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The Densest Packing of 13 Congruent Circles in a Circle

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Abstract. The densest packings of n congruent circles in a circle are known for $n \leq 12$ and $n = 19$. In this article we examine the case of 13 congruent circles. We show that the optimal configurations are identical to Kravitz's conjecture. We use a technique developed from a method of Bateman and Erdős, which proved fruitful in investigating the cases $n = 12$ and 19.

MSC 2000: 52C15
Keywords: circle packing, density, optimal packing

1. Preliminaries and Results

We shall denote the points of the Euclidean plane $\mathbf{E}^2$ by capitals, sets of points by script capitals, and the distance of two points by $d(P,Q)$. We use PQ for the line through P, Q, and $\overline{PQ}$ for the segment with endpoints P, Q. $\angle POQ$ denotes the angle determined by the three points P,O,Q in this order. $C(r)$ means the closed disk of radius r with center O while $c(r)$ denotes its perimeter. By an annulus $r < \rho \leq s$ we mean all points P such that $r < d(P,O) \leq s$. We utilize the linear structure of $\mathbf{E}^2$ by identifying each point P with the vector $\vec{OP}$, where O is the origin. For a point P and a vector $\vec{a}$ by $P + \vec{a}$ we always mean the vector $\vec{OP} + \vec{a}$.

In this paragraph we give a brief account of the history of the subject. This review is based on the doctoral dissertation of Melissen [13]. The problem of finding the densest packing of congruent circles in a circle arose in the 1960s. The question was to find the smallest circle in which we can pack n congruent unit circles, or equivalently, the smallest circle in which we can place n points with mutual distances at least 1. Dense circle packings were first given by Kravitz [11] for $n = 2, \dots, 16$. Pirl [14] proved that these arrangements are optimal for $n \leq 9$ and he also found the optimal configuration for $n = 10$. Pirl also conjectured dense configurations for $11 \leq n \leq 19$. For $n \leq 6$ proofs were given independently by Graham [3]. A proof for $n = 6$ and 7 was also given by Crilly and Suen [4]. Later, improvements were presented by Goldberg [8] for $n = 14, 16$, and 17. He also found a new packing with 20 circles. In 1975 Reis [15] used a mechanical argument to generate good packings up to 25 circles. Recently, Graham et al. [9, 10], using computers, established packings with more than 100 circles and improved the packing of 25 circles. In 1994 Melissen [12] proved Pirl's conjecture for $n = 11$ and the author [6, 7] proved it for $n = 19$ and $n = 12$. The problem of finding the densest packing of equal circles in a circle is also mentioned as an unsolved problem in the book of Croft, Falconer and Guy [5]. Packings of congruent circles in the hyperbolic plane were treated by K. Bezdek [2]. Analogous results of packing n equal circles in an equilateral triangle and square can be traced down in the doctoral dissertation of Melissen [13].

In this article we shall find the optimal configurations for $n = 13$. We shall prove the following theorem.

Theorem 1. *The radius of the smallest circle in which we can pack* 13 *points with mutual distances at least* 1 *is* $r_{13} = (2 \sin 36°)^{-1} = \frac{1+\sqrt{5}}{2}$. *The* 13 *points in* $C(r_{13})$ *must form one of the configurations shown in Figure* 1.

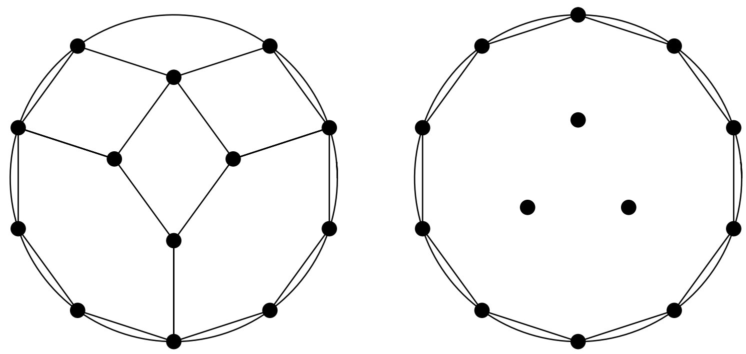

Figure 1. The optimal configurations for $n = 13$

In Figure 1 two points are connected with a straight line segment precisely when their distance is 1. Note that in the second configuration in Figure 1 the three points in the middle are free to move. On the other hand, the first configuration is rigid; no point can move. We shall discuss the first configuration in greater detail subsequently.

We shall prove Theorem 1 in the following way. We shall divide $C(r_{13})$ into a smaller circle $C(1)$ and an annulus $1 < \rho \leq r_{13}$. We are going to show that there can be at most 4 points in $C(1)$ and at most 10 points in $1 < \rho \leq R$. Next we shall prove that if there are 4 points in $C(1)$, then they must form the first configuration in Figure 1. If there are 3 points in $C(1)$, then they must be in the second configuration shown in Figure 1.

