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Holonomicity in Synthetic Differential Geometry of Jet Bundles

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Abstract. In the repetitive approach to the theory of jet bundles there are three methods of repetition, which yield non-holonomic, semi-holonomic, and holonomic jet bundles respectively. However the classical approach to holonomic jet bundles failed to be truly repetitive, for it must resort to such a non-repetitive coordinate-dependent construction as Taylor expansion. The principal objective in this paper is to give a purely repetitive definition of holonomicity by using microsquares (double tangents), which spells the flatness of the Cartan connection on holonomic infinite jet bundles without effort. The definition applies not only to formal manifolds but to microlinear spaces in general, enhancing the applicability of the theory of jet bundles considerably. The definition is shown to degenerate into the classical one in case of formal manifolds.

Introduction

The flatness of the Cartan connection lies smack dab at the center of the theory of infinite jet bundles and its comrade called the geometric theory of nonlinear partial differential equations. Indeed it is the flatness of the Cartan connection that enables the de Rham complex to decompose two-dimensionally into the variational bicomplex, from which the algebraic topology of nonlinear partial differential equations has emerged by means of spectral sequences (cf. Bocharov et al. [1]), just as the algebraic topology of smooth manifolds is centered on the unique concept of de Rham complex.

In our previous paper (Nishimura [14]) we have approached the theory of infinite jet bundles from a standpoint of synthetic differential geometry by regarding a 1-jet as a decomposition of the tangent space to the space at the point at issue (cf. Saunders [16, Theorem

4.3.2]) and then looking on higher-order jets as repeated 1-jets (cf. Saunders [16, §5.2 and §5.3]). As is well known, three different ways of repetition of 1-jets yield higher-order jet bundles of three different kinds, namely, *non-holonomic*, *semi-holonomic* and *holonomic* ones. In Nishimura [14] we have happened to adopt the semi-holonomic version of infinite jet bundle as our favorite, but we have failed to distinguish it from the holonomic one.

The principal objective in this paper is to give a purely combinatorial definition of holonomicity, which is undoubtedly coordinate-free and applicable to microlinear spaces in general. Once we have deciphered holonomicity from a synthetic coign of vantage and we have switched from *semi-holonomic* infinite jet bundles to *holonomic* ones, the flatness of the Cartan connection is easily seen to obtain as an immediate meed, just as in the classical case of finite-dimensional manifolds. These matters belong in the first section of this paper.

If our new definition of holonomicity claims to deserve its name, it should be shown to degenerate into the classical coordinate one in case of formal manifolds, thereby tallying with Kock's [6] synthetic paraphrase of the classical approach. This is the topic of Section 4. Sections 2 and 3 are devoted to laconic reviews of formal manifolds and Kock's [6] synthetic approach to jet bundles respectively.

Our idea is presented within our favorite framework of synthetic differential geometry, but apart from the notion of tangent vector being synthetic, the idea could so easily be translated into classical analytical terms as to be applicable to much a wider context than the classical theory of jet bundles was originally intended for, say, differential spaces. It is the high formality that endues our approach to jet bundles with much wider applicability than such a former generalization of the classical theory as Libermann's [11] Banach-space-modelled extension enjoys.

Now we fix our notation and terminology. We denote by $\mathbf{R}$ the extended set of real numbers with cornucopia of infinitesimals, which is expected to satisfy the so-called general Kock axiom (cf. Lavendhomme [10, §2.1]). An $\mathbf{R}$-module is called *Euclidean* (*in the extended sense*) if it satisfies the general Kock axiom. Obviously $\mathbf{R}^n$ is a Euclidean $\mathbf{R}$-module. We denote by D the totality of elements of $\mathbf{R}$ whose squares vanish. Given a microlinear space M and $\mathbf{x} \in M$, we denote by $\mathbf{T}^n(M)$ the set of all mappings γ from D^n to M, while we denote by $\mathbf{T}^n_{\mathbf{x}}(M)$ the set of all mappings γ from D^n to M with $\gamma(0,\dots,0) = \mathbf{x}$. A mapping $\pi : E \to M$ of microlinear spaces is called a *bundle over* M, in which E is called the *total space* of π, M is called the *base space* of π, and $E_x = \pi^{-1}(x)$ is called the *fiber* over $x \in M$. Given $\mathbf{x} \in E$, we denote by $\mathbf{V}^n_{\mathbf{x}}(\pi)$ the subset of $\mathbf{T}^n_{\mathbf{x}}(E)$ consisting of all $\gamma \in \mathbf{T}^n_{\mathbf{x}}(E)$ such that $\pi \circ \gamma(d_1,\dots,d_n) = \pi(\mathbf{x})$ for all $(d_1,\dots,d_n) \in D^n$. Given two bundles $\pi : E \to M$ and $\pi' : E' \to M$ over the same microlinear space M , a mapping $f : E \to E'$ is said to be *over* M if it induces the identity mapping of M. In this case f is called a *morphism of bundles over* M *from* π *to* π' .

