two distinct blocks B_1 and B_2 in $\mathcal{B}^*$.

In a circular planar near-ring, a block $B = aN^{\#}+b$ is called a *circle with centre b and radius a.*

7.2.2. Let $q=0$, $r=s^2, s>0$. Once again $(V,+,\cdot)$ is a field and $(V,+,\odot,*)$ is a double planar near-ring with V_2 circular. In V_1 we have $\mathcal{R}_1 = \{(x,y) \mid x^2+y^2s^2=1\}$, i.e. all the points on an ellipse and for $a \neq 0$, $(\mathcal{B}_1)_a$ consists of all the points on a ray as in 7.2.1 above. In V_2 we have $\mathcal{R}_2$ as in 7.2.1 and for $a \neq 0$, the associated basic block (= circle) is given by $(\mathcal{B}_2)_a = \{(x,y) \mid x^2+y^2s^2=u^2\}$ where $u^2=a_1^2+a_2^2s^2$, i.e. all the points on the ellipse

$$\frac{x^2}{u^2}+\frac{y^2}{(u/s)^2}=1.$$

The planar near-ring V_2 is circular with circles (blocks) these ellipses. But the circularity is not just by virtue of the blocks being ellipses. Rather, it is because nature takes care of itself very well! One could not just have arbitrary ellipses and expect to get a circular planar near-ring. The reason being that in general two ellipses may intersect in four distinct points. However, in the example above, this cannot occur since the ellipses all have a fixed scaling between the axes – meaning for example if $s>1$, that all the ellipses have their horizontal axes as their major axis.

7.2.3. Let $q=r=1$, i.e. $f(x)=x^2+x+1$. Once again $\triangle < 0$ and $(V,+,.)$ is a field. In V_1 of the double planar near-ring $(V,+,\odot,*)$ we have the right identities $\mathcal{R}_1 = \{(x,y) \mid x^2-xy+y^2=1\}$ and in V_2, for $a \neq 0$, a basic block is given by $(\mathcal{B}_2)_a = \{(x,y) \mid x^2-xy+y^2=u^2\}$ where $u^2=a_1^2-a_1a_2+a_2^2$. This is just a rotation of an ellipse with the major axis on the line $y=x$. A block is then just a translation of such an ellipse and V_2 is a circular planar near-ring.

In all three of the above examples, we had $\triangle < 0$. Below examples with $\triangle > 0$ will be given, but firstly we should mention the anticipated:

Proposition 8. *If $f(x)=x^2+qx+r$ and $g(x)=x^2+px+s$ are two monic polynomials of degree two over $\mathbb{R}$, both with discriminants $\triangle(f)<0$ and $\triangle(g)<0$, then the two associated double planar near-rings $(V_f,+,\odot_f,*_f)$ and $(V_g,+,\odot_g,*_g)$ are isomorphic.*

Proof. Without loss of generality we will assume $p=0$ and $s=1$ and then define a bijection $\phi : V_g \longrightarrow V_f$ which preserves all operations.
Firstly note that in V_g we have $(a,b)\cdot_g(c,d)=(ac-bd, ad+bc)$ and $\delta_g(a,b)=\sqrt{a^2+b^2}$. In V_f we have $(a,b)\cdot_f(c.d)=(ac-bdr, ad+bc-bdq)$ and $\delta_f(a,b)=\sqrt{a^2-abq+b^2r}$.

By assumption, $\triangle(f) = q^2 - 4r < 0$. Let $k := \sqrt{-\triangle(f)}$. Define ϕ by $\phi(a,b) := (a + \frac{q}{k}b, \frac{2}{k}b)$. Then ϕ is clearly injective, it preserves the addition and $\phi(c - \frac{q}{2}d, \frac{k}{2}d) = (c,d)$ shows that ϕ is surjective. Since $k^2 = -\triangle(f) = 4r - q^2$, one can verify that $\delta_f(c + \frac{q}{k}d, \frac{2}{k}d) = \sqrt{(c + \frac{q}{k}d)^2 - (c + \frac{q}{k}d)\frac{2}{k}dq + (\frac{2}{k}d)^2 r} = \sqrt{c^2 + d^2}$.

Hence for $(c,d) \neq 0$,

$$\begin{aligned}\phi((a,b) \odot_g (c,d) &= \phi((a,b)\delta_g(c,d)) \\ &= \left((a + \frac{q}{k}b)\sqrt{c^2+d^2},\ \frac{2}{k}b\sqrt{c^2+d^2}\right)\end{aligned}$$

and

$$\begin{aligned}\phi(a,b) \odot_f \phi(c,d) &= (a + \frac{q}{k}b, \frac{2}{k}b)\ \delta_f(c + \frac{q}{k}d, \frac{2}{k}d) \\ &= \left((a + \frac{q}{k}b)\sqrt{c^2+d^2},\ \frac{2}{k}b\sqrt{c^2+d^2}\right).\end{aligned}$$

Lastly

$$\begin{aligned}\phi((a,b) *_g (c,d)) &= \phi\left(\frac{ac-bd}{\sqrt{c^2+d^2}}, \frac{ad+bc}{\sqrt{c^2+d^2}}\right) \\ &= \left(\frac{ac-bd}{\sqrt{c^2+d^2}} + \frac{q}{k}\frac{(ad+bc)}{\sqrt{c^2+d^2}},\ \frac{2}{k}\frac{(ad+bc)}{\sqrt{c^2+d^2}}\right)\end{aligned}$$

and

$$\begin{aligned}\phi(a,b) *_f \phi(c,d) &= \left(a + \frac{q}{k}b,\ \frac{2}{k}b\right) *_f \left(c + \frac{q}{k}d,\ \frac{2}{k}d\right) \\ &= \left(a + \frac{q}{k}b,\ \frac{2}{k}b\right) \cdot_f \left(c + \frac{q}{k}d,\ \frac{2}{k}d\right)\ \delta_f \left(c + \frac{q}{k}d,\ \frac{2}{k}d\right)^{-1} \\ &= \phi\left((a,b) *_g (c,d)\right)\end{aligned}$$

as straightforward calculations will show. $\square$

This results shows that "circles" need not be round, reminding us of a limerick written by one of Paul Erdös colleagues to humor his papers published in some unknown journals (as quoted in [5]):

A conjecture both deep and profound
Is whether the circle is round
In a paper of Erdös
Written in Kurdish
A counterexample is found.