In the course of our proof we shall use the following two statements. Lemma 1, which is slightly modified, originates from Bateman and Erdős [1]. Lemma 2 was used in [6, 7].

Lemma 1. [1] *Let $r, s, (s \geq \frac{1}{2})$ be two positive real numbers and suppose that we have two points P and Q which lie in the annulus $r \leq \rho \leq s$ and which have mutual distance at least 1. Then the minimum $\overline{\phi}(r,s)$ of the angle $\angle POQ$ has the following values:*

$$\overline{\phi}(r,s) = \arccos \frac{s^2 + r^2 - 1}{2rs}, \quad \textit{if} \quad 0 < s - 1 \leq r \leq s - 1/s;$$

$$\overline{\phi}(r,s) = 2 \arcsin \frac{1}{2s}, \quad \textit{if} \quad 0 < s - 1/s \leq r \leq s \quad \textit{or} \quad s \leq 1.$$

Lemma 2. [6] *Let $\mathcal{S}$ be a set of n $(n \geq 2)$, points in the plane and $\mathcal{C}$ the smallest circle containing $\mathcal{S}$. Let $\vec{a}$ be a vector. There exist two points P_1, P_2 on the boundary of $\mathcal{S}$, such that $d(P_1 + \vec{a}, O) + d(P_2 + \vec{a}, O) \geq 2r$, where r is the radius and O is the center of $\mathcal{C}$.*

2. Proofs

We will use the following notation subsequently $\phi(x,y) = \arccos \frac{x^2+y^2-1}{2xy}$. Notice that $r_{13} - \frac{1}{r_{13}} = 1$. It is known that $r_7 = 1$ and the 7 points can be packed in $C(1)$ in a unique way: one point is at O and the other 6 points form a regular hexagon of unit side length with vertices on $c(1)$. Since $\phi(r_{13}, r_{13}) = 36°$, any point from the annulus $1 < \rho \leq r_{13}$ creates a central angle of at least 72° if it is situated in an angular sector determined by two consecutive points on $c(1)$. This is not possible because such an angular sector must have a 60° central angle.

The argument for 6 points is similar. It is known that $r_6 = 1$ and the 6 points either form a regular hexagon of unit side length with vertices on $c(1)$, or a pentagon with vertices on $c(1)$ and a sixth point at O. In the first case the argument is the same as for 7 points. In the second case note that there cannot be 3 or more points from the annulus $1 < \rho \leq r_{13}$ in an angular sector determined by two consecutive points on $c(1)$ because $4\phi(r_{13}, r_{13}) + 4\phi(1,1) = 4 \cdot 36° + 4 \cdot 60° = 384°$. This means that two angular sectors must contain exactly two points each from the annulus $1 < \rho \leq r_{13}$. Then $6\phi(r_{13}, r_{13}) + 3\phi(1,1) = 6 \cdot 36° + 3 \cdot 60° = 396°$.

Also note that, since $\phi(r_{13}, r_{13}) = 36°$, there cannot be more than 10 points in the annulus $1 < \rho \leq r_{13}$. We shall use $d_i = d(P_i, O),\ i = 1, \ldots, 13$ subsequently. Now we proceed to prove the following lemma.

Lemma 3. *There cannot be exactly 5 points in $C(1)$.*

Proof. Suppose, on the contrary, that there are exactly 5 points in $C(1)$. We know that $r_5 = (2 \cos 54°)^{-1} = 0.85065\ldots$ and the minimal radius is realized by a regular pentagon of unit side length. According to Lemma 2 there must be two points, say P_1 and P_2, in $C(1)$ such that $d_1 \geq d_2$ and $d_1 + d_2 \geq 2r_5$.

Claim 1. The points $P_2, \ldots, P_5$ cannot all be in $C(0.812)$.

Assume, on the contrary, that $P_2, \ldots, P_5$ are in $C(0.812)$. The sum of the 5 consecutive central angles determined by the points in $C(1)$ is $3\phi(0.812, 0.812) + 2\phi(1, 0.812) = 3 \cdot 76°.014 + 2 \cdot 66°.046 = 360°.13\ldots$. Therefore we may suppose that $d_2 \geq 0.812$.

Note that the function $2\phi(d_1, r_{13}) + 2\phi(d_2, r_{13})$ takes on its minimum when $d_1 = 2r_5 - 0.812$ and $d_2 = 0.812$; the minimum is $2\phi(0.8893, r_{13}) + 2\phi(0.812, r_{13}) = 2 \cdot 33°.17 + 2 \cdot 29°.924 = 126.188°$.