1. Nishimura's synthetic approach revisited

Let $\pi : E \to M$ and $\pi' : E' \to M$ be bundles over the same microlinear space M, which shall be fixed throughout this section. A *preconnection over* the bundle $\pi : E \to M$ *at* $\mathbf{x} \in E$ is a mapping $\nabla_{\mathbf{x}} : \mathbf{T}^1_{\pi(\mathbf{x})}(M) \to \mathbf{T}^1_{\mathbf{x}}(E)$ such that for any $t \in \mathbf{T}^1_{\pi(\mathbf{x})}(M)$ and any $\alpha \in \mathbf{R}$, we have the following:

$$\pi \circ \nabla_{\mathbf{x}}(\gamma) = t \tag{1.1}$$

$$\nabla_{\mathbf{x}}(\alpha t) = \alpha \nabla_{\mathbf{x}}(t) \tag{1.2}$$

We note in passing that condition (1.2) implies that $\nabla_{\mathbf{x}}$ is linear by dint of Proposition 10 of Lavendhomme [10, §1.2]. We denote by $\mathbf{J}^1_{\mathbf{x}}(\pi)$ the totality of preconnections $\nabla_{\mathbf{x}}$ over the bundle $\pi : E \to M$ at $\mathbf{x} \in E$. We denote by $\mathbf{J}^1(\pi)$ the set-theoretic union of $\mathbf{J}^1_{\mathbf{x}}(\pi)'s$ for all $x \in E$. The canonical projection $\mathbf{J}^1(\pi) \to E$ is denoted by $\pi_{1,0}$ with $\pi_1 = \pi \circ \pi_{1,0}$.

Let f be a morphism of bundles over M from π to π' over the same base space M. We say that a preconnection $\nabla_{\mathbf{x}}$ over π at a point $\mathbf{x}$ of E *is f-related to* a preconnection $\nabla_{\mathbf{y}}$ over π' at a point $\mathbf{y} = f(\mathbf{x})$ of E' (*in the sense of Nishimura*) provided that

$$f \circ \nabla_{\mathbf{x}}(t) = \nabla_{\mathbf{y}}(t) \tag{1.3}$$

for any $t \in \mathbf{T}^1_{\mathbf{a}}(M)$ with $\mathbf{a} = \pi(\mathbf{x}) = \pi'(\mathbf{y})$.

Theorem 1.1. (cf. Saunders [16, Theorem 4.3.2]). *Given a preconnection $\nabla_{\mathbf{x}}$ over the bundle $\pi : E \to M$ at a point $\mathbf{x}$ of E, the set $\mathbf{H}_{\mathbf{x}}(\pi, \nabla_{\mathbf{x}}) = \{\nabla_{\mathbf{x}}(t) \mid t \in \mathbf{T}^1_{\pi(\mathbf{x})}(M)\}$ is an $\mathbf{R}$-submodule of the $\mathbf{R}$-module $\mathbf{T}^1_{\mathbf{x}}(E)$, and the $\mathbf{R}$-module $\mathbf{T}^1_{\mathbf{x}}(E)$ is the direct sum of $\mathbf{R}$-submodules $\mathbf{H}_{\mathbf{x}}(\pi, \nabla_{\mathbf{x}})$ and $\mathbf{V}^1_{\mathbf{x}}(\pi)$ of $\mathbf{T}^1_{\mathbf{x}}(E)$. Conversely, given an $\mathbf{R}$-submodule H of $\mathbf{T}^1_{\mathbf{x}}(E)$ with $\mathbf{T}^1_{\mathbf{x}}(E) = H \oplus \mathbf{V}^1_{\mathbf{x}}(\pi)$, there exists a unique preconnection $\nabla_{\mathbf{x}}$ with $\mathbf{H}_{\mathbf{x}}(\pi, \nabla_{\mathbf{x}}) = H$ provided that there exists a preconnection over π at $\mathbf{x}$ at all.*

In short, providing that there exists a preconnection over π at $\mathbf{x}$ at all, the assignment $\nabla_{\mathbf{x}} \mapsto \mathbf{H}_{\mathbf{x}}(\pi, \nabla_{\mathbf{x}})$ gives a bijective correspondence between preconnections over π at $\mathbf{x}$ and $\mathbf{R}$-submodules of $\mathbf{T}^1_{\mathbf{x}}(E)$ complementary to $\mathbf{V}^1_{\mathbf{x}}(\pi)$.

Proof. Essentially the same as in the proof of Theorem 2.1 of Nishimura [13]. □

Vectors in $\mathbf{H}_{\mathbf{x}}(\pi, \nabla_{\mathbf{x}})$ in the above theorem are called *horizontal.* The canonical projections of $\mathbf{T}^1_{\mathbf{x}}(E)$ into $\mathbf{H}_{\mathbf{x}}(\pi, \nabla_{\mathbf{x}})$ and $\mathbf{V}^1_{\mathbf{x}}(\pi)$ with respect to the decomposition $\mathbf{T}^1_{\mathbf{x}}(E) = \mathbf{H}_{\mathbf{x}}(\pi, \nabla_{\mathbf{x}}) \oplus \mathbf{V}^1_{\mathbf{x}}(\pi)$ in the above theorem are denoted respectively by $h_{\nabla_{\mathbf{x}}}$ and $v_{\nabla_{\mathbf{x}}}$. Note that $h_{\nabla_{\mathbf{x}}}(t) = \nabla_{\mathbf{x}}(\pi \circ t)$ for any $t \in \mathbf{T}^1_{\mathbf{x}}(E)$.

The above theorem has its counterpart for morphisms of bundles over M.

