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Smooth Lines on Projective Planes over Two-Dimensional Algebras and Submanifolds with Degenerate Gauss Maps

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Abstract. The authors study smooth lines on projective planes over the algebra $\mathbb{C}$ of complex numbers, the algebra $\mathbb{C}^1$ of double numbers, and the algebra $\mathbb{C}^0$ of dual numbers. In the space $\mathbb{R}P^5$, to these smooth lines there correspond families of straight lines forming point three-dimensional submanifolds X^3 with degenerate Gauss maps of rank $r \leq 2$. The authors study focal properties of these submanifolds and prove that they represent examples of different types of submanifolds X^3 with degenerate Gauss maps. Namely, the submanifold X^3, corresponding in $\mathbb{R}P^5$ to a smooth line γ of the projective plane $\mathbb{C}P^2$, does not have real singular points, the submanifold X^3, corresponding in $\mathbb{R}P^5$ to a smooth line γ of the projective plane $\mathbb{C}^1P^2$, bears two plane singular lines, and finally the submanifold X^3, corresponding in $\mathbb{R}P^5$ to a smooth line γ of the projective plane $\mathbb{C}^0P^2$, bears one singular line. It is also proved that in all three cases, the rank of X^3 is equal to the rank of the curvature of the line γ.

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1. Introduction

The theory of projective planes over algebras is the subject belonging to the geometry and the algebra, and this subject attracts the attention of both algebraists and geometers. This theory was considered in Pickert's book [14], and in the single chapters of the books [6], [15], and [16].

However, not so much is known about the differential geometry of such projective planes. Some questions in this direction were considered in the paper [1]. In that paper the author studied smooth lines in projective planes over the matrix algebra and over some of its subalgebras. In this study, he used the mapping of the projective plane MP^2 over the algebra M of $(n \times n)$-matrices onto the Grassmannian $G(n-1, 3n-1)$ of subspaces of dimension $n-1$ of a real projective space $\mathbb{R}P^{3n-1}$.

In the current paper, we continue the investigations of Akivis [1]. However, we restrict ourselves to the study of smooth lines on projective planes over the algebra $\mathbb{C}$ of complex numbers, the algebra $\mathbb{C}^1$ of double numbers, and the algebra $\mathbb{C}^0$ of dual numbers (for description of these algebras, see, for example, [13] or [18] or [16]). In the space $\mathbb{R}P^5$, to these smooth lines there correspond families of straight lines forming point three-dimensional submanifolds X^3 with degenerate Gauss maps of rank $r \leq 2$ (see [2], Ch. 4). We study focal properties of these submanifolds and prove that they represent examples of different types of submanifolds X^3 with degenerate Gauss maps. Namely, the submanifold X^3, corresponding in $\mathbb{R}P^5$ to a smooth line γ of the projective plane $\mathbb{C}P^2$, does not have real singular points, the submanifold X^3, corresponding in $\mathbb{R}P^5$ to a smooth line γ of the projective plane $\mathbb{C}^1P^2$, bears two plane singular lines, and finally the submanifold X^3, corresponding in $\mathbb{R}P^5$ to a smooth line γ of the projective plane $\mathbb{C}^0P^2$, bears one double singular line.

In the last case, the submanifold X^3 is a generalization of submanifolds with degenerate Gauss maps without singularities in the four-dimensional Euclidean space R^4 constructed by Sacksteder in [17] and recently considered by Bourgain. Note that Bourgain's hypersurface was described in the papers [19] and [9, 10, 11], and the authors of the current paper proved (see [3]) that the hypersurfaces of Sacksteder and Bourgain are locally equivalent.

The authors introduce the notion of the curvature of a smooth line γ in the plane $\mathbb{A}P^2, \mathbb{A} = \mathbb{C}$, $\mathbb{C}^1$, $\mathbb{C}^0$, and prove that in all three cases, the rank of X^3 is equal to the rank of the curvature of the line γ.

Note also that it follows from the the results of the paper [1] that in the projective plane MP^2 over the algebra M of (2×2)-matrices, there are no smooth lines different from straight lines. A family of straight lines in $\mathbb{R}P^5$ corresponding to those straight lines is the Grassmannian $G(1, 3)$ of straight lines lying in a three-dimensional subspace $\mathbb{R}P^3$ of the space $\mathbb{R}P^5$.