We now consider polynomials with $\triangle > 0$.

7.2.4. (continuation of Example 7.2) Suppose $q = 0$ and $r = -1$. Then $f(x) = x^2 - 1$ and $\triangle > 0$. Here $(V, +, \cdot)$ is not a field, $T = \{(x, y) \mid x^2 - y^2 > 0\}$ and $\delta(x, y) = \sqrt{x^2 - y^2}$. The associated double planar near-ring has $\mathcal{R}_1 = \{(x, y) \mid x^2 - y^2 = 1\}$. If $(a, b) \in T$, then $(\mathcal{B}_2)_{(a,b)}$ $= \left\{(x, y) \,|\, \frac{x^2}{u^2} - \frac{y^2}{u^2} = 1\right\}$ where $u^2 = a^2 - b^2$ which is just a hyperbola. This planar near-ring $V_2 = (V, +, *)$ was originally constructed by Karzel in an unsuccessful attempt to define an infinite circular planar near-ring different from the one in Example 7.2.1. His approach, as described by Clay [3], was completely different to the procedure given above. It is not known if it was realized at that time that V_2 is partner in a double planar near-ring. We will return to a variant of this example below, and for this reason we need to say something more about its structure. For any nonzero (a, b) in V_2, the associated basic block is

$$(a, b) * V^{\#} = \begin{cases} \{(x, y) \mid x^2 - y^2 = u^2\} \text{ if } u^2 = a^2 - b^2 > 0 \ (ie \text{ if } (a, b) \in T) \\ \{(x, y) \mid y^2 - x^2 = v^2\} \text{ if } v^2 = b^2 - a^2 > 0 \\ \{(x, y) \mid y = x, x \neq 0\} \text{ if } a = b \\ \{(x, y) \mid y = -x, x \neq 0\} \text{ if } a = -b \end{cases}.$$

The blocks then are just translations of these hyperbolas and straight lines. Since the asymptotic lines of these hyperbolas have fixed gradients, it can be verified that any two blocks have at most two points in common, which means that the first requirement for circularity is satisfied. To consider the validity of the second condition, take any two distinct points (a, b) and (c, d) in V. If the gradient of the straight line through these two points is not ± 1, then there are at least two hyperbolas (=blocks) which contain both these points. If this gradient is ± 1, then there is just one block, namely a straight line with gradient either $+1$ or -1, which contains these two points. Hence (C2) is not satisfied and V_2 is not circular.

7.2.5. Let $q = 1, r = 0$. Then $f(x) = x^2 + x$, $\triangle > 0$ and $T = \{(x, y) \mid x^2 - xy > 0\}$ with $\delta(x, y) = \sqrt{x^2 - xy}$. A basic block determined by $(a, b) \in T$ in V_2 is given by $(\mathcal{B}_2)_{(a,b)} = \left\{(x, y) \mid x \neq 0, y = x - \frac{u^2}{x}\right\}$ where $u = \sqrt{a^2 - ab}$. This is just a rotation of a hyperbola.

As was the case with $\triangle < 0$, the next result can be expected:

Proposition 9. *If $f(x) = x^2 + qx + r$ and $g(x) = x^2 + px + s$ are two monic polynomials of degree two over $\mathbb{R}$, both with discriminates $\triangle(f) > 0$ and $\triangle(g) > 0$, then the two associated double planar near-rings are isomorphic.*

Proof. Without loss of generality, take $p = 0$ and $s = -1$. Let $k := \sqrt{\triangle(f)}$ and define $\phi : V_g \longrightarrow V_f$ by $\phi(a, b) = \left(a + \frac{q}{k} b, \frac{2}{k} b\right)$. As in the proof of Proposition 8, it can be shown that ϕ is a bijection which preserves all operations. □

7.2.6. (continuation of Example 7.2) With $V = \frac{\mathbb{R}[x]}{<f(x)>}$ as above, one may define different δ's and still get double planar near-rings. For example: Let $q = -1, r = 0$. Then $f(x) = x^2 - x, \triangle > 0$ and $V = \frac{\mathbb{R}[x]}{<f(x)>}$ is a commutative ring with identity where $(a, b)(c, d) = (ac, ad + bc + bd)$. Let $S = (\mathbb{R}^*, \cdot)$ which we identify with $\{(a, 0) \in V \mid a \neq 0\}$ and let $T = \{(a, b) \in V \mid a^2 + ab > 0\}$. Then T is a subgroup of $\mathcal{U}(V) = \{(a, b) \mid a^2 + ab \neq 0\}$.