Theorem 1.2. *Let f be a morphism of bundles from $\pi : E \to M$ to $\pi' : E' \to M$ over M. A preconnection $\nabla_{\mathbf{x}}$ over π at $\mathbf{x} \in E$ is f-related to a preconnection $\nabla_{\mathbf{y}}$ over π' at $\mathbf{y} = f(\mathbf{x})$ iff $h_{\nabla_{\mathbf{y}}}(f \circ t) = f \circ h_{\nabla_{\mathbf{x}}}(t)$ for any $t \in \mathbf{T}^1_{\mathbf{x}}(E)$, or equivalently, iff $v_{\nabla_{\mathbf{y}}}(f \circ t) = f \circ v_{\nabla_{\mathbf{x}}}(t)$ for any $t \in \mathbf{T}^1_{\mathbf{x}}(E)$, or equivalently again, iff the differential of f at $\mathbf{x}$ preserves horizontal vectors.*

Proof. See the proof of Proposition 1.2 of Nishimura [14]. □

By convention we let $\tilde{\mathbf{J}}^0(\pi) = \hat{\mathbf{J}}^0(\pi) = \mathbf{J}^0(\pi) = E$ with $\tilde{\pi}_{0,0} = \hat{\pi}_{0,0} = \pi_{0,0} = id_E$ and $\tilde{\pi}_0 = \hat{\pi}_0 = \pi_0 = \pi$. We let $\tilde{\mathbf{J}}^1(\pi) = \hat{\mathbf{J}}^1(\pi) = \mathbf{J}^1(\pi)$ with $\tilde{\pi}_{1,0} = \hat{\pi}_{1,0} = \pi_{1,0}$ and $\tilde{\pi}_1 = \hat{\pi}_1 = \pi_1$.

Now we are going to define $\tilde{\mathbf{J}}^{k+1}(\pi)$, $\hat{\mathbf{J}}^{k+1}(\pi)$ and $\mathbf{J}^{k+1}(\pi)$ together with mappings $\tilde{\pi}_{k+1,k}$: $\tilde{\mathbf{J}}^{k+1}(\pi) \to \tilde{\mathbf{J}}^{k}(\pi)$, $\hat{\pi}_{k+1,k} : \hat{\mathbf{J}}^{k+1}(\pi) \to \hat{\mathbf{J}}^{k}(\pi)$ and $\pi_{k+1,k} : \mathbf{J}^{k+1}(\pi) \to \mathbf{J}^{k}(\pi)$ by induction on $k \geq 1$. These are intended for non-holonomic, semi-holonomic and holonomic jet bundles in order. We let $\tilde{\pi}_{k+1} = \tilde{\pi}_k \circ \tilde{\pi}_{k+1,k}$, $\hat{\pi}_{k+1} = \hat{\pi}_k \circ \hat{\pi}_{k+1,k}$ and $\pi_{k+1} = \pi_k \circ \pi_{k+1,k}$. First we deal with $\tilde{\mathbf{J}}^{k+1}(\pi)$, which is defined to be $\mathbf{J}^1(\tilde{\pi}_k)$ with $\tilde{\pi}_{k+1,k} = (\tilde{\pi}_k)_{1,0}$.

Next we deal with $\hat{\mathbf{J}}^{k+1}(\pi)$, which is defined to be the subspace of $\mathbf{J}^1(\hat{\pi}_k)$ consisting of $\nabla_{\mathbf{x}}$'s with $\mathbf{x} = \nabla_{\mathbf{y}} \in \hat{\mathbf{J}}^{k}(\pi)$ abiding by the following condition:

$$\nabla_{\mathbf{x}} \text{ is } \hat{\pi}_{k,k-1}\text{-related to } \nabla_{\mathbf{y}}. \tag{1.4}$$

Finally we deal with $\mathbf{J}^{k+1}(\pi)$, which is defined to be the subspace of $\mathbf{J}^1(\pi_k)$ consisting of $\nabla_{\mathbf{x}}$'s with $\mathbf{x} = \nabla_{\mathbf{y}} \in \mathbf{J}^{k}(\pi)$ abiding by the following conditions:

1) $\nabla_{\mathbf{x}}$ is $\pi_{k,k-1}$-related to $\nabla_{\mathbf{y}}$.

2) Let $d_1, d_2 \in D$ and γ a microsquare on M with $\gamma(0,0) = \pi_k(\mathbf{x})$. Let it be that

$$\mathbf{z} = \nabla_{\mathbf{y}}(\gamma(\cdot,0))(d_1)$$

$$\mathbf{w} = \nabla_{\mathbf{y}}(\gamma(0,\cdot))(d_2)$$

$$\nabla_{\mathbf{z}} = \nabla_{\mathbf{x}}(\gamma(\cdot,0))(d_1)$$

$$\nabla_{\mathbf{w}} = \nabla_{\mathbf{x}}(\gamma(0,\cdot))(d_2)$$

Then we have

$$\nabla_{\mathbf{z}}(\gamma(d_1,\cdot))(d_2) = \nabla_{\mathbf{w}}(\gamma(\cdot,d_2))(d_1). \tag{1.5}$$

Now we define the infinite jet bundles $\tilde{\mathbf{J}}^{\infty}(\pi)$, $\hat{\mathbf{J}}^{\infty}(\pi)$ and $\mathbf{J}^{\infty}(\pi)$ of the bundle $\pi : E \to M$ as the inverse limits of the following sequences in order:

$$\tilde{\mathbf{J}}^0(\pi) \xleftarrow{\tilde{\pi}_{1,0}} \tilde{\mathbf{J}}^1(\pi) \xleftarrow{\tilde{\pi}_{2,1}} \tilde{\mathbf{J}}^2(\pi) \xleftarrow{\tilde{\pi}_{3,2}} \tilde{\mathbf{J}}^3(\pi) \ldots \tag{1.6}$$

$$\hat{\mathbf{J}}^0(\pi) \xleftarrow{\hat{\pi}_{1,0}} \hat{\mathbf{J}}^1(\pi) \xleftarrow{\hat{\pi}_{2,1}} \hat{\mathbf{J}}^2(\pi) \xleftarrow{\hat{\pi}_{3,2}} \hat{\mathbf{J}}^3(\pi) \ldots \tag{1.7}$$

$$\mathbf{J}^0(\pi) \xleftarrow{\pi_{1,0}} \mathbf{J}^1(\pi) \xleftarrow{\pi_{2,1}} \mathbf{J}^2(\pi) \xleftarrow{\pi_{3,2}} \mathbf{J}^3(\pi) \ldots \tag{1.8}$$