In [4], the authors found the basic types of submanifolds with degenerate Gauss maps and proved the structure theorem for such submanifolds: an arbitrary submanifold with a degenerate Gauss map is either irreducible or if it is reducible, it is foliated by submanifolds of basic types. The finding of examples of submanifolds with degenerate Gauss maps of basic and not basic types is important. Such examples can be found in [2, 3, 4], [5], [8], [9, 10, 11], [12], [19]. In particular, in [10], Ishikawa found real algebraic cubic nonsingular hypersurfaces with a degenerate Gauss map in $\mathbb{R}P^n$ for $n = 4, 7, 13, 25$. These hypersurfaces have the structure of homogeneous spaces of groups $\mathbf{SO}(3), \mathbf{SU}(3), \mathbf{Sp}(3)$, and F_4, respectively, and their projective duals are linear projections of Veronese embeddings of projective planes $\mathbf{K}P_2$ for $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}, \mathbf{O}$, where $\mathbf{O}$ represents Cayley's octonions.

The examples we have constructed in this paper are of the same nature as Ishikawa's examples but they are much simpler.

2. Two-dimensional algebras and their representation

There are known three two-dimensional algebras: the algebra of complex numbers $z = x + iy$, where $i^2 = -1$, the algebra of double (or split complex) numbers $z = x + ey$, where $e^2 = 1$, and the algebra of dual numbers $z = x + \varepsilon y$, where $\varepsilon^2 = 0$. Here everywhere $x, y \in \mathbb{R}$. Usually these three algebras are denoted by $\mathbb{C}$, $\mathbb{C}^1$, and $\mathbb{C}^0$ (see [16], §1.1). These algebras are commutative and associative, and any two-dimensional algebra is isomorphic to one of them.

Each of these three algebras admits a representation by means of the real (2×2)-matrices:

$$z = x + iy \rightarrow \begin{pmatrix} x & -y \\ y & x \end{pmatrix}, \tag{1}$$

$$z = x + ey \rightarrow \begin{pmatrix} x & y \\ y & x \end{pmatrix}, \tag{2}$$

and

$$z = x + \varepsilon y \rightarrow \begin{pmatrix} x & 0 \\ y & x \end{pmatrix}. \tag{3}$$

In what follows, we identify the algebras $\mathbb{C}, \mathbb{C}^1$, and $\mathbb{C}^0$ with their matrix representations.

The algebras $\mathbb{C}, \mathbb{C}^1$, and $\mathbb{C}^0$ are subalgebras of the complete matrix algebra M formed by all real (2×2)-matrices

$$\begin{pmatrix} x_0^0 & x_1^0 \\ x_0^1 & x_1^1 \end{pmatrix}, \tag{4}$$

which is associative but not commutative.

The algebra $\mathbb{C}$ does not have zero divisors while the algebras $\mathbb{C}^1, \mathbb{C}^0$, and M have such divisors. In the matrix representation, zero divisors of these algebras are determined by the condition

$$\det \begin{pmatrix} x_0^0 & x_1^0 \\ x_0^1 & x_1^1 \end{pmatrix} = 0.$$

For the algebra $\mathbb{C}^1$ the last condition takes the form

$$x^2 - y^2 = 0,$$

for the algebra $\mathbb{C}^0$ the form $x = 0$, and for the algebra M the form

$$x_0^0 x_1^1 - x_0^1 x_1^0 = 0. \tag{5}$$

The elements of the algebras $\mathbb{C}^1$ and $\mathbb{C}^0$, as well as the regular complex numbers (the elements of the algebra $\mathbb{C}$), can be represented by the points on the plane xOy. In this representation, the zero divisors of the algebra $\mathbb{C}^1$ are represented by the points of the straight lines $y = \pm x$, and the zero divisors of the algebra $\mathbb{C}^0$ by the points of the y-axis.

The elements of the algebra M are represented by the points of a four-dimensional vector space, and its zero divisors by the points of the cone (5) whose signature is $(2, 2)$. Thus, to the algebra M, there corresponds a four-dimensional pseudo-Euclidean space R^4_2 of signature 2 with the isotropic cone (5). This cone bears two family of plane generators defined by the equations

$$\frac{x^0_0}{x^1_0} = \frac{x^0_1}{x^1_1} = \lambda, \quad \frac{x^0_0}{x^0_1} = \frac{x^1_0}{x^1_1} = \mu, \tag{6}$$

where λ and μ are real numbers.

3. The projective planes over the algebras $\mathbb{C}$, $\mathbb{C}^1$, $\mathbb{C}^0$, and M

Denote by A one of the algebras $\mathbb{C}$, $\mathbb{C}^1$, $\mathbb{C}^0$, or M and consider a projective plane $\mathbb{A}P^2$ over the algebra $\mathbb{A}$ (see [6]). A point $X \in \mathbb{A}P^2$ has three matrix coordinates X^0, X^1, X^2 that have respectively the form (1), (2), (3), or (4). Since it is convenient to write point coordinates as a column-matrix, we write

$$X = (X^0, X^1, X^2)^T. \tag{7}$$

The matrix X in (7) has 6 rows and 2 columns. Of course, the columns of this matrix must be linearly independent. The coordinates X^α, $\alpha = 0, 1, 2$, are defined up to a multiplication from the right by an element P of the algebra $\mathbb{A}$ which is not a zero divisor. So, we have $X' \sim XP$, $P \in \mathbb{A}$.