Define $\delta : T \longrightarrow S$ by

$$\delta(x, y) = \frac{x^2}{x + y}.$$

All the requirements of Proposition 6 are satisfied and consequently $(V, +, \odot, *)$ is a double planar near-ring. The associated geometry is given by: $T = \{(x, y) \mid (x > 0 \text{ and } y > -x) \text{ or } (x < 0 \text{ and } y < -x)\}$, $\mathcal{R}_1 = \{(x, y) \mid y = x^2 - x, x > 0\}$ and if $(a, b) \in T$, then $(\mathcal{B}_1)_{(a,b)} = \{(x, y) \,|y = \frac{b}{a}x, x \neq 0\}$. In V_2 one has $\mathcal{R}_2 = \{(x, y) \mid y = 0, x \neq 0\}$ and for $(a, b) \in T$, $(\mathcal{B}_2)_{(a,b)} = \{(x, y) \,|y = \frac{1}{u}x^2 - x, ux > 0\}$ where $u = \delta(a, b) = \frac{a^2}{a+b}$.
Another such example is:

7.2.7. Let V be the direct sum of two copies of the field of real numbers $\mathbb{R}$. Then $\mathcal{U}(V) = \{(a, b) \mid ab \neq 0\}$. Let $T = \{(a, b) \mid ab > 0\}$ and $S = (\mathbb{R}^*, \cdot)$. Define $\delta : T \to S$ by $\delta(a, b) = \frac{a^2}{b}$. The requirements of Proposition 6 are satisfied and we obtain a double planar near-ring $(V, +, \odot, *)$. The associated geometry is: $\mathcal{R}_1 = \{(x, y) \mid y = x^2, x > 0\}, \mathcal{R}_2 = \{(x, x) \mid x \neq 0\}$ and for $(a, b) \neq 0$, the corresponding basic blocks in V_1 and V_2 respectively are:

$$(a, b) \odot V^{\#} = \begin{cases} \{(x, \frac{b}{a}x) \mid x \neq 0\} \text{ if } a \neq 0 \\ \{(0, y) \mid y \neq 0\} \text{ if } a = 0 \end{cases} \text{ and}$$

$$(a, b) * V^{\#} = \begin{cases} \{(x, \frac{b}{a^2}x^2) \mid bx > 0\} \text{ if } ab > 0 \\ \{(0, y) \mid yb > 0\} \text{ if } a = 0 \\ \{(x, 0) \mid ax > 0\} \text{ if } b = 0 \\ \{(x, \frac{b}{a^2}x^2) \mid bx < 0\} \text{ if } ab < 0 \end{cases}.$$

We will now generalize the procedure described in Example 7.2. Let F be a field and let $f(x) = f_\circ + f_1x + \cdots + f_{n-1}x^{n-1} + x^n$ be a monic polynomial of degree n $(n \geq 2)$ over F. Let V be the commutative ring with identity $V = \frac{F[x]}{<f(x)>}$. Then V is isomorphic to $\{a_\circ + at + \cdots + a_{n-1}t^{n-1} | \; a_i \in F, f(t) = 0\}$ which we identify with the n-tuple $V = (F, F, \ldots, F) = \{(a_0, a_1, \ldots ., a_n) \mid a_i \in F\}$. We will not distinguish between $a_\circ + a_1t + \cdots + a_{n-1}t^{n-1}$ and $(a_\circ, a_1, \ldots, a_{n-1})$ and use whichever is more convenient in the particular case. We identify the field F with $(F, 0, 0, \ldots, 0)$. The addition in V is componentwise and the multiplication the usual for polynomials subject to $t^n = -(f_\circ + f_1t + \cdots + f_{n-1}t^{n-1})$. Let $\mathcal{U}(V)$ be the group of units of V. To facilitate the calculations involving the multiplication, we will find it convenient to associate with every $a = (a_\circ, a_1, \ldots, a_{n-1}) \in V$ a uniquely determined $n \times n$ matrix $M(a) \in M_n(F)$. With respect to this matrix, the product ab in V can be written as $ab = (a_\circ, a_1, \ldots, a_{n-1})\ (b_\circ, b_1, \ldots, b_{n-1})$

$$= \left(M(a) \begin{bmatrix} b_\circ \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix} \right)^*$$

where $(...)^*$ denotes the transposed matrix. We will not distinguish between the n-tuple $(b_\circ, b_1, \ldots, b_{n-1})$ and the $1 \times n$ matrix $[b_\circ \; b_1 \; \cdots \; b_{n-1}]$. To describe the matrix $M(a)$, we identify the powers of t in V with $n \times 1$ column matrices:

$$t^{\circ} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and for $j = 0, 1, 2, \dots, 2n-3$, if

$$t^{j} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} \text{ then}$$

$$t^{j+1} = \begin{bmatrix} 0 \\ c_0 \\ \vdots \\ c_{n-2} \end{bmatrix} - c_{n-1} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{bmatrix} = \begin{bmatrix} -f_0 c_{n-1} \\ c_0 - f_1 c_{n-1} \\ \vdots \\ c_{n-2} - f_{n-1} c_{n-1} \end{bmatrix}.$$

With this identification and $t^{j} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$, the $(j+1)$-th column of $M(a)$, $0 \leq j \leq n-1$, is given by

$$\begin{aligned} at^{j} &= (a_0 + a_1 t + \cdots + a_{n-1} t^{n-1}) t^{j} \\ &= a_0 t^{j} + a_1 t^{j+1} + \cdots + a_{n-1} t^{j+n-1} \\ &= a_0 \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} + a_1 \begin{bmatrix} -f_0 c_{n-1} \\ c_0 - f_1 c_{n-1} \\ \vdots \\ c_{n-2} - f_{n-1} c_{n-1} \end{bmatrix} + \cdots + a_{n-1} \begin{bmatrix} \cdot \\ \cdot \\ \vdots \\ \cdot \end{bmatrix}. \end{aligned}$$