Therefore a point $\mathbf{x}$ of $\mathbf{J}^{\infty}(\pi)$ is represented by a sequence $\{\mathbf{x}_i\}_{i \geq k}$ with $x_i \in \mathbf{J}^k(\pi)$ and $\pi_{i+1,i}(\mathbf{x}_{i+1}) = \mathbf{x}_i$. We define a mapping $\pi_{\infty,n} : \mathbf{J}^{\infty}(\pi) \to \mathbf{J}^n(\pi)$ to be $\pi_{\infty,n}(\{\mathbf{x}_i\}_{i \geq k}) = \pi_{m,n}(\mathbf{x}_m)$ for some $m \geq \max\{k,n\}$. This definition is surely independent of our choice of m, and so is well defined. We define a mapping $\pi_{\infty} : \mathbf{J}^{\infty}(\pi) \to M$ to be $\pi_{\infty}(\{\mathbf{x}_i\}_{i \geq k}) = \pi_k(\mathbf{x}_k)$. These remarks and definitions apply to $\tilde{\mathbf{J}}^{\infty}(\pi)$ and $\hat{\mathbf{J}}^{\infty}(\pi)$ with due but obvious modifications.

We conclude this section by remarking that the assumption [$\mathbf{J}^{\infty}$II] of our previous paper (Nishimura [14]) is redundant, so long as we turn our attention from the semi-holonomic infinite jet bundle $\hat{\mathbf{J}}^{\infty}(\pi)$ to the holonomic infinite jet bundle $\mathbf{J}^{\infty}(\pi)$.

Now we are going to define a connection on the bundle $\pi_\infty : \mathbf{J}^\infty(\pi) \to M$ to be called the *Cartan connection* and to be denoted by ∇^∞:

$$\nabla^\infty(t, \mathbf{x})(d) = \{\nabla^{i+1}_{\mathbf{x}_i}(t)(d)\}_{i \geq k} \tag{1.9}$$

for any $t \in \mathbf{T}^1_{\pi_\infty(x)}(M)$, any $d \in D$ and any $x = \{\mathbf{x}_i\}_{i\geq k} = \{\nabla^{i+1}_{\mathbf{x}_i}\}_{i\geq k} \in \mathbf{J}^\infty(\pi)$.

The existence of the Cartan connection ∇^∞ on $\mathbf{J}^\infty(\pi)$ makes the infinite jet bundle $\mathbf{J}^\infty(\pi)$ prodigiously predominant over higher-order jet bundles $\mathbf{J}^i(\pi)$'s in theory and applications. The proof of the following theorem is simple, but we could not exaggerate its importance in the theory of infinite jet bundles.

Theorem 1.3. *For any microsquare γ on M, any $d_1, d_2 \in D$ and any $\mathbf{x} \in \mathbf{J}^\infty(\pi)$ with $\pi_\infty(\mathbf{x}) = \gamma(0,0)$, we have*

$$\nabla^\infty(\gamma(\cdot, d_2), \nabla^\infty(\gamma(0,\cdot), \mathbf{x})(d_2))(d_1) = \nabla^\infty(\gamma(d_1, \cdot), \nabla^\infty(\gamma(\cdot, 0), \mathbf{x})(d_1))(d_2). \tag{1.10}$$

Proof. Let it be that

$$\begin{aligned}
\mathbf{x} &= \{\mathbf{x}_i\}_{i\geq 0} = \{\nabla^i_{\mathbf{x}}\}_{i\geq 1} \\
\mathbf{y} &= \nabla^\infty(\gamma(0,\cdot), \mathbf{x})(d_2) = \{\mathbf{y}_i\}_{i\geq 0} = \{\nabla^{i+1}_{\mathbf{y}_i}\}_{i\geq 0} = \{\nabla^{i+1}_{\mathbf{x}_i}(\gamma(0,\cdot))(d_2)\}_{i\geq 0} \\
\mathbf{z} &= \nabla^\infty(\gamma(\cdot,0), \mathbf{x})(d_1) = \{\mathbf{z}_i\}_{i\geq 0} = \{\nabla^{i+1}_{\mathbf{z}_i}\}_{i\geq 0} = \{\nabla^{i+1}_{\mathbf{x}_i}(\gamma(\cdot,0))(d_1)\}_{i\geq 0}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\nabla^\infty(\gamma(\cdot, d_2), \nabla^\infty(\gamma(0,\cdot), \mathbf{x})(d_2))(d_1) \\
&= \nabla^\infty(\gamma(\cdot, d_2), \{\nabla^{i+1}_{\mathbf{x}_i}(\gamma(0,\cdot))(d_2)\}_{i\geq 0})(d_1) \\
&= \{\nabla^{i+1}_{\mathbf{y}_i}(\gamma(\cdot, d_2))(d_1)\}_{i\geq 0}.
\end{aligned} \tag{1.11}$$

On the other hand we have

$$\begin{aligned}
&\nabla^\infty(\gamma(d_1, \cdot), \nabla^\infty(\gamma(\cdot, 0), \mathbf{x})(d_1))(d_2) \\
&= \nabla^\infty(\gamma(d_1, \cdot), \{\nabla^{i+1}_{\mathbf{x}_i}(\gamma(\cdot, 0))(d_1)\}_{i\geq 0})(d_2) \\
&= \{\nabla^{i+1}_{\mathbf{z}_i}(\gamma(d_1, \cdot))(d_2)\}_{i\geq 0}.
\end{aligned} \tag{1.12}$$

Therefore it follows from (1.5), (1.11) and (1.12) that (1.10) obtains. □

All the results and their gimmicks of our previous paper (Nishimura [14]) persist through their holonomicization, except that the assumption [$\mathbf{J}^\infty$II] is now seen to be redundant.