In particular, for $X \in \mathbb{C}P^2$, $X \in \mathbb{C}^1P^2$, and $X \in \mathbb{C}^0P^2$, we have

$$X = \begin{pmatrix} x^0_0 & -x^1_0 \\ x^1_0 & x^0_0 \\ x^2_0 & -x^3_0 \\ x^3_0 & x^2_0 \\ x^4_0 & -x^5_0 \\ x^5_0 & x^4_0 \end{pmatrix}, \quad X = \begin{pmatrix} x^0_0 & x^1_0 \\ x^1_0 & x^0_0 \\ x^2_0 & x^3_0 \\ x^3_0 & x^2_0 \\ x^4_0 & x^5_0 \\ x^5_0 & x^4_0 \end{pmatrix}, \quad X = \begin{pmatrix} x^0_0 & 0 \\ x^1_0 & x^0_0 \\ x^2_0 & 0 \\ x^3_0 & x^2_0 \\ x^4_0 & 0 \\ x^5_0 & x^4_0 \end{pmatrix},$$

respectively.

The columns of the matrix X can be considered as coordinates of the points x_0 and x_1 of a projective space $\mathbb{R}P^5$, and the straight line $x_0 \wedge x_1$ in the space $\mathbb{R}P^5$ corresponds to the matrix X. So, we can set $X = x_0 \wedge x_1$. The set of all straight lines of the space $\mathbb{R}P^5$ forms the Grassmannian $\mathbb{R}G(1, 5)$, whose dimension is equal to 8, dim $\mathbb{R}G(1, 5) = 2 \cdot 4 = 8$.

Note that $\mathbb{R}G(1, 5)$ is a differentiable manifold. Thus, $\mathbb{A}P^2$ is also a differentiable manifold over $\mathbb{R}$.

4. Equation of a straight line

A straight line U in the plane $\mathbb{A}P^2$ is defined by the equation

$$U_0 X^0 + U_1 X^1 + U_2 X^2 = 0,$$

where $U_\alpha \in \mathbb{A}$, $\alpha = 0, 1, 2$. The coordinates U_α admit a multiplication from the left by an element $P \in \mathbb{A}$, which is not a zero divisor.

In general, two *skewed* straight lines in $\mathbb{R}P^5$ correspond to points $X, Y \in \mathbb{A}P^2$. These straight lines define a subspace $\mathbb{R}P^3$ corresponding to the unique straight line in $\mathbb{A}P^2$ passing through the points X and Y.

Two points X and Y are called *adjacent* if more than one straight line passes through them in $\mathbb{A}P^2$. To such points, there correspond *intersecting* straight lines $x^0 \wedge x^1$ and $y^0 \wedge y^1$ in $\mathbb{R}P^5$. Through adjacent points $X, Y \in \mathbb{A}P^2$, there passes a two-parameter family of straight lines in $\mathbb{A}P^2$, since through a plane $\mathbb{R}P^5$, there passes a 2-parameter family of subspaces $\mathbb{R}P^3 \subset \mathbb{R}P^5$.

If

$$X = (X^0, X^1, X^2)^T, \;\; Y = (Y^0, Y^1, Y^2)^T$$

are adjacent points, then the rank of the (6×4)-matrix composed of the matrix coordinates of X and Y is less than 4. If the rank of this matrix is 4, then through the points X and Y, there passes a unique straight line.

On a plane $\mathbb{A}P^2$, there are three basis points E_0, E_1, E_2 with coordinates

$$E_0 = (E, 0, 0)^T, \;\; E_1 = (0, E, 0)^T, \;\; E_2 = (0, 0, E)^T,$$

where $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the unit matrix, and 0 is the (2×2) 0-matrix. A point $X \in \mathbb{A}P^2$ can be represented in the form

$$X = E_0 X^0 + E_1 X^1 + E_2 X^2. \tag{8}$$

However, as we noted earlier, the coordinates X_α of this point admit a multiplication from the right by an element $P \in \mathbb{A}$ which is not a zero divisor.