Claim 2. *The points P_3, P_4, P_5 cannot all be in $C(0.73)$.*

Assume, on the contrary, that P_3, P_4, P_5 are all in $C(0.73)$. If P_1 *and* P_2 *are adjacent*, then the sum of the five consecutive central angles determined by the points in $C(1)$ is $\phi(1,1) + 2\phi(0.73, 0.73) + 2\phi(1, 0.73) = 60° + 2 \cdot 86°.46 + 2 \cdot 68°.592 = 370° \ldots$.

If P_1 *and* P_2 *are not adjacent*, then the sum is $\phi(0.73, 0.73) + 4\phi(1, 0.73) = 360°.8 \ldots$. Therefore one of P_3, P_4, P_5, say P_3, is such that $d_3 \geq 0.73$. Let us also observe that $d_3 \leq 0.762$. If $d_3 > 0.762$ then the sum of the 8 central angles determined by the points in the annulus $1 < \rho \leq r_{13}$ is $180° + 126°.188 + 2\phi(0.762, r_{13}) = 306°.188 + 2 \cdot 26°.92 = 360°.028 \ldots$.

Claim 3. *The points P_4 and P_5 cannot be both in $C(0.71)$.*

Assume, on the contrary, that both P_4 and P_5 are in $C(0.71)$. We must consider two cases.

If P_1 and P_2 are adjacent, then the sum of the five consecutive central angles determined by the points in $C(1)$ is either $\phi(1,1)+\phi(1,0.762)+\phi(0.71,0.71)+\phi(0.762,0.71)+\phi(0.71,1) = 60° + 67°.6 + 89°.5 + 85°.5 + 69°.2 = 371° \ldots$ or $\phi(1,1) + 2\phi(0.762, 0.71) + 2\phi(0.71, 1) = 60° + 2 \cdot 85°.5 + 2 \cdot 69°.2 = 369° \ldots$.

If P_1 *and* P_2 *are not adjacent*, then the sum is either $2\phi(1, 0.762) + \phi(0.71, 0.71) + 2\phi(0.71, 1) = 2 \cdot 67°.6 + 89°.5 + 2 \cdot 69°.2 = 363° \ldots$ or $\phi(1, 0.762) + \phi(0.71, 0.762) + 3\phi(0.71, 1) = 67°.6 + 85°.5 + 3 \cdot 69°.2 = 360°.7 \ldots$. Therefore one of P_4 and P_5, say P_4, is such that $d_4 \geq 0.71$.

Finally, the sum of the 8 central angles determined by consecutive points in the annulus $1 < \rho \leq r_{13}$ is $4 \cdot 36° + 2\phi(d_1, r_{13}) + 2\phi(d_2, r_{13}) + 2\phi(0.73, r_{13}) + 2\phi(0.71, r_{13}) = 144° + 126°.188 + 48°.8 + 45° = 363° \ldots$. □

Thus we know that there are either 4 or 3 points in $C(1)$. We shall examine the first case in detail subsequently. In the second case there must 10 points in the annulus $1 < \rho \leq r_{13}$, and since $\phi(r_{13}, r_{13}) = 36°$, these points must be all on $c(r_{13})$. Therefore the 13 points must form the second configuration shown in Figure 1. The 10 points on $c(r_{13})$ are in a unique position up to rotation. The positions of the 3 points in $C(1)$ are not unique.

Proposition 1. *Let $f(r) = \phi(r, r_{13}) + \phi(r, s)$ and $\frac{\sqrt{2}}{2} \leq s \leq 1$ fixed. If $r_{13} - 1 \leq r \leq \min\{0.77, s\}$, then $f(r)$ is an increasing function of r.*

Proof. We shall evaluate the derivative of $f(r)$. Our goal is to show that

$$f'(r) = \frac{\frac{r_{13} - r^2}{r}}{\sqrt{(2rr_{13})^2 - (r^2 + r_{13})^2}} - \frac{\frac{r^2 - s^2 + 1}{r}}{\sqrt{(2rs)^2 - (s^2 + r^2 - 1)^2}} > 0.$$

After rearrangement of the terms we obtain

$$\sqrt{\frac{(2rs)^2 - (s^2 + r^2 - 1)^2}{(2rr_{13})^2 - (r^2 + r_{13})^2}} > \frac{r^2 - s^2 + 1}{r_{13} - r^2}.$$