2. Formal manifolds

We denote by $D(n)_k$ the set

$$\{(d^1, \dots, d^n) \in \mathbf{R}^n \mid \text{ any product of } k+1 \text{ or more of the } d^i\text{'s is } 0\}.$$

Trivially $D(n)_k \subset D(n)_{k+1}$. We denote by $D(n)_\infty$ the set-theoretic union of all $D(n)_k$'s. It is easy to see that

$$D(n)_k \times D(m)_1 \subset D(n+m)_{k+1} \tag{2.1}$$

$$D(n+m)_k \subset D(n)_k \times D(m)_k \tag{2.2}$$

so that

$$D(n+m)_\infty = D(n)_\infty \times D(m)_\infty \tag{2.3}$$

A subspace Y of a space X is called *étale* provided that for any commutative square

$$\begin{array}{ccc} 1 & \longrightarrow & K \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array} \tag{2.4}$$

with an infinitesimal space K and the canonical injection $Y \to X$ there is a unique diagonal fill-in $K \to$Y.

A space M is called a *formal* n-dimensional *manifold* if for each $\mathbf{x} \in M$ there exists an étale subspace of M containing $\mathbf{x}$ and isomorphic to $D(n)_\infty$, which is usually denoted schematically by $\mathbf{x} + D(n)_\infty$ and is called the ∞-*monad* around $\mathbf{x}$. Intuitively speaking, a formal n-dimensional manifold is a space which is of n-dimensional coordinates infinitesimally. We will often identify a point of M and its coordinates. In particular, it is not difficult to see that $\mathbf{x} + D(n)_k$ is independent of our choice of an infinitesimal coordinate system and is called the k-*monad* around $\mathbf{x}$.

A *formal bundle* is a mapping π of a formal $(n+m)$-dimensional manifold E into a formal n-dimensional manifold M such that for any $\mathbf{x} \in E$ there exist an étale subspace of E isomorphic to $D(n+m)_\infty = D(n)_\infty \times D(m)_\infty$ and containing $\mathbf{x}$ and an étale subspace of M isomorphic to $D(n)_\infty$ and containing $\pi(\mathbf{x})$ such that the restriction of π to the above étale subspace of E is represented by the canonical projection $D(n+m)_\infty \to D(n)_\infty$ mapping (x^i, u^p) to $(x^i)(1 \leq i \leq n,\ 1 \leq p \leq m)$, in which the formal bundle π is said to be of *dimension* (n, m). In other words the mapping $\pi : E \to M$ is a formal bundles iff it is *infinitesimally trivializable.*

Now we choose formal (n, m)-dimensional bundles $\pi : E \to M$ and $\pi' : E' \to M$ once and for all.

3. Kock's synthetic approach summarized

Kock's synthetic approach to jet bundles is based on the following fundamental theorem ([4, Theorem 4.3]).

Theorem 3.1. *Let $f : \mathbf{a} + D(n)_\infty \to X$, where $\mathbf{a} \in \mathbf{R}^n$ and X is any Euclidean $\mathbf{R}$-module. Then we have*

$$f(\mathbf{a} + (d_1, \ldots, d_n)) = \Sigma_{p=0}^{k} \frac{1}{p!} \Sigma_{j_1=1}^{n} \cdots \Sigma_{j_p=1}^{n} \mathbf{D}_{j_1} \cdots \mathbf{D}_{j_p} f(\mathbf{a}) d_{j_1} \cdots d_{j_p} \tag{3.1}$$

for any $(d_1, \ldots, d_n) \in D(n)_k$, where $\mathbf{D}_i$ denotes the i-th directional derivative.

This theorem has justified Kock [6] to define an *k-jet of* π *at* $\mathbf{a} \in M$, using an infinitesimal coordinate, as a mapping $\gamma : \mathbf{a} + D(n)_k \to E$ such that $\pi \circ \gamma$ is the identity mapping. We denote by $\mathcal{J}^k(\pi)$ the totality of k-jets of π. Since $D(n)_k \subset D(n)_{k+1}$, there is a canonical projection $\mathcal{J}^{k+1}(\pi) \to \mathcal{J}^k(\pi)$, which we denote by $\bar{\pi}_{k+1,k}$. We denote by $\bar{\pi}_k$ the canonical mapping $\mathcal{J}^k(\pi) \to$ M. It is easy to see that

Proposition 3.2. *$\mathcal{J}^k(\pi)$ is a formal manifold of dimension $n+m\binom{n+k}{k}$, and $\bar{\pi}_k : \mathcal{J}^k(\pi) \to M$ is a formal bundle of dimension $(n, m\binom{n+k}{k})$.*

Given a morphism f of bundles over M from π to π', we say that a k-jet γ of π at $\mathbf{a} \in M$ *is f-related to* a k-jet γ' of π' at $\mathbf{a}$ (*in the sense of Kock*) provided that $\gamma' = \pi \circ \gamma$.