A point X is in general position with the straight line $E_\alpha \wedge E_\beta$, $\alpha, \beta = 0, 1, 2$, if and only if its coordinate X^γ, $\gamma \neq \alpha, \beta$, is not a zero divisor. Let, for instance, a point X be in general position with the straight line $E_1 \wedge E_2$. Then its coordinate X^0 is not a zero divisor, and all its coordinates can be multiplied from the right by $(X^0)^{-1}$. Then expression (8) of the point X takes the form

$$X = E^0 + E_1 \widetilde{X}^1 + E_2 \widetilde{X}^2, \tag{9}$$

where $\widetilde{X}^1 = X^1 (X^0)^{-1}$, $\widetilde{X}^2 = X^2 (X^0)^{-1}$. Now the (4×2)-matrix $(\widetilde{X}^1, \widetilde{X}^2)^T$ is defined uniquely and is called the *matrix coordinate* of the point X as well as of the straight line $x_0 \wedge x_1$ defined in the space $\mathbb{R}P^5$ by the point X (see [16], Sect. 2.4.1, and also [15], Ch. 3, §3).

For the plane MP^2, the matrix coordinate has 8 real components. Hence dim $MP^2 = 8$. Since dim $MP^2 =$ dim $G(1,5)$, the plane MP^2 can be bijectively mapped onto the Grassmannian $\mathbb{R}G(1,5)$.

For the planes $\mathbb{C}P^2, \mathbb{C}^1P^2$, and $\mathbb{C}^0P^2$, the matrix coordinates of points have 4 real components. Hence the real dimension of these planes is 4,

$$\dim \mathbb{C}P^2 = \dim \mathbb{C}^1P^2 = \dim \mathbb{C}^0P^2 = 4.$$

Therefore, the family of straight lines $x_0 \wedge x_1$ in the space $\mathbb{R}P^5$ for each of these planes depends on 4 parameters, i.e., it forms a *congruence* in the space $\mathbb{R}P^5$. We denote these congruences by K, K^1, and K^0, respectively.

5. Moving frames in projective planes over algebras

A moving frame in a projective plane $\mathbb{A}P^2$ over an algebra $\mathbb{A}$ is a triple of points A_α, $\alpha = 0, 1, 2$, that are mutually not adjacent. Any point $X \in \mathbb{A}P^2$ can be written as

$$X = A_0X^0 + A_1X^1 + A_2X^2,$$

where $X^\alpha \in \mathbb{A}$ are the coordinates of this point with respect to the frame $\{A_0, A_1, A_2\}$. The coordinates of a point X are defined up to a multiplication from the right by an element P of the algebra $\mathbb{A}$ which is not a zero divisor. If a point X is in general position with the straight line $A_1 \wedge A_2$, then its coordinate X^0 is not a zero divisor. Thus, the point X can be written as

$$X = A^0 + A_1\widetilde{X}^1 + A_2\widetilde{X}^2,$$

where $\widetilde{X}^1 = X^1(X^0)^{-1}, \widetilde{X}^2 = X^2(X^0)^{-1}$. The matrix $(\widetilde{X}^1, \widetilde{X}^2)^T$ is the matrix coordinate of the point X with respect to the moving frame $\{A_\alpha\}$, and this coordinate is defined uniquely.

The plane $\mathbb{A}P^2$ admits a representation on the Grassmannian $\mathbb{R}G(1,5)$ formed by the straight lines of the space $\mathbb{R}P^5$. Under this representation, to the vertices of the frame $\{A_\alpha\}$, there correspond the straight lines

$$A_0 = a_0 \wedge a_1, \quad A_1 = a_2 \wedge a_3, \quad A_2 = a_4 \wedge a_5 \tag{10}$$

in $\mathbb{R}P^5$; here a_i, $i = 0, \dots, 5$, are points of the space $\mathbb{R}P^5$.

The equations of infinitesimal displacement of the moving frame $\{A_0, A_1, A_2\}$ have the form

$$dA_\alpha = A_\beta \Omega^\beta_\alpha, \quad \alpha, \beta = 0, 1, 2, \tag{11}$$

where Ω^β_α are 1-forms over the algebra $\mathbb{A}$. In the representation of the algebra $\mathbb{A}$ by (2×2)-matrices, these forms are expressed as the transposed matrices (1), (2), (3), and (4). Their entries are not the numbers. The entries are real 1-forms:

$$\Omega^\beta_\alpha = \begin{pmatrix} \omega^{2\beta}_{2\alpha} & \omega^{2\beta+1}_{2\alpha} \\ \omega^{2\beta}_{2\alpha+1} & \omega^{2\beta+1}_{2\alpha+1} \end{pmatrix}. \tag{12}$$