Notice that $\frac{r^2 - s^2 + 1}{r_{13} - r^2}$ takes on its maximum if $r = s = 0.77$, and the maximum is less than 1. On the left hand side we can decrease $(2rs)^2 - (s^2 + r^2 - 1)^2$ if we replace s by r because $(2rs)^2 - (s^2 + r^2 - 1)^2$ is an increasing function of s. Now, we are going to show that

$$\frac{(2r^2)^2 - (2r^2 - 1)^2}{(2rr_{13})^2 - (r^2 + r_{13})^2} > 1.$$

This inequality is equivalent to

$$(2r^2)^2 - (2r^2-1)^2 - ((2rr_{13})^2 - (r^2+r_{13})^2) = r^4 - 2r_{13}r^2 + r_{13} > 0.$$

The polynomial on the left hand side is zero if $r = \sqrt{r_{13}-1} = 0.78\ldots$, and for $r \in [r_{13}-1, 0.77]$ it is non-negative. Thus the inequality holds. $\square$

Let the 4 points in $C(1)$ be labeled as $P_1, \ldots, P_4$ in clockwise direction such that d_1 is the largest of $d_i, i = 1, \ldots, 4$. Let us label the points in the annulus $1 < \rho \leq r_{13}$ by $P_5, \ldots, P_{13}$ in a clockwise direction such that P_5 is in P_1OP_2 and P_5 is adjacent to P_1O.

Proposition 2. *In each sector determined by two adjacent points in $C(1)$ there must be at least 2 points from the annulus $1 < \rho \leq r_{13}$.*

Proof. First, note that $d_i \leq 0.77, i = 2, 3, 4$ because $4\phi(0.77, r_{13}) + 7\cdot 36° = 4\cdot 27°.464 + 7\cdot 36° = 361°.8\ldots$. Suppose, on the contrary, that there is one sector which contains only one point from the annulus $1 < \rho \leq r_{13}$.

Case 1. This sector is P_1OP_2.

We may assume that $d_2 \leq d_1$. The sum of the 9 central angles determined by consecutive points in the annulus is not less than $7\cdot 36° + \phi(d_1, r_{13}) + \phi(d_1, d_2) + \phi(d_2, r_{13})$. By Proposition 1 the total angle is not less than $252° + \phi(d_1, r_{13}) + \phi(d_1, r_{13}-1) + \phi(r_{13}-1, r_{13})$. This function takes on its minimum at $d_1 = 1$, where it is exactly 360°. Since P_1OP_2 contains only P_5, we know that $\angle P_2OP_6 > 0°$. Thus the total angle exceeds 360°.

Case 2. This sector is P_4OP_1.

We may assume that $d_4 \leq d_1$ and we may repeat the previous argument and obtain the same formula for the total angle.

Case 3. This sector is either P_2OP_3 or P_3OP_4.

Both cases work the same way so we assume that we are talking about P_2OP_3. We may suppose that $d_2 \geq d_3$. The sum of the 9 central angles determined by the points in the annulus is not less than $7\cdot 36° + \phi(d_2, r_{13}) + \phi(d_2, d_3) + \phi(d_3, r_{13})$. By Proposition 1 the total angle is not less than $252° + \phi(d_2, r_{13}) + \phi(d_2, r_{13}-1) + \phi(r_{13}-1, r_{13})$. This function takes on its minimum at $d_2 = r_{13}-1$, where it is 360°. Since $\angle P_3OP_i > 0°$, where P_i is the point in the annulus adjacent to P_3 and not in P_2OP_3, the total angle exceeds 360°. $\square$

We shall examine the positions of the $P_1, \ldots, P_4$. We shallprove that d_1 has to be equal to 1. Let C_i denote the unit disk centered at P_i. Let $\mathcal{R}$ be the region of $C(r_{13})$ that is not covered by $C_i, i = 1, 5, \ldots, 13$. This is where P_2, P_3, P_4 can be situated. The line P_1O cuts $\mathcal{R}$ into two subregions, $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ as shown in Figure 2. We shall show that if $0.745 \leq d_1 < 1$, then $\mathrm{diam}(\mathcal{R}_1) < 1$. Let $Q_6 = c_1 \cap c_6 \cap C(1), Q_7 = c_6 \cap c_7 \cap C(1), Q_8 = c_8 \cap c_9 \cap C(1), Q_0 = P_1O \cap c_9 \cap C(1), Q_1 = c_1 \cap P_1O \cap C(1)$.