4. The equivalence of the two approaches

The principal objective of this section is to give a classical coordinate description of $\mathbf{J}^k(\pi)$ after the manner of Saunders [16, Chaper 5], thereby establishing the equivalence between Kock's [6] and our synthetic approaches to jet bundles within the framework of formal bundles. Let us begin with 1-jets. We are going to define an isomorphism $\Psi^1_\pi : \mathcal{J}^1(\pi) \to \mathbf{J}^1(\pi)$. Let $\gamma \in \mathcal{J}^1(\pi)$, so that it is defined on $(x^i) + D(n)_1$. We have to define a 1-preconnection $\nabla_{\mathbf{x}}$ over the bundle π at $\mathbf{x} = \gamma((x^i))$, which we intend for $\Psi^1_\pi(\gamma)$. Let $t \in \mathbf{T}^1_{(x^i)}(M)$, so that it is of the form

$$d \in D \mapsto (x^i) + d(a^i), \tag{4.1}$$

where $(a^i) \in \mathbf{R}^n$. Since $d(a^i) \in D(n)_1$, we can define $\nabla_{\mathbf{x}}(t)$ to be $d \in D \mapsto \gamma((x^i) + d(a^i))$. It is not difficult to see that the mapping $\nabla_{\mathbf{x}} : t \in \mathbf{T}^1_{(x^i)}(M) \mapsto \nabla_{\mathbf{x}}(t)$ is indeed a 1-preconnection over π at $\mathbf{x}$, so that we have a mapping $\Psi^1_\pi : \mathcal{J}^1(\pi) \to \mathbf{J}^1(\pi)$ with $\Psi^1_\pi(\gamma) = \nabla_{\mathbf{x}}$. It is not difficult to see that

Theorem 4.1. *$\mathcal{J}^1(\pi)$ and $\mathbf{J}^1(\pi)$ can naturally be identified under Ψ^1_π.*

In particular, given $\mathbf{x} = (x^i, u^p) \in E$ and $\nabla_{\mathbf{x}} \in \mathbf{J}^1(\pi)$, there exist unique u^p_i's of $\mathbf{R}$ ($1 \leq i \leq n$, $1 \leq p \leq m$) such that $\nabla_{\mathbf{x}}(t)$ is of the form

$$d \in D \mid\rightarrow (x^i + da^i, u^p + d\Sigma^n_{j=1} a^j u^p_j) \tag{4.2}$$

for any $t \in \mathbf{T}^1_{(x^i)}(M)$ of the form (4.1). This gives a coordinate description $(x^i, u^p; u^p_i)$ of $\nabla_{\mathbf{x}}$, which we will usually identify with $\nabla_{\mathbf{x}}$ itself.

It is easy to see that

Proposition 4.2. *Let f be a morphism of bundles over M from $\pi : E \to M$ to $\pi' : E' \to M$. Let $(x^i) \in M$. Let $\gamma \in \mathcal{J}^1(\pi)$ and $\gamma' \in \mathcal{J}^1(\xi)$ with $f(\gamma(x^i)) = \gamma'((x^i))$. Let $\nabla_{\mathbf{x}} = \Psi^1_\pi(\gamma)$ and $\nabla_{\mathbf{y}} = \Psi^1_{\pi'}(\gamma')$, where $\mathbf{x} = \gamma((x^i))$ and $\mathbf{y} = \gamma'((x^i))$. Then γ is f-related to γ' in the sense of Kock iff $\nabla_{\mathbf{x}}$ is f-related to $\nabla_{\mathbf{y}}$ in the sense of Nishimura.*

Now we will deal with 2-jets. Let $t \in \mathbf{T}^1_{(x^i)}(M)$ be of the form (4.1). Let $\nabla_{\mathbf{x}} = (x^i, u^p; u^p_i; u^p_{;j}, u^p_{i;j}) \in \tilde{\mathbf{J}}^2(\pi) = \mathbf{J}^1(\pi_1)$ and $\nabla_{\mathbf{y}} = (x^i, u^p; u^p_i) \in \mathbf{J}^1(\pi)$ with $\mathbf{x} = (x^i, u^p; u^p_i)$ and $\mathbf{y} = (x^i, u^p)$, so that $\nabla_{\mathbf{y}}(t)$ is of the form (4.2) while $\nabla_{\mathbf{x}}(t)$ is of the form

$$d \in D \mapsto (x^i + da^i, u^p + d\Sigma^n_{j=1} a^j u^p_{;j}; u^p_i + d\Sigma^n_{j=1} a^j u^p_{i;j}), \tag{4.3}$$

for which we have

Proposition 4.3. $(x^i, u^p; u^p_i; u^p_{;j}, u^p_{i;j}) \in \hat{\mathbf{J}}^2(\pi)$ *iff* $u^p_i = u^p_{;i}$ *for all* $1 \leq p \leq m$ *and all* $1 \leq i \leq n$.

Proof. It follows from (4.2) and (4.3) that $\nabla_{\mathbf{x}}$ is $\pi_{1,0}$-related to $\nabla_{\mathbf{y}}$ iff

$$u^p + d\Sigma^n_{j=1} a^j u^p_{;j} = u^p + d\Sigma^n_{j=1} a^j u^p_j \tag{4.4}$$

for all $(a^1, \dots, a^n) \in \mathbf{R}^n$ and all $d \in D$, which is tantamount to saying that

$$u^p_{;j} = u^p_j \tag{4.5}$$

for all $1 \leq p \leq m$ and all $1 \leq j \leq n$. This completes the proof. $\square$

Thus the coordinate $(x^i, u^p; u^p_i; u^p_{;j}, u^p_{i;j}) \in \hat{\mathbf{J}}^2$ can be simplified to $(x^i, u^p, u^p_i, u^p_{i;j})$, so that we have

Corollary 4.4. $\hat{\mathbf{J}}^2(\pi)$ *is a formal manifold, and* $\hat{\pi}_2 : \hat{\mathbf{J}}^2(\pi) \to M$ *is a formal bundle.*

Now we take a step forward.