Thus, for the plane $\mathbb{C}P^2$, the entries of the matrix Ω_α^β satisfy the equations

$$\omega_{2\alpha}^{2\beta} = \omega_{2\alpha+1}^{2\beta+1}, \quad \omega_{2\alpha}^{2\beta+1} = -\omega_{2\alpha+1}^{2\beta}, \tag{13}$$

for the plane $\mathbb{C}^1P^2$ the equations

$$\omega_{2\alpha}^{2\beta} = \omega_{2\alpha+1}^{2\beta+1}, \quad \omega_{2\alpha}^{2\beta+1} = \omega_{2\alpha+1}^{2\beta}, \tag{14}$$

and for the plane $\mathbb{C}^0P^2$ the equations

$$\omega_{2\alpha}^{2\beta} = \omega_{2\alpha+1}^{2\beta+1}, \quad \omega_{2\alpha+1}^{2\beta} = 0. \tag{15}$$

If now the frame $\{A_\alpha\}$ moves in the plane $\mathbb{A}P^2$, then the points $a_i \in \mathbb{R}P^5$ also move. The equations of infinitesimal displacement of the moving frame $\{a_i\}$ can be written in the form

$$da_i = a_j \omega_i^j, \quad i, j = 0, 1, \dots, 5, \tag{16}$$

where by (10), the forms ω_i^j coincide with the corresponding forms (12). The forms ω_j^i satisfy the structure equations of the projective space $\mathbb{R}P^5$:

$$d\omega_j^i = -\omega_k^i \wedge \omega_j^k, \tag{17}$$

where d is the symbol of exterior differential, and $\wedge$ denotes the exterior multiplication of the linear differential forms (see for example, [2], Sec. 1.3).

6. Focal properties of the congruences K, K^1, and K^0

Now we consider the congruences K, K^1, and K^0 of the space $\mathbb{R}P^5$, representing the planes $\mathbb{C}P^2, \mathbb{C}^1P^2$, and $\mathbb{C}^0P^2$ in this space, and investigate their focal properties.

Theorem 1. *The projective planes $\mathbb{C}P^2, \mathbb{C}^1P^2$, and $\mathbb{C}^0P^2$ admit a bijective mapping onto the linear congruences K, K^1, and K^0 of the real space $\mathbb{R}P^5$. These congruences are respectively elliptic, hyperbolic, and parabolic.*

Proof. To each of these congruences, we associate a family of projective frames in such a way that the points a_0 and a_1 are located on a moving straight line of the congruence.

For the congruence K, equations of infinitesimal displacement of the points a_0 and a_1 can be written in the form

$$\left\{\begin{aligned} da_0 &= \omega_0^0 a_0 + \omega_0^1 a_1 + \omega_0^2 a_2 + \omega_0^3 a_3 + \omega_0^4 a_4 + \omega_0^5 a_5, \\ da_1 &= -\omega_0^1 a_0 + \omega_0^0 a_1 - \omega_0^3 a_2 + \omega_0^2 a_3 - \omega_0^5 a_4 + \omega_0^4 a_5. \end{aligned}\right. \tag{18}$$

By (14), for the congruence K^1, similar equations take the form

$$\left\{\begin{aligned} da_0 &= \omega_0^0 a_0 + \omega_0^1 a_1 + \omega_0^2 a_2 + \omega_0^3 a_3 + \omega_0^4 a_4 + \omega_0^5 a_5, \\ da_1 &= \omega_0^1 a_0 + \omega_0^0 a_1 + \omega_0^3 a_2 + \omega_0^2 a_3 + \omega_0^5 a_4 + \omega_0^4 a_5. \end{aligned}\right. \tag{19}$$

Finally, by (15), for the congruence K^0, similar equations take the form

$$\left\{ \begin{array}{ll} da_0 = \omega_0^0 a_0 + \omega_0^1 a_1 + \omega_0^2 a_2 + \omega_0^3 a_3 + \omega_0^4 a_4 + \omega_0^5 a_5, \\ da_1 = \qquad \omega_0^0 a_1 \qquad + \omega_0^2 a_3 \qquad\qquad + \omega_0^4 a_5. \end{array} \right. \tag{20}$$

Let $x = a_1 + \lambda a_0$ be an arbitrary point of the straight line $a_0 \wedge a_1$. This point is a focus of this straight line if for some displacement, its differential dx also belongs to this straight line.

Let us start from the congruence K^1, since the focal images for this congruence are real and look more visual. By (19), for this congruence we have

$$dx \equiv (\omega_0^3 + \lambda\omega_0^2)a_2 + (\omega_0^2 + \lambda\omega_0^3)a_3 + (\omega_0^5 + \lambda\omega_0^4)a_4 + (\omega_0^4 + \lambda\omega_0^5)a_5 \pmod{a_0 \wedge a_1}, \tag{21}$$

and as a result, for its focus x, the following equations must be satisfied:

$$\left\{ \begin{array}{ll} \omega_0^2 + \lambda\omega_0^3 = 0, & \omega_0^4 + \lambda\omega_0^5 = 0, \\ \lambda\omega_0^2 + \omega_0^3 = 0, & \lambda\omega_0^4 + \omega_0^5 = 0. \end{array} \right. \tag{22}$$