For example, the first column of $M(a)$ (*i.e.* when $j = 0$) is

$$at^{\circ} = a_{\circ} t^{\circ} + a_1 t + \cdots + a_{n-1} t^{n-1} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

and the second column is

$$at = a_0 t + a_1 t^2 + \cdots + a_{n-1} t^{n} = \begin{bmatrix} -a_{n-1} f_{\circ} \\ a_{\circ} - f_1 a_{n-1} \\ \vdots \\ a_{n-2} - f_{n-1} a_{n-1} \end{bmatrix}.$$

Now

$$\begin{aligned} ab &= a\left(b_0 + b_1 t + \cdots + b_{n-1}t^{n-1}\right) \\ &= ab_0 t^0 + ab_1 t + \cdots + ab_{n-1}t^{n-1} \end{aligned}$$

which can be identified with

$$\left(\begin{bmatrix} at^0 & at & at^2 & \dots & at^{n-1} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix} \right)^*$$

$$= \left(M(a) \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix} \right)^* .$$

We will now record some of the properties of $M(a)$:

(i) $a \in \mathcal{U}(V)$ if and only if $\det(M(a)) \neq 0$. Indeed:

$a \in \mathcal{U}(V)$

$\Longleftrightarrow$there is a unique $b = (b_0, b_1, \dots, b_{n-1}) \in V$ with $ab = 1 = (1, 0, \dots, 0)$

$\Longleftrightarrow$ there is a unique $b = (b_0, b_1, \dots, b_{n-1}) \in V$ with $[b_0\ b_1 \cdots b_{n-1}]\ M(a)^* = [1\,0\ \dots\ 0]$

$\Longleftrightarrow$ det $M(a)^* \neq 0$

$\Longleftrightarrow$ det $M(a) \neq 0$.

(ii) $M(1) = M(1, 0, \dots, 0) = I_n$, the $n \times n$ identity matrix.

(iii) For $a \in V$ and $s \in F$, $M(as) = M(a)s$ (the product on the right is just the usual scalar product): The $(j+1)$-th column of $M(as)$ is given by

$$\begin{aligned} (as)t^j &= a_\circ st^j + a_1 st^{j+1} + \cdots + a_{n-1}st^{j+n+1} \\ &= (a_\circ t^j + a_1 t^{j+1} + \cdots + a_{n-1}t^{j+n-1})s \end{aligned}$$

which is just the $(j+1)$-th column of $M(a)$ with each entry multiplied by s.

(iv) It is worthwhile to draw attention to the two representations of $at^j, a \in V$: On the one hand, thinking of a as $a = a_0 + a_1 t + \cdots + a_{n-1}t^{n-1}$ and each t^i as a column matrix, we get $at^j = a_0 t^j + a_1 t^{j+1} + \cdots + a_{n-1}t^{j+n-1}$ which gives the $(j+1)$-th column of $M(a)$. On the other hand, thinking of a as $a = (a_0, a_1, \dots, a_{n-1})$ and t^j as $t^j = 0 + 0t + 0t^2 + \cdots + 1t^j + \cdots + 0t^{n-1}$ which we identify with $(0, \dots, 0, 1, 0, \dots, 0)$ where 1 is in the $(j+1)$-th position, we get

$$
\begin{aligned}
at^j &= (a_0, a_1, \ldots, a_{n-1})(0, \ldots, 0, 1, 0, \ldots, 0) \\
&= \left(M(a) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right)^* \\
&= [0 \ldots 0\,1\,0 \ldots 0] M(a)^*
\end{aligned}
$$

which is also just the $(j+1)$-th column of $M(a)$.

(v) For all $a, b \in V$, $M(ab) = M(a)M(b)$: The $(j+1)$-th column of $M(ab)$ is $(ab)t^j$. The $(j+1)$-th column of $M(a)M(b)$

$$
\begin{aligned}
&= [0 \ \ldots \ 0 \ 1 \ 0 \ \ldots \ 0]\,(M(a)M(b))^* \\
&= ([0 \ \ldots \ 0 \ 1 \ 0 \ \ldots \ 0]\,M(b)^*)\,M(a)^* \\
&= \left(M(b) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right)^* M(a)^* \\
&= (M(a).bt^j)^* \\
&= a(bt^j) \\
&= (ab)t^j
\end{aligned}
$$

(iv) For any $a \in U(V), M(a^{-1}) = M(a)^{-1}$.

Next we want to define the conjugate of an element in V. For any $a \in V$, the adjoint matrix of $M(a)$, denoted by $Adj(M(a))$, is given by

$$
Adj(M(a)) = \begin{bmatrix} M(a)_{11} & M(a)_{21} & \cdots & M(a)_{n1} \\ M(a)_{12} & M(a)_{22} & \cdots & M(a)_{n2} \\ \vdots & & & \\ M(a)_{1n} & M(a)_{2n} & \cdots & M(a)_{nn} \end{bmatrix}
$$

where $M(a)_{ij}$ is the (i,j)-th cofactor of $M(a)$.

For $a \in \mathcal{U}(V)$, $M(a)^{-1} = \dfrac{1}{\det(M(a))} \cdot Adj(M(a))$. Suppose $a^{-1} = (b_0, b_1, \ldots, b_{n-1})$. From

$aa^{-1} = 1$ we get

$$\left(M(a) \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix} \right)^* = [1\ 0\ \ \dots\ 0]$$

i.e.

$$\begin{aligned} [b_0\ b_1\ \dots\ b_{n-1}] &= [1\ 0\ \dots\ 0](M(a)^*)^{-1} \\ &= \frac{1}{\det(M(a))}\left[M(a)_{11}\ M(a)_{12}\dots\ M(a)_{1n}\right], \end{aligned}$$

i.e. $a^{-1} = \dfrac{1}{\det(M(a))}(M(a)_{11}, M(a)_{12}, \dots, M(a)_{1n})$.