Proposition 4.5. *Let* $(x^i, u^p, u^p_i, u^p_{i;j}) \in \hat{\mathbf{J}}^2(\pi)$. *Then* $(x^i, u^p, u^p_i, u^p_{i;j}) \in \mathbf{J}^2(\pi)$ *iff* $u^p_{i;j} = u^p_{j;i}$ *for all* $1 \leq p \leq m$ *and all* $1 \leq i, j \leq n$.

Proof. Let $\gamma \in \mathbf{T}^2_{(x^i)}(M)$. Then γ is of the form

$$\begin{aligned}
&(d_1, d_2) \in D^2 \mapsto \\
&\quad (x^1, \dots, x^n) + d_1(a^1, \dots, a^n) + d_2(b^1, \dots, b^n) + d_1 d_2(c^1, \dots, c^n) \\
&= (x^1 + d_1 a^1, \dots, x^n + d_1 a^n) + d_2(b^1 + d_1 c^1, \dots, b^n + d_1 c^n) \\
&= (x^1 + d_2 b^1, \dots, x^n + d_2 b^n) + d_1(a^1 + d_2 c^1, \dots, a^n + d_2 c^n).
\end{aligned}$$

Let $\nabla_{\mathbf{x}_1} = (x^i, u^p, u^p_i, u^p_{i;j})$, $\nabla_{\mathbf{x}_0} = \mathbf{x}_1 = (x^i, u^p, u^p_i)$ and $\mathbf{x}_0 = (x^i, u^p)$. Then we have

$$\begin{aligned}
\nabla_{\mathbf{x}_0}(\gamma(\cdot, 0))(d_1) &= (x^i + d_1 a^i, u^p + d_1 \Sigma^n_{j=1} a^j u^p_j) \\
\nabla_{\mathbf{x}_1}(\gamma(\cdot, 0))(d_1) &= (x^i + d_1 a^i, u^p + d_1 \Sigma^n_{j=1} a^j u^p_j, u^p_i + d_1 \Sigma^n_{j=1} a^j u^p_{i;j}).
\end{aligned}$$

Let $\mathbf{y}_0 = \nabla_{\mathbf{x}_0}(\gamma(\cdot, 0))(d_1)$ and $\nabla_{\mathbf{y}_0} = \nabla_{\mathbf{x}_1}(\gamma(\cdot, 0))(d_1)$. Then we have

$$\begin{aligned}
&\nabla_{\mathbf{y}_0}(\gamma(d_1, \cdot))(d_2 \\
&= (x^i + d_1 a^i + d_2 b^i + d_1 d_2 c^i, u^p + d_1 \Sigma_{j=1}^n a^j u_j^p \\
&\quad + d_2 \Sigma_{k=1}^n (b^k + d_1 c^k)(u_k^p + d_1 \Sigma_{j=1}^n a^j u_{k;j}^p)) \\
&= (x^i + d_1 a^i + d_2 b^i + d_1 d_2 c^i, u^p + d_1 \Sigma_{j=1}^n a^j u_j^p \\
&\quad + d_2 \Sigma_{k=1}^n b^k u_k^p + d_1 d_2 \Sigma_{k=1}^n c^k u_k^p + d_1 d_2 \Sigma_{j=1}^n \Sigma_{k=1}^n a^j b^k u_{k;j}^p).
\end{aligned} \tag{4.6}$$

On the other hand we have

$$\begin{aligned}
\nabla_{\mathbf{x}_0}(\gamma(0, \cdot))(d_2) &= (x^i + d_2 b^i, u^p + d_2 \Sigma_{j=1}^n b^j u_j^p) \\
\nabla_{\mathbf{x}_1}(\gamma(0, \cdot))(d_2) &= (x^i + d_2 b^i, u^p + d_2 \Sigma_{j=1}^n b^j u_j^p, u_i^p + d_2 \Sigma_{j=1}^n b^j u_{i;j}^p).
\end{aligned}$$

Let $\mathbf{z}_0 = \nabla_{\mathbf{x}_0}(\gamma(0, \cdot))(d_2)$ and $\nabla_{\mathbf{z}_0} = \nabla_{\mathbf{x}_1}(\gamma(0, \cdot))(d_2)$. Then we have

$$\begin{aligned}
&\nabla_{\mathbf{z}_0}(\gamma(\cdot, d_2))(d_1) = \\
&= (x^i + d_1 a^i + d_2 b^i + d_1 d_2 c^i, u^p + d_2 \Sigma_{j=1}^n b^j u_j^p + \\
&\quad d_1 \Sigma_{k=1}^n (a^k + d_2 c^k)(u_k^p + d_2 \Sigma_{j=1}^n b^j u_{k;j}^p)) \\
&= (x^i + d_1 a^i + d_2 b^i + d_1 d_2 c^i, u^p + d_1 \Sigma_{j=1}^n a^j u_j^p + \\
&\quad d_2 \Sigma_{k=1}^n b^k u_k^p + d_1 d_2 \Sigma_{k=1}^n c^k u_k^p + d_1 d_2 \Sigma_{j=1}^n \Sigma_{k=1}^n a^k b^j u_{k;j}^p) \\
&= (x^i + d_1 a^i + d_2 b^i + d_1 d_2 c^i, u^p + d_1 \Sigma_{j=1}^n a^j u_j^p + \\
&\quad + d_2 \Sigma_{k=1}^n b^k u_k^p + d_1 d_2 \Sigma_{k=1}^n c^k u_k^p + d_1 d_2 \Sigma_{j=1}^n \Sigma_{k=1}^n a^j b^k u_{j;k}^p).
\end{aligned} \tag{4.7}$$