Lemma 4. *For a fixed value of $d_1 \in [\frac{\sqrt{2}}{2}, 1]$ the arc of c_1 between Q_1 and Q_6 is the longest if $\angle P_1OP_6 = 36° + \psi$, where $\psi = \phi(d_1, r_{13})$.*

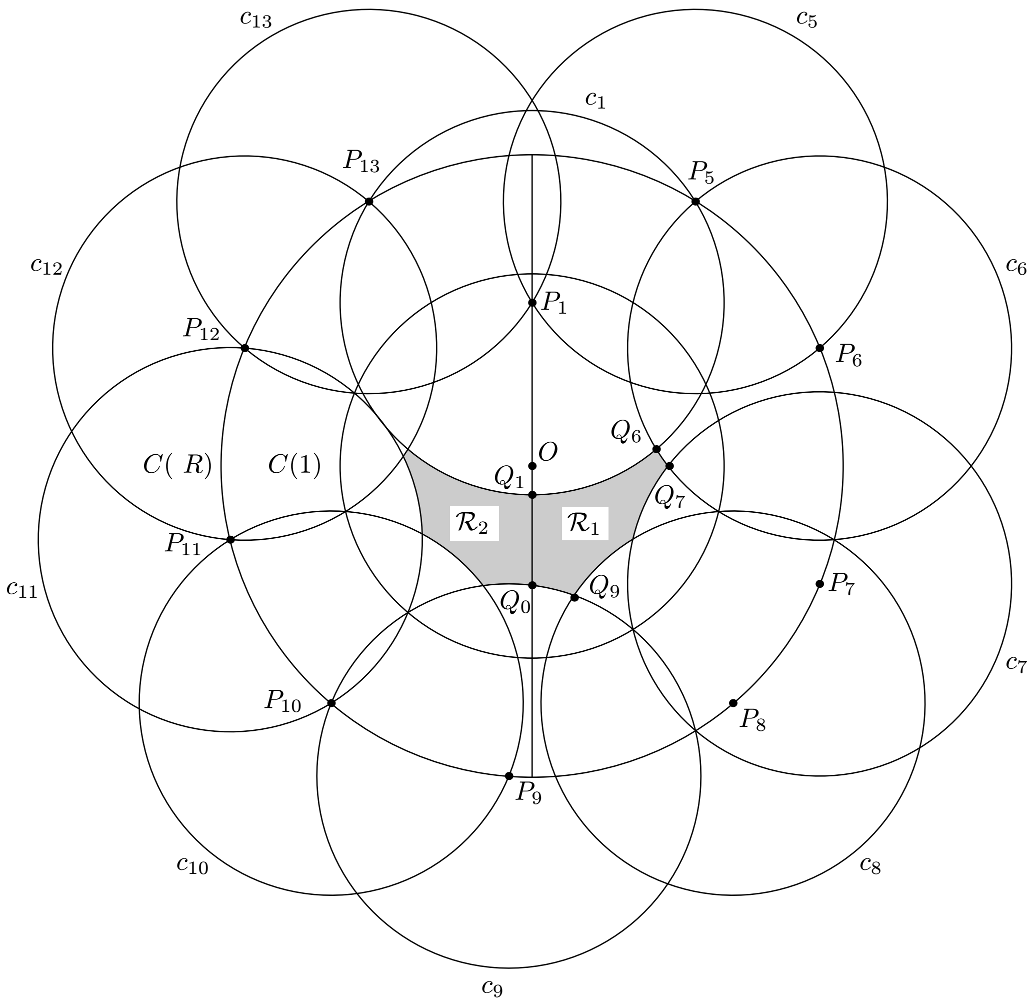

Figure 2.

Proof. Notice that of two unit circles centered on $C(r_{13})$ the one whose center makes the smaller central angle with P_1O provides the longer arc on c_1 if their intersection is in c_1. For a fixed d_1, $36° + \psi \leq \angle P_1OP_6 \leq 108° - \psi$. It is enough to show that the intersection of the two unit circles in the extreme positions is in c_1, see Figure 3. Note that for this intersection point T, $\angle P_1OT = 72°$ holds for all values of d_1. Let $S = c_1 \cap OT$ and let $s = d(S, O)$. It is enough to show that $d^2(S, P_6) = r_{13}^2 + s^2 - 2r_{13}s\cos(36° - \psi) \leq 1$. The equation $r_{13}^2 + s^2 - 2r_{13}s\cos(36° - \psi) = 1$ can be written, after some transformations, as a polynomial equation for d_1 as follows:

$(3\sqrt{5}+7)x^{12}+(-31-13\sqrt{5})x^{10}+(6\sqrt{5}+18)x^8+(12\sqrt{5}+22)x^6+(45+103\sqrt{5})x^4+(-166-74\sqrt{5})x^2+47+21\sqrt{5}=0$

The polynomial on the left hand side of the equation has only one root in $[\frac{\sqrt{2}}{2}, 1]$, and this root is 1. □

Now we choose the positions of $P_1, \ldots, P_{13}$ such that they maximize $\text{diam}(\mathcal{R}_1)$. We shall assume that $P_5, \ldots, P_{13} \in c(r_{13})$. We shall also assume that $\angle P_1OP_9 = 216° - \psi$. This ensures