The necessary and sufficient condition of consistency of this system is

$$\left| \begin{array}{cc} 1 & \lambda \\ \lambda & 1 \end{array} \right|^2 = 0.$$

It follows that the values $\lambda = \pm 1$ are double roots of this equation. Thus, each line $a_0 \wedge a_1$ of the congruence K^1 has two double foci

$$f_1 = a_1 + a_0, \quad f_2 = a_1 - a_0.$$

Equations (14) imply that the differentials of the focus f_1 are expressed only in terms of the points $a_0 + a_1, a_2 + a_3$, and $a_4 + a_5$. The differentials of these points are expressed in terms of the same points. As a result, the plane

$$\pi_1 = (a_0 + a_1) \wedge (a_2 + a_3) \wedge (a_4 + a_5)$$

remains fixed when the straight line $a_0 \wedge a_1$ describes the congruence K^1 in the space $\mathbb{R}P^5$. In a similar way, one can prove that the focus f_2 describes the plane

$$\pi_2 = (a_0 - a_1) \wedge (a_2 - a_3) \wedge (a_4 - a_5).$$

Thus, the congruence K^1 is a four-parameter family of straight lines of the space $\mathbb{R}P^5$ intersecting its two planes π_1 and π_2 that are in general position. Hence K^1 is a *hyperbolic* line congruence.

In a similar way, we can prove that each straight line $a_0 \wedge a_1$ of the congruence K bears two double complex conjugate foci

$$f_1 = a_1 + ia_0, \quad f_2 = a_1 - ia_0,$$

and these foci describe two complex conjugate two-dimensional planes π_1 and π_2, $\pi_2 = \overline{\pi}_1$. Hence K is an *elliptic* line congruence in the space $\mathbb{R}P^5$. The straight lines of K do not have singular points in $\mathbb{R}P^5$.

Finally, consider the congruence K^0 in the space $\mathbb{R}P^5$. We look for the foci of its straight lines in the same form

$$x = a_1 + \lambda a_0.$$

Differentiating this expression by means of (20), we find that

$$dx \equiv \lambda\omega_0^2 a_2 + (\lambda\omega_0^3 + \omega_0^2)a_3 + \lambda\omega_0^4 a_4 + (\lambda\omega_0^5 + \omega_0^4)a_5 \pmod{a_0 \wedge a_1}.$$

Thus, the focus x must satisfy the following equations:

$$\left\{\begin{array}{llll} \lambda\omega_0^2 & = 0, & \lambda\omega_0^4 & = 0, \\ \omega_0^2 \; +\lambda\omega_0^3 & = 0, & \omega_0^4 \; +\lambda\omega_0^5 & = 0. \end{array}\right. \tag{23}$$

This system is consistent if and only if

$$\begin{vmatrix} \lambda & 0 \\ 1 & \lambda \end{vmatrix}^2 = 0.$$

It follows that the value $\lambda = 0$ is a quadruple root of this equation. Thus, each line $a_0 \wedge a_1$ of the congruence K^0 has a real quadruple singular point $f = a_1$. Applying equations (15), it is easy to prove that when the straight line $a_0 \wedge a_1$ describes the congruence K^0, this focus describes the plane $\pi = a_1 \wedge a_3 \wedge a_5$. Hence K^0 is a *parabolic* line congruence. □

7. Smooth lines in projective planes

On a projective plane $\mathbb{A}P^2$, where $\mathbb{A}$ is one of the algebras $\mathbb{C}$, $\mathbb{C}^1$, and $\mathbb{C}^0$, consider a smooth point submanifold γ of real dimension three. Such a submanifold is called an $\mathbb{A}$-*smooth line* if at any of its points X, it is tangent to a straight line U passing through X.

With an A-smooth line γ, we associate a family of projective frames $\{A_0A_1A_2\}$ in such a way that $A_0 = X$ and A_1 lies on the tangent U to γ at X. Then on the line γ, the first equation of (11) takes the form

$$dA_0 = A_0\Omega_0^0 + A_1\Omega_0^1. \tag{24}$$

It follows that $\mathbb{A}$-smooth lines on a plane $\mathbb{A}P^2$ are defined by the equation

$$\Omega_0^2 = 0. \tag{25}$$

The 1-form Ω_0^1 in equation (24) defines a displacement of the point A_0 along the curve γ. So, this form is a basis form on γ.