Therefore it follows from (4.6) and (4.7) that

$$\nabla_{\mathbf{y}_0}(\gamma(d_1, \cdot))(d_2) = \nabla_{\mathbf{z}_0}(\gamma(\cdot, d_2))(d_1)$$

for all $\gamma \in \mathbf{T}^2_{(x^i)}(M)$ and all $(d_1, d_2) \in D^2$ iff $u_{j;k}^p = u_{k;j}^p$ for all $1 \leq p \leq m$ and all $1 \leq j, k \leq n$. This completes the proof. □

Corollary 4.6. $\mathbf{J}^2(\pi)$ *is a formal manifold of dimension* $n + m\binom{n+2}{2}$, *and* $\pi_2 : \mathbf{J}^2(\pi) \to M$ *is a formal bundle of dimension* $(n, m\binom{n+2}{2})$.

With due regard to Proposition 4.5 we have

Theorem 4.7. $\mathcal{J}^2(\pi)$ *and* $\mathbf{J}^2(\pi)$ *can naturally be identified.*

The explicit construction of the isomorphism $\Psi_\pi^2 : \mathcal{J}^2(\pi) \to \mathbf{J}^2(\pi)$ goes as follows. Let $\gamma \in \mathcal{J}^2(\pi)$, so that it is defined on $(x^i) + D(n)_2$. We have to define a preconnection $\nabla_{\mathbf{x}}$ over the bundle $\pi_1 : \mathbf{J}^1(\pi) \to M$ at $\mathbf{x} = \Psi_\pi^1(\gamma|_{(x^i)+D(n)_1})$, which we intend for $\Psi_\pi^2(\gamma)$. Let $t \in \mathbf{T}^1_{(x^i)}(M)$, so that it is of the form $d \in D \mapsto (x^i) + d(a^i)$, where $(a^i) \in \mathbf{R}^n$. Since $d(a^i) \in D(n)_1$ and $D(n)_1 + D(n)_1 \subset D(n)_2$, we can define $\nabla_{\mathbf{x}}(t)$ to be $d \in D \mapsto \Psi_\pi^1(\gamma|_{(x^i)+d(a^i)+D(n)_1})$. It is not difficult to see that the mapping $\nabla_{\mathbf{x}} : t \in \mathbf{T}^1_{(x^i)}(M) \mapsto \nabla_{\mathbf{x}}(t)$ is indeed a 1-preconnection over π_1 at $\mathbf{x}$, so that we have the desired isomorphism $\Psi_\pi^2 : \mathcal{J}^2(\pi) \to \mathbf{J}^2(\pi)$ with $\Psi_\pi^2(\gamma) = \nabla_{\mathbf{x}}$.

The above argument leading to Theorem 4.7 can easily be generalized so as to yield

Theorem 4.8. *$\mathcal{J}^k(\pi)$ and $\mathbf{J}^k(\pi)$ can naturally be identified for any natural number k.*

Corollary 4.9. *$\mathbf{J}^k(\pi)$ is a formal manifold of dimension $n + m\binom{n+k}{k}$, and $\pi_k : \mathbf{J}^k(\pi) \to M$ is a formal bundle of dimension $(n, m\binom{n+k}{k})$.*

The isomorphism Ψ_π^k from $\mathcal{J}^k(\pi)$ to $\mathbf{J}^k(\pi)$ is explicitly constructed by induction on k. We have already constructed $\Psi_\pi^1 : \mathcal{J}^1(\pi) \to \mathbf{J}^1(\pi)$ and $\Psi_\pi^2 : \mathcal{J}^2(\pi) \to \mathbf{J}^2(\pi)$. Now we proceed inductively. The explicit construction of the isomorphism $\Psi_\pi^{k+1} : \mathcal{J}^{k+1}(\pi) \to \mathbf{J}^{k+1}(\pi)$ goes as follows. Let $\gamma \in \mathcal{J}^{k+1}(\pi)$, so that it is defined on $(x^i) + D(n)_{k+1}$. We have to define a preconnection $\nabla_{\mathbf{x}}$ over the bundle $\pi_k : \mathbf{J}^k(\pi) \to M$ at $\mathbf{x} = \Psi_\pi^k(\gamma|_{(x^i)+D(n)_k})$, which we intend for $\Psi_\pi^{k+1}(\gamma)$. Let $t \in \mathbf{T}^1_{(x^i)}(M)$, so that it is of the form $d \in D \mapsto (x^i) + d(a^i)$, where $(a^i) \in \mathbf{R}^n$. Since $d(a^i) \in D(n)_1$ and $D(n)_1 + D(n)_k \subset D(n)_{k+1}$, we can define $\nabla_{\mathbf{x}}(t)$ to be $d \in D \mapsto \Psi_\pi^k(\gamma|_{(x^i)+d(a^i)+D(n)_k})$. It is not difficult to see that the mapping $\nabla_{\mathbf{x}} : t \in \mathbf{T}^1_{(x^i)}(M) \mapsto \nabla_{\mathbf{x}}(t)$ is indeed a preconnection over π_k at $\mathbf{x}$, so that we have the desired isomorphism $\Psi_\pi^{k+1} : \mathcal{J}^{k+1}(\pi) \to \mathbf{J}^{k+1}(\pi)$ with $\Psi_\pi^{k+1}(\gamma) = \nabla_{\mathbf{x}}$.

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