We define the *conjugate* of any $a \in V$, denoted by $\overline{a}$, by $\overline{a} := (M(a)_{11}, M(a)_{12}, \dots, M(a)_{1n})$. If $a \in \mathcal{U}(V)$, then $\overline{a} = \det(M(a))\ a^{-1}$, hence for such an a, $a\overline{a} = \det(M(a))$ (remember, we identify F with $(F, 0, 0, \dots, 0)$ in V).

(vii) For any $a \in V$ and $s \in F$, $\overline{as} = \overline{a}s^{n-1}$.

(viii) For $a, b \in \mathcal{U}(V)$, $\overline{ab} = \overline{a}\overline{b}$.

(ix) If $a \in \mathcal{U}(V)$, then $\overline{a} \in \mathcal{U}(V)$ and for such a, $\overline{\overline{a}} = (\det(M(a)))^{n-2}a$.

We suppose that $(F^*, \cdot)$ has a subgroup S such that if $s_1, s_2 \in S$ and $s_1^n = s_2^n$, then $s_1 = s_2$. Let $T = \{a \in \mathcal{U}(V) \mid a\overline{a} = \det(M(a)) = s^n \text{ for some } s \in S\}$ and define $\delta : T \longrightarrow S$ by

$$\delta(a) = \sqrt[n]{a\overline{a}} = \sqrt[n]{\det(M(a))}$$

where $\sqrt[n]{\det(M(a))}$ denotes the unique $s \in S$ with $\det(M(a)) = s^n$ for $a \in T$. Using the properties above, one can show that T is a subgroup of $\mathcal{U}(V), TS \subseteq T$ and δ is an S-homogeneous group homomorphism (the action of S on V is given by the multiplication in V where as usual we identify S with the subset $(S, 0, 0, \dots, 0)$ of V).

If conditions (i),(ii) and (iii) of Proposition 6 are satisfied, then we obtain a double planar near-ring (or if $(V, +, \cdot)$ is a field, we only have to check (i)).

Example 10. We conclude with four applications of this generalization.

10.1. Let F be the field of constructible real numbers. Then $\mathbb{Q} \subseteq F \subseteq \mathbb{R}$ and $f(x) = x^2 + 1 \in F[x]$ is irreducible over F. Hence $V = \dfrac{F[x]}{\langle f(x) \rangle}$ is a countable field (see, for example, Proposition 1.19 in Chapter 2 of Karpilovsky [6]). Let $S = (F^+, \cdot)$. Here $M(a) = \begin{bmatrix} a_0 & -a_1 \\ a_1 & a_0 \end{bmatrix}$ if $a = (a_0, a_1) \in V$. Hence $\det(M(a)) \neq 0 \iff (a_0, a_1) \neq 0$. Since the square root of any constructible number is constructible, $T = \{a \in \mathcal{U}(V) \mid \det(M(a)) = s^2 \text{ for some } s \in S\} = \mathcal{U}(V)$. Now $\delta(a) = \sqrt{\det(M(a))} = \sqrt{a_0^2 + a_1^2}$. The conditions of Proposition 6 are satisfied and we obtain a double planar near-ring $(V, +, \odot, *)$. Next we show that $V_2 = (V, +, *)$ is

circular (and certainly not isomorphic to the circular planar near-ring in Example 7.2.1). For $(a, b) \neq 0$, the associated basic block $(\mathcal{B}_2)_{(a,b)}$ in V_2 is given by $\{(x, y) \in V \mid x^2 + y^2 = u^2\}$ where $u^2 = a^2 + b^2$.
Any two circles in the Euclidean plane can intersect in at most two points; hence any two blocks in V_2 can intersect in at most two points. Let (a, b) and (c, d) be two distinct points in V. Then a, b, c and d are constructible and thus $(\frac{1}{2}(a+c), \frac{1}{2}(b+d)) \in V$ and r, with $r > 0$ and $r^2 = (\frac{1}{2}(a+c) - a)^2 + (\frac{1}{2}(b+d) - b)^2$, is constructible. Both the points (a, b) and (c, d) are on the block $(x - \frac{1}{2}(a+c))^2 + (y - \frac{1}{2}(b+d))^2 = r^2$ in V_2. A different block in V_2 containing both these points is given by $(x - p)^2 + (y - q)^2 = s^2$ where $s = \sqrt{2}r$ and (p, q) is any one of the two points where the circle described above intersects the perpendicular bisector of the straight line joining (a, b) and (c, d) (obviously (p, q) is in V). We note that a block in V_2 contains infinitely many points, for example, for any $n \in \mathbb{N}$ we have $\left(\frac{r}{n}, \frac{r}{n}\sqrt{n^2 - 1}\right) \in V$ and it is on the block $\{(x, y) \in V \mid x^2 + y^2 = r^2\}$ where $r \in F^+$. But on the other hand, this block in V_2 is missing infinitely many points from the corresponding circle in the Euclidean plane. Indeed, since π is not constructible, neither is $\frac{\pi}{m}$ where $m \in \mathbb{N}$ with $\frac{\pi}{m} < r$. Hence $\left(\frac{\pi}{m}, \frac{1}{m}\sqrt{r^2m^2 - \pi^2}\right) \notin V$ but $\left(\frac{\pi}{m}, \frac{1}{m}\sqrt{r^2m^2 - \pi^2}\right)$ is on the circle $x^2 + y^2 = r^2$ in the Euclidean plane. Such a block in V_2 can thus be called a "near-ring" in the true sense of the word!

In the example above, we had a field $(V, +, \cdot)$ and T consisted of all the nonzero elements of V. But one may have T a proper subset of V^* (ie the planar near-rings V_1 and V_2 need not be integral) and still have V_2 an infinite circular planar near-ring. This is the reason for our next example.