By equations (12), we have

$$\Omega_0^1 = \begin{pmatrix} \omega_0^2 & \omega_0^3 \\ \omega_1^2 & \omega_1^3, \end{pmatrix}, \quad \Omega_0^2 = \begin{pmatrix} \omega_0^4 & \omega_0^5 \\ \omega_1^4 & \omega_1^5 \end{pmatrix},$$

where ω_i^j are real 1-forms. For the algebras $\mathbb{C}, \mathbb{C}^1$, and $\mathbb{C}^0$, they are related respectively by equations (13), (14), and (15). As a result, on the line $\gamma \subset \mathbb{A}P^2$, the following differential equations will be satisfied:

$$\omega_0^4 = 0, \quad \omega_0^5 = 0. \tag{26}$$

These equations are equivalent to equations (25).

Since Ω_0^1 is a basis form on the line $\gamma \subset \mathbb{A}P^2$, the real forms ω_0^2 and ω_0^3 are linearly independent. The families of straight lines in the space $\mathbb{R}P^5$ corresponding to these lines depend on 2 parameters and form a three-dimensional ruled submanifold. Denote this submanifold by S. These submanifolds belong to the congruences K, K^1, and K^0 if $\gamma \subset \mathbb{C}P^2, \gamma \subset \mathbb{C}^1P^2$, and $\gamma \subset \mathbb{C}^0P^2$, respectively.

Theorem 2. *The tangent subspace $T_x(S)$ to the ruled submanifold S corresponding in the space $\mathbb{R}P^5$ to a smooth line in the planes $\mathbb{C}P^2, \mathbb{C}^1P^2$, and $\mathbb{C}^0P^2$ is fixed at all points of its rectilinear generator L, and the submanifold S is a submanifold with a degenerate Gauss map of rank $r \leq 2$.*

Proof. Consider a rectilinear generator $L = a_0 \wedge a_1$ of the submanifold S. By (26), the differentials of the points a_0 and a_1 are written in the form

$$\begin{cases} da_0 = \omega_0^0 a_0 + \omega_0^1 a_1 + \omega_0^2 a_2 + \omega_0^3 a_3, \\ da_1 = \omega_0^1 a_0 + \omega_1^1 a_1 + \omega_1^2 a_2 + \omega_1^3 a_3. \end{cases} \tag{27}$$

It follows that at any point $x \in a_0 \wedge a_1$, the tangent subspace $T_x(X)$ belongs to a three-dimensional subspace $\mathbb{R}P^3 \subset \mathbb{R}P^5$ defined by the points a_0, a_1, a_2, and a_3. Thus, the subspace $T_x(X)$ remains fixed along the rectilinear generator $L = a_0 \wedge a_1$, and the submanifold S is a submanifold with a degenerate Gauss map of rank $r \leq 2$. □

8. Singular points of submanifolds corresponding to smooth lines in the projective spaces over two-dimensional algebras

We will prove the following theorem.

Theorem 3. *To smooth lines in the projective planes $\mathbb{C}P^2, \mathbb{C}^1P^2$, and $\mathbb{C}^0P^2$ over the algebras of complex, double, and dual numbers, there correspond three-dimensional submanifolds X^3 with degenerate Gauss maps of rank $r \leq 2$ in the space $\mathbb{R}P^5$. For the algebra $\mathbb{C}$, such a submanifold does not have real singular points, for the algebra $\mathbb{C}^1$, such a submanifold is the join formed by the straight lines connecting the points of two plane curves that are in general position, and for the algebra $\mathbb{C}^0$, such a submanifold is a subfamily of the family of straight lines intersecting a plane curve. In all these cases, a submanifold S depends on two functions of one variable.*

A rectilinear generator $L = a_0 \wedge a_1$ of a submanifold S of rank 2 bears two foci. Let us find these foci for the submanifolds S corresponding to the lines γ in the planes $\mathbb{C}P^2, \mathbb{C}^1P^2$, and $\mathbb{C}^0P^2$. We assume that these foci have the form $x = a_1 + \lambda a_0$.

If a line $\gamma \subset \mathbb{C}^1 P^2$, then equations (14) and (27) are satisfied. They imply that

$$dx \equiv (\omega_0^3 + \lambda\omega_0^2)a_2 + (\omega_0^2 + \lambda\omega_0^3)a_3 \pmod{a_0 \wedge a_1},$$

and for the focus x, we have

$$\omega_0^3 + \lambda\omega_0^2 = 0, \;\; \omega_0^2 + \lambda\omega_0^3 = 0.$$

This system is consistent if and only if

$$\begin{vmatrix} 1 & \lambda \\ \lambda & 1 \end{vmatrix} = 0,$$

i.e., if $\lambda = \pm 1$. Thus, the foci of the straight line $a_0 \wedge a_1$ are the points $a_1 + a_0$ and $a_1 - a_0$. These points belong to the focal planes π_1 and π_2 of the congruence K^1 and describe lines γ_1 and γ_2. Such manifolds S are called *joins.* Since each of the lines γ_1 and γ_2 on the planes π_1 and π_2 is defined by means of one function of one variable, a submanifold S depends on two functions of one variable. The same result could be obtained by applying the Cartan test (see [7]) to the system of equations (14) and (26).