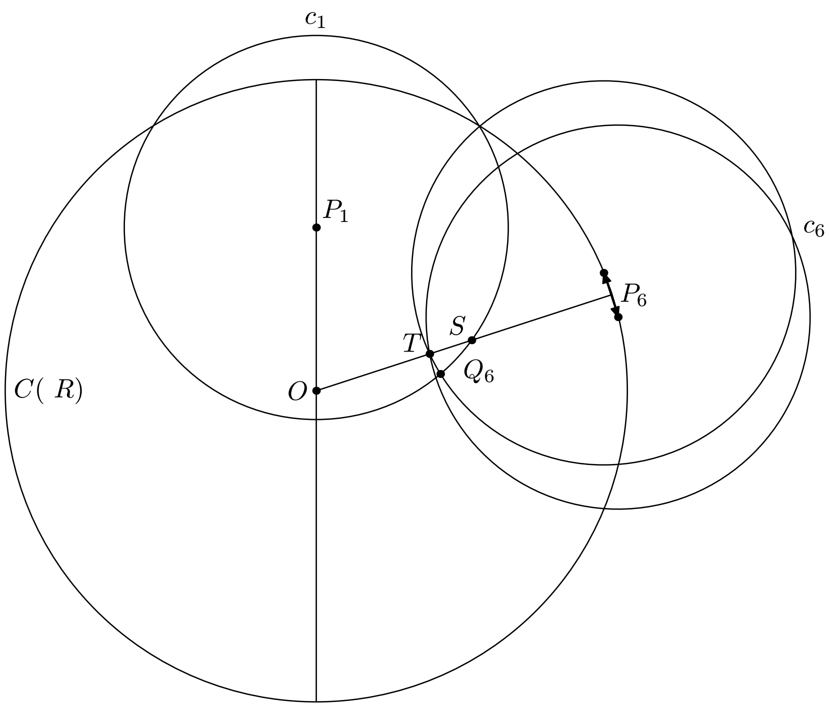

Figure 3.

that the part of P_1O which bounds $\mathcal{R}_1$ is maximal in length. Moreover, let $\angle P_1OP_7 = 144° - \psi$ and $\angle P_1OP_8 = 108° + \psi$. This guarantees that the arcs of c_7, c_9, and $\overline{Q_0Q_1}$ are the longest possible. Notice that in such position $d(P_7, P_8)$ may become less than 1.

Lemma 5. *The diameter of* $\mathcal{R}_1$ *is less than* 1 *if* $d_1 \in [0.745, 1)$.

Proof. Notice that $\angle P_1OQ_7 = 90°, \angle P_1OQ_9 = 162°$, and $d(Q_7, O) = d(Q_9, O)$ is a decreasing function of d_1. Also, $d(Q_0, O)$ is a monotonically decreasing function of d_1. $\mathcal{R}_1$ is bounded by arcs of $c_1, c_6, \ldots, c_9$, and the segment $\overline{Q_0Q_1}$. Its vertices are $Q_0, Q_1, Q_6, \ldots Q_9$ as shown in Figure 2.

Under these circumstances the diameter of $\mathcal{R}_1$ can only be realized by one of the segments $\overline{Q_0Q_6}, \overline{Q_0Q_7}, \overline{Q_9Q_6}, \overline{Q_9Q_7}$. Note that $d(Q_7, Q_9)$ and $d(Q_0, Q_7)$ are both decreasing functions of d_1 and by direct substitution we can see that they do not exceed 1 if $d_1 \in [0.745, 1)$.

Using the coordinates of Q_0 and Q_6 we may write $d^2(Q_0, Q_6) = 1$ as an algebraic equation for d_1. Furthermore, after a sufficient number of transformations, it may be written as a polynomial equation $p(Q_0, Q_6) = 0$ for d_1. The Cartesian coordinates of Q_0 and Q_6 are the following

$Q_6 = (r_{13}(\sin(36° + \psi) - \sin\psi), d_1 + r_{13}(\cos(36° + \psi) - \cos\psi))$;

$Q_0 = (0, -r_{13}\cos(36° - \psi) + \sqrt{r_{13}^2\cos^2(36° - \psi) - r_{13}})$.