10.2. The polynomial $f(x) = x^2 + 1$ is irreducible over $\mathbb{Q}$ and $(V = \frac{\mathbb{Q}[x]}{\langle f(x)\rangle}, +, \cdot)$ is a field. We identify V with $\mathbb{Q} \times \mathbb{Q}$ where addition is componentwise and multiplication is given by $(a, b)(c, d) = (ac - bd, ad + bc)$. Let $S = (\mathbb{Q}^+, \cdot)$ which we identify with the subset $\{(s, 0) \mid s > 0\}$ and let $T = \{(x, y) \mid x^2 + y^2 = s^2$ for some $s \in S\}$. Define $\delta : T \to S$ by $\delta(a, b) = s$ where s is the unique element of S with $a^2 + b^2 = s^2$. All the requirements of Proposition 6 are satisfied and we get a double planar near-ring $(V, +, \odot, *)$. We will show that V_2 is circular. For any $(a, b) \neq 0$ in V, the basic block $(a, b) * V^\# = \{(x, y) \in V \mid x^2 + y^2 = a^2 + b^2\}$. Indeed, if $(c, d) \in V^\#$, then $(a, b) * (c, d) = \left(\frac{ac-bd}{\delta(c,d)}, \frac{ad+bc}{\delta(c,d)}\right) \in V$ and $\left(\frac{ac-bd}{\delta(c,d)}\right)^2 + \left(\frac{ad+bc}{\delta(c,d)}\right)^2 = a^2 + b^2$. Conversely, let $(x, y) \in V$ with $x^2 + y^2 = a^2 + b^2$. Then $(c, d) := \left(\frac{xa+yb}{a^2+b^2}, \frac{ya-xb}{a^2+b^2}\right) \in T$, $\delta(c, d) = 1$ and $(x, y) = (a, b) * (c, d) \in (a, b) * V^\#$. For any $(p, q) \in V$, the block $(a, b) * V^\# + (p, q) = \{(x, y) \in V \mid (x - p)^2 + (y - q)^2 = a^2 + b^2\}$, ie all the points with rational coordinates on the circle with center (p, q) and radius $\sqrt{a^2 + b^2}$ in the Euclidean plane. This block contains infinitely many points from V: the point $(a + p, b + q)$ is on the block and for any $m \in \mathbb{Q}$, the straight line with gradient m and through the point $(a + p, b + q)$ will intersect the circle in a rational point (ie both the coordinates of the point are rational numbers). Since two circles in the Euclidean plane can intersect in at most two points, the same is certainly true for two blocks in V_2. Let (a, b) and (c, d) be two distinct points in V_2. Then $0 \neq (\frac{1}{2}(c - a), \frac{1}{2}(d - b)) \in V$, $0 \neq (\frac{1}{2}(a + b + c - d), \frac{1}{2}(-a + b + c + d)) \in V$ and both the points (a, b) and (c, d) are on the two distinct blocks

$$(\tfrac{1}{2}(c - a), \tfrac{1}{2}(d - b)) * V^\# + (\tfrac{1}{2}(a + c), \tfrac{1}{2}(b + d)) \quad \text{and}$$
$$(\tfrac{1}{2}(a - b - c + d), \tfrac{1}{2}(a + b - c - d)) * V^\# + (\tfrac{1}{2}(a + b + c - d), \tfrac{1}{2}(-a + b + c + d))$$

in V_2. Thus V_2 is another example of an infinite circular planar near-ring.

We have seen earlier (Example 7.2.4) that V_2 in the double planar near-ring V as determined according to the above procedure by the polynomial $f(x) = x^2 - 1$ over $\mathbb{R}$, has blocks the hyperbolas and the straight lines with gradient ± 1. This planar near-ring is not circular, because any two distinct points on a straight line with gradient ± 1 can only be on one block. Is it possible to remove these points on the asymptotic lines to get a circular planar near-ring? The next example will address this question.

10.3. The polynomial $f(x) = x^2 - 2$ is irreducible over $\mathbb{Q}$ and $(V = \frac{\mathbb{Q}[x]}{\langle f(x)\rangle}, +, \cdot)$ is a field. We identify V with $\mathbb{Q} \times \mathbb{Q}$ where addition is componentwise and multiplication is given by $(a, b)(c, d) = (ac + 2bd, ad + bc)$. Let $S = (Q^+, \cdot)$ which we identify with the subset $\{(s, 0) \mid s \in S\}$ and let $T = \{(x, y) \mid x^2 - 2y^2 = s^2 \text{ for some } s \in S\}$. Define $\delta : T \to S$ by $\delta(a, b) = s$ where s is the unique element of S with $a^2 - 2b^2 = s^2$. All the requirements of Proposition 6 are satisfied and we get a double planar near-ring $(V, +, \odot, *)$ (condition (i) is satisfied, for example, with $t_0 = (\frac{3}{5}, \frac{2}{5})$). For any $(a, b) \neq 0$ in V, we have $a^2 - 2b^2 \neq 0$ and the basic block $(a, b) * V^{\#}$ is given by $\{(x, y) \in V \mid x^2 - 2y^2 = a^2 - 2b^2\}$. Indeed, let $(c, d) \in V^{\#} = T$. Then $(a, b) * (c, d) = \left(\frac{ac+2bd}{\delta(c,d)}, \frac{ad+bc}{\delta(c,d)}\right) \in V$ and $\left(\frac{ac+2bd}{\delta(c,d)}\right)^2 - 2\left(\frac{ad+bc}{\delta(c,d)}\right)^2 = a^2 - 2b^2$. Conversely, let $(x, y) \in V$ with $x^2 - 2y^2 = a^2 - 2b^2$. Then $(c, d) := \left(\frac{xa-2yb}{a^2-2b^2}, \frac{ya-xb}{a^2-2b^2}\right) \in T$, $\delta(c, d) = 1$ and $(x, y) = (a, b) * (c, d) \in (a, b) * V^{\#}$. The blocks of V_2 are thus given by:

$$(a, b) * V^{\#} + (p, q) = \{(x, y) \in V \mid (x - p)^2 - 2(y - q)^2 = a^2 - 2b^2\}$$

$$= \begin{cases} \{(x, y) \in V \mid \frac{(x-p)^2}{u^2} - \frac{(y-q)^2}{(u/\sqrt{2})^2} = 1\} \text{ if } u^2 = a^2 - 2b^2 > 0 \\ \{(x, y) \in V \mid \frac{(y-q)^2}{(v/\sqrt{2})^2} - \frac{(x-p)^2}{v^2} = 1\} \text{ if } v^2 = 2b^2 - a^2 > 0 \end{cases}$$

ie all the rational points on the corresponding hyperbolas in the Euclidean plane. The asymptotic lines of these hyperbolas all have gradient $\pm\frac{1}{\sqrt{2}}$. We note that if (a, b) and (c, d) are two distinct points in V_2, then the gradient of the straight line through these two points cannot be $\pm\frac{1}{\sqrt{2}}$ (since $\frac{d-b}{c-a} \in \mathbb{Q}$ for $a \neq c$). Two distinct hyperbolas of the type above can intersect in at most two points in the Euclidean plane, hence the same is true for two blocks in V_2. We now show that the second requirement for circularity is also satisfied. Let (a, b) and (c, d) be two distinct points in V_2. We distinguish three cases:

(i) Suppose $a = c$. Then $b \neq d$, say $b < d$. Choose $p \in \mathbb{Q}^+$ such that $p^2 > 2(\frac{b-d}{2})^2$. Then

$$(p, \tfrac{1}{2}(b - d)) * V^{\#} + (a - p, \tfrac{1}{2}(b + d))$$

is a block in V_2 which contains both the points (a, b) and (c, d). By varying p, we get many different blocks containing these two points.

(ii) Suppose $b = d$. Then $a \neq c$, say $a < c$. This case is similar to (i) above, just use the north-south hyperbolas (in contrast to the east-west hyperbolas used in (i)).

(iii) Suppose $a \neq c$ and $b \neq d$, say $a < c$ and $b < d$. Then

$$\left(\frac{(a-c)^2-2(d-b)^2}{2(a-c)},\ 0\right) * V^{\#} + \left(\frac{a^2-c^2+2(d-b)^2}{2(a-c)},\ b\right)$$

and

$$\left(0,\ \tfrac{(a-c)^2-2(d-b)^2}{4(d-b)}\right) * V^{\#} + \left(c,\ \tfrac{(a-c)^2-2b^2+2d^2}{2(d-b)}\right)$$

are two distinct blocks in V_2, both containing the points (a,b) and (c,d).
Of course one should ask about the number of points on these blocks. But this should not be of any concern, as each of the basic blocks $(a,b) * V^{\#}$ in V_2, $(a,b) \neq 0$, contains infinitely many points from V. Indeed, this block contains the points (a,b), $(a,-b)$, $(-a,b)$ and $(-a,-b)$ which gives at least two distinct points on the block (a or b could be 0). It is known from elementary number theory that Pell's Equation $x^2 - 2y^2 = 1$ has infinitely many integer solutions (p_{2k-1}, q_{2k-1}), $k = 1, 2, 3, \ldots$ where $\frac{p_j}{q_j}$ is the j-th convergent of the continued fraction of $\sqrt{2}$ (see for example [6], Theorem 11.10). Recall, $\sqrt{2} = 1 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}}$ $= [1; 2, 2, 2, \ldots]$ in the notation of [8]. Then the j-th convergent $\frac{p_j}{q_j}$ is $[1; 2, 2, \ldots, 2]$ where 2 appears j times. For example, $\frac{p_1}{q_1} = \frac{3}{2}$ and $\frac{p_3}{q_3} = \frac{17}{12}$. Then, if (p,q) is an integer solution of Pell's Equation and (X,Y) is a rational solution of $x^2 - 2y^2 = a^2 - 2b^2$, it can be verified that both $(pX + 2qY, qX + pY)$ and $(pX - 2qY, qX - pY)$ are solutions of $x^2 - 2y^2 = a^2 - 2b^2$ in V. Alternatively, if (X,Y) is a rational point on a hyperbola as above, it can be verified that any straight line through this point and with rational slope will intersect the hyperbola in a rational point.

One has the obvious geometric interpretation of the basic blocks in the Euclidean plane as the rational points on the corresponding hyperbola. A nicer geometric interpretation to bring the circularity of the near-ring to the fore is as follows: Consider the stereographic projection of the Euclidean plane onto the Riemann number sphere. Under this mapping, a basic block becomes the rational points on a figure eight on the sphere with the double point of the figure eight at the north pole (which is not a point on the block).

Our last example is over a finite field.