The polynomial equation is as follows:

$p(Q_0, Q_6) = (-10 - 4\sqrt{5})d_1^4 + (-5 - 7\sqrt{5})d_1^6 + 14\sqrt{5}d_1^8 + (55 + 13\sqrt{5})d_1^{10} + (25 + 15\sqrt{5})d_1^{12} + (10 + 4\sqrt{5})d_1^{14} = 0.$

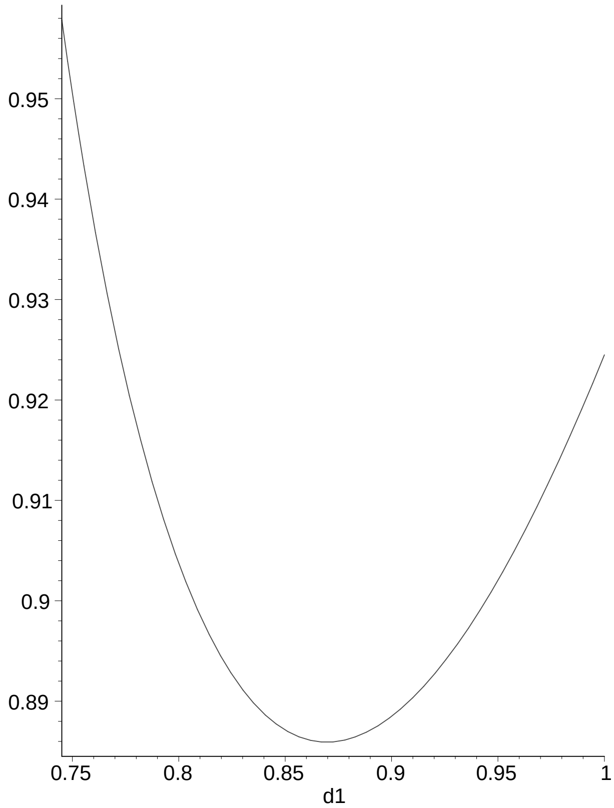

Figure 4.

This equation has two roots in the interval $[\frac{\sqrt{2}}{2}, 1]$, $0.744\ldots$ and 1. By direct substitution we can check that $p(Q_0, Q_6) < 1$ in $[0.745, 1)$, and $p(Q_0, Q_6) = 1$ at $d_1 = 1$. In a similar manner we may write $d^2(Q_6, Q_9) = 1$ as a polynomial equation for d_1 and check for roots in the designated interval. Note that $d(Q_9, O) = r_{13}\cos(56° - \psi) - \sqrt{r_{13}^2\cos^2(56° - \psi) - r_{13}}$. The graph of $d(Q_6, Q_9)$ is shown in Figure 4. This function has no zeros in the interval $[0.745, 1]$. □

Lemma 6. *If* $d_1 \in [\frac{\sqrt{2}}{2}, 0.745]$*, then there cannot be three points from the annulus* $1 < \rho \leq r_{13}$ *in either one of the sectors* P_2OP_3, P_3OP_4.

Proof. We assume, on the contrary, that there are three points, P_9, P_{10} and P_{11}, in P_3OP_4. The case when there are three points in P_2OP_3 is similar. For every value of d_1 there is a number d_m such that $d_i \geq d_m$ for $i = 2, 3, 4$. We may obtain d_m from the the following equation

$$\phi(d_1, d_1) + \phi(d_1, d_m) = 180°.$$

It turns out that $d_m = \frac{1}{d_1} - d_1$. Note that d_m is a monotonically decreasing function of d_1 in $[\frac{\sqrt{2}}{2}, 0.745]$. Furthermore, let $d_M = \sqrt{1 - d_1^2}$. Note that $\phi(d_M, d_1) = 90°$. We claim that either $d_3 \geq d_M$ or $d_4 \geq d_M$. This may be shown by examining the sum of the four consecutive central angles determined by $P_1, \ldots, P_4$. The sum is $s(d_1) = \phi(d_1, d_1) + 2\phi(d_M, d_1) + \phi(d_M, d_M)$. Then

$$\frac{ds(d_1)}{dd_1} = -2\frac{1}{d_1^2\sqrt{4-1/d_1^2}} + 2\frac{d_1}{(1-d_1^2)^{3/2}\sqrt{4-\frac{1}{1-d_1^2}}}. \qquad (1)$$

It is a simple exercise to show that $\frac{ds(d_1)}{dd_1} \geq 0$ if $d_1 \in [\frac{\sqrt{2}}{2}, 1]$. The function $s(d_1)$ has a minimum at $d_1 = \frac{\sqrt{2}}{2}$, and the minimum is 360° which proves our claim.