10.4. Let F be the field $\mathbb{Z}_7 = \{0,1,2,3,4,5,6\}$. Then $f(x) = x^2 + 1$ is irreducible over F and $V = \dfrac{F[x]}{\langle f(x) \rangle}$ is a field where $(a,b)(c,d) = (ac + 6bd, ad + bc)$. For $a = (a_0, a_1) \in V$, $M(a) = \begin{bmatrix} a_0 & -a_1 \\ a_1 & a_0 \end{bmatrix}$ and $\det(M(a)) = a_0^2 + a_1^2$. Let $S = \{1,2,4\}$. Then S is a subgroup of $(F^*, \cdot)$ which satisfies $s_1^2 = s_2^2 \Longrightarrow s_1 = s_2$ ($s_i \in S$). Let $T = \{a \in V \mid \det(M(a)) = s^2$ for some $s \in S\}$. Then $T = \{(0,1), (0,2), (0,3), (0,4)\ (0,5), (0,6), (1,0), (1,1), (1,6), (2,0), (2,2), (2,5), (3,0), (3,3), (3,4), (4,0), (4,3), (4,4), (5,0), (5,2), (5,5), (6,0), (6,1), (6,6)\}$. Define $\delta : T \longrightarrow S$ by $\delta(a) = s$ where $\det(M(a)) = s^2$, $s \in S$. The requirements of Proposition 6 are satisfied and we get a double planar near-ring $(V, +, \odot, *)$. We will show that $V_2 = (V, +, *)$ is circular: In view of Theorem 5.5 in Clay [3], one only has to show that $|B_1 \cap B_2| \leq 2$ for two distinct blocks $B_1 = (a,b) * V^{\#} + (c,d)$ and $B_2 = (r,s) * V^* + (u,v)$ with (a,b) and (r,s) both nonzero. We note that $(a,b) * V^{\#} = \{(a,b), (b,6a), (6a,6b), (6b,a), (2a+2b, 5a+2b), (5a+2b, 5a+5b), (5a+5b, 2a+5b), (2a+5b, 2a+2b)\}$ and that there are only three distinct basic blocks namely $(0,1) * V^{\#}$, $(0,2) * V^{\#}$ and $(0,3) * V^{\#}$. To check that all the intersections of two distinct blocks B_1 and B_2 in V_2 has at most two points in common is a tedious and time-consuming task. But with the necessary motivation and perseverance, it can be done to conclude that V_2 is indeed a circular planar near-ring.

We have seen earlier (Example 3.2) that a vectorspace over a field is a rich source of planar near-rings. Our examples of double planar near-rings, and also Proposition 6, suggest that for double planar near-rings one should look at algebras over fields.

As most of the known planar near-rings are field (or ring) generated, we rephrase Propositions 2 and 6 in terms of this construction and Ferrero pairs. Let $(V,+,\cdot)$ be a field with S a subgroup of $(V^*,\cdot)$ such that $|S|\geq 2$. For every $s\in S$, we have an automorphism θ_s of $(V,+)$ defined by $\theta_s(v)=vs$ for all $v\in V$. We identify S with the subgroup $\Phi := \{\theta_s \mid s\in S\}$ of $(Aut(V,+),\circ)$. Clearly S acts faithfully on V from the right. Let $\mathcal{O}=\{\Phi(a)\mid 0\neq a\in V\}$ be the class of all nonzero orbits and choose $\mathcal{C}$ a non-empty subset of $\mathcal{O}$. Fix a set $E=\{e_i\mid i\in I\}$ of representatives for the orbits in $\mathcal{C}$, ie $\mathcal{C}=\{\Phi(e_i)\mid i\in I\}$. For each i, $\Phi(e_i)=e_iS$ and if $b\in\cup\mathcal{C}$, then there are unique $i_b\in I$ and $s_b\in S$ with $b=\theta_{s_b}(e_{i_b})=e_{i_b}s_b$. The multiplication

$$a\diamond b=\begin{cases}\theta_{s_b}(a) \text{ if } b\in\cup\mathcal{C}\\ 0 \text{ if } b\notin\cup\mathcal{C}\end{cases}=\begin{cases} abe_{i_b}^{-1} \text{ if } b\in\cup\mathcal{C}\\ 0 \text{ if } b\notin\cup\mathcal{C}\end{cases}$$

gives the well-known field generated planar near-ring $(V,+,\diamond)$. If we let $T:=\cup\mathcal{C}$ and define $\delta: T\to S$ by $\delta(b)=s_b$ we have $T\subseteq V^*$, $TS\subseteq T$ and δ is S-homogeneous (cf Proposition 2). Then $a\diamond b=a\odot b$ for all $a,b\in V$. In Proposition 6 we need T to be a subgroup of $(V^*,\cdot)$. For this it is sufficient to require that E is a subgroup of $(V^*,\cdot)$, suppose $e_{i_0}=1$, and for future use we also suppose $|E|\geq 2$. It can be verified that δ is a group homomorphism. By our requirements on the sizes of E and S, we can choose a $t_0\in\Phi(e_i), i\neq i_0$ and $t_0\neq e_i$. This means condition (i) of Proposition 6 is satisfied and we have a double planar near-ring $(V,+,\odot,*)$ where

$$a*b=\begin{cases} ab\delta(b)^{-1} \text{ if } b\in T\\ 0 \text{ if } b\notin T\end{cases}=\begin{cases} ae_{i_b} \text{ if } b\in e_{i_b}S \text{ for some } i_b\in I\\ 0 \text{ if } b\notin T\end{cases}.$$

When $(V,+,\cdot)$ is only a ring with identity, the obvious adaptions can be made to describe the ring generated double planar near-ring.

In conclusion, the following may be worthy of more investigations: The planar near-ring V_2 obtained from the polynomial $f(x)=x^2-1$ over $\mathbb{R}$ has some interesting geometrical properties. In particular, it gives rise to affine configurations with two pencils [3]. Are there similar or other interesting geometrical properties in the other examples considered above? One could also consider other definitions for basic blocks (eg $\{a,-a\}N$) and this may lead to interesting geometric interpretations. Most of the examples above were defined in the Euclidean plane. Are these new algebraic operations compatible with the Euclidean topology? Are there other natural topologies associated with these double near-rings?

And finally, the obvious should be mentioned: the theory of near-rings provides a very convenient setting to describe a variety of geometric shapes solely in terms of algebraic operations.

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