Now we are going to add up the four angles, $\angle P_1OP_2, \angle P_2OP_3, \angle P_3OP_4$, and $\angle P_4OP_1$. The sum is $\phi(d_1,d_2)+\phi(d_2,d_3)+\phi(d_3,r_{13})+72°+\phi(d_4,r_{13})+\phi(d_4,d_1)$. Note that $\phi(d_2,d_1) \geq \phi(d_1,d_1)$ and $\phi(d_2,d_3)+\phi(d_3,r_{13})$ takes on its minimum when $d_2 = d_1$ and $d_3 = d_m$. The same is true for $\phi(d_4,r_{13})+\phi(d_4,d_1)$ so we may assume that $d_4 = d_M$. Whence the sum is $\phi(d_1,d_1)+\phi(d_m,d_1)+\phi(d_m,r_{13})+72°+\phi(d_M,r_{13})+\phi(d_M,d_1)$. After simplification we obtain $\phi(d_M,r_{13})+\phi(d_m,r_{13})+342°$ which is a decreasing function of d_1 in $[\frac{\sqrt{2}}{2}, 0.735]$ and its value at $d_1 = \frac{\sqrt{2}}{2}$ is $2\phi(\frac{\sqrt{2}}{2}, r_{13})+342° = 386°\ldots$.

In $[0.735, 0.745]$ we are going to write the total angle differently. Notice that if $d_1 \geq \frac{1-r_{13}+\sqrt{6-r_{13}}}{2} = 0.7376\ldots$, then $d_m \leq r_{13}-1$ so we may omit $\phi(d_m, r_{13})$.

Case 1. $d_3 \geq 0.62$. The sum of the angles is not less than $\phi(d_1,d_1)+\phi(d_1,0.62)+\phi(0.62,r_{13})+\phi(d_M,r_{13})+\phi(d_M,d_1)+72°$, which is equal to $\phi(d_1,d_1)+\phi(0.62,d_1)+\phi(0.62,r_{13})+\phi(d_M,r_{13})+162°$. This is a decreasing function of d_1, and its value at $d_1 = 0.745$ is $= \phi(0.745,0.745)+\phi(0.745,0.62)+\phi(0.62,r_{13})+\phi(0.667,r_{13}) = 84°.31+93°.759+3°.58+17°.4+162° = 361°\ldots$.

Case 2. $d_3 < 0.62$. Notice that one of d_2 and d_4 is larger than 0.732 or $2\phi(0.62,0.732)+2\phi(0.732,0.745) = 2\cdot 95°.042+2\cdot 85°.22 = 360°.5\ldots$. Moreover, $d_2, d_4 \geq 0.72$ or $\phi(0.745, 0.745)+\phi(0.745,0.62)+\phi(0.62,0.72)+\phi(0.72,0.745) = 84°.31+93°.759+96°.25+86°.075 = 360°.4\ldots$. In this case the total angle is not less than $2\phi(0.72,r_{13})+2\phi(0.732,r_{13})+2\phi(d_1,r_{13})+216° = 2\cdot 23°.517+2\cdot 24°.596+2\cdot 24°.851+216° = 361°.9\ldots$. □

Lemma 7. *If* $d_1 \in [\sqrt{2}/2, 0.745]$, *then there cannot be three points from the annulus* $1 < \rho \leq r_{13}$ *in either one of the sectors* P_1OP_2, P_4OP_1.

Proof. We assume, on the contrary, that there are three points, P_{10}, P_{11}, and P_{13}, in P_4OP_1. The case when there are three points in P_1OP_2 is similar. The sum of the four consecutive angles determined by $P_1,\ldots,P_4$ is not less than $2\phi(d_1,d_1)+\phi(d_m,d_1)+\phi(d_m,r_{13})+72°+\phi(d_1,r_{13})$, which is not less than $72°+2\phi(d_1,d_1)+\phi(d_1,r_{13})+\phi(r_{13}-1,d_1)$ in $[0.737, 0.745]$. This is a decreasing function, and at $d_1 = 0.745$ its value is $72°+2\phi(d_1,d_1)+\phi(d_1,r_{13})+\phi(r_{13}-01,d_1) = 72°+2\cdot 84°.31+25°.662+93°.923 = 360°.2\ldots$. Note that $\phi(d_1,d_1)+\phi(d_1,r_{13})$ is a decreasing function in the designated interval.

In $[\sqrt{2}/2, 0.737]$ the total angle is larger than or equal to $252°+\phi(d_1,d_1)+\phi(d_m,r_{13})+\phi(d_1,r_{13})$ which is a decreasing function of d_1 and at $d_1 = 0.737$ its value is $252°+\phi(0.737, 0.737)+\phi(0.6198,r_{13})+\phi(0.737,r_{13}) = 252°+85°.441+3°.448+25°.018 = 365°.9\ldots$. Note that $\phi(d_m,r_{13})+\phi(d_1,r_{13})$ is a decreasing function of d_1 in $[\frac{\sqrt{2}}{2}, 0.737]$. □

Now the only possibility is that $d_1 = 1$. We saw in Lemma 5 that in this case the diameter of $\mathcal{R}_1$ and $\mathcal{R}_2$ is equal to 1 and this diameter is realized by Q_0 and Q_6. Therefore the four points in $C(1)$ must be in the configuration shown in the first part of Figure 1.

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