If a line $\gamma \subset \mathbb{C}P^2$, then we can prove that a rectilinear generator $L = a_0 \wedge a_1$ of the ruled submanifold S corresponding to γ bears two complex conjugate foci belonging to complex conjugate focal planes π_1 and $\pi_2 = \overline{\pi}_1$ of the congruence K. Hence *in the real space* $\mathbb{R}P^5$*, the submanifold* S *does not have singular points.*

In the complex plane π_1, the focus f_1 can describe an arbitrary differentiable line. But such a line is defined by means of two functions of one real variable. Therefore, in this case the submanifold S also depends on two functions of one real variable.

Finally, consider a submanifold $S \subset \mathbb{R}P^5$ corresponding to a line $\gamma \subset \mathbb{C}^0 P^2$. Such a submanifold is defined in $\mathbb{R}P^5$ by differential equations (15) and (26). Using the same method as above, we can prove that a rectilinear generator $L = a_0 \wedge a_1$ of the ruled submanifold S corresponding to γ bears a double real focus $f = a_1$ belonging to the focal plane π of the congruence K^0 and describing in this plane an arbitrary line.

We prove that in this case a submanifold S is also defined by two functions of one variable. But now in order to prove this, we apply the Cartan test.

Taking exterior derivatives of equations (26) and applying equations (15), we obtain the following exterior quadratic equations:

$$\omega_0^2 \wedge \omega_2^4 = 0, \;\; \omega_0^2 \wedge \omega_2^5 + \omega_0^3 \wedge \omega_2^4 = 0. \tag{28}$$

It follows from (28) that

$$\omega_2^4 = p\omega_0^2, \;\; \omega_2^5 = q\omega_0^2 + p\omega_0^3. \tag{29}$$

We apply the Cartan test to the system of equations (26), (28), and (29). In addition to the basis forms ω_0^2 and ω_0^3, equations (28) contain two more forms ω_2^4 and ω_2^5. Thus, we have $q = 2$. The number of independent equations in (28) is also 2, i.e., $s_1 = 2$. As a result, $s_2 = q - s_1 = 0$, and the Cartan number

$$Q = s_1 + 2s_2 = 2.$$

Equations (29) show that the number N of parameters on which the general two-dimensional integral element depends is also 2, $N = 2$. Since $Q = N$, the system of equations (26) is involution, and its solution depends on two functions of one variable. $\square$

9. Curvature of smooth lines over algebras

Differentiating equation (25) defining a smooth line γ in the plane $\mathbb{A}P^2$, where $\mathbb{A} = \mathbb{C}$, $\mathbb{C}^1$, $\mathbb{C}^0$, and applying Cartan's lemma, we obtain

$$\Omega_1^2 = R\Omega_0^1, \tag{30}$$

where $R \in \mathbb{A}$. The quantity R is called the *curvature* of the line $\gamma \subset \mathbb{A}P^2$.

For a line γ in the plane $\mathbb{C}^1 P^2$, in formula (30) we have

$$\Omega_0^1 = \begin{pmatrix} \omega_0^2 & \omega_0^3 \\ \omega_0^3 & \omega_0^2 \end{pmatrix}, \quad \Omega_1^2 = \begin{pmatrix} \omega_2^4 & \omega_2^5 \\ \omega_2^5 & \omega_2^4 \end{pmatrix}, \quad R = \begin{pmatrix} p & q \\ q & p \end{pmatrix},$$

and $\det R = p^2 - q^2$. If $\operatorname{rank} R = 2$, then the quantity R is not a zero divisor, and the rank of the ruled manifold X that corresponds in $\mathbb{R}P^5$ to the line γ, is also equal to 2. If $\operatorname{rank} R = 1$, then R is a zero divisor, $R \neq 0$, and the rank of the manifold X is equal to 1. Finally, if $R = 0$, then a line γ is a straight line in the plane $\mathbb{C}^1 P^2$, and the manifold X corresponding to γ in $\mathbb{R}P^5$ is a subspace $\mathbb{R}P^3$.

For a line γ in the plane $\mathbb{C}P^2$, in formula (30) we have

$$\Omega_0^1 = \begin{pmatrix} \omega_0^2 & \omega_0^3 \\ -\omega_0^3 & \omega_0^2 \end{pmatrix}, \quad \Omega_1^2 = \begin{pmatrix} \omega_2^4 & \omega_2^5 \\ -\omega_2^5 & \omega_2^4 \end{pmatrix}, \quad R = \begin{pmatrix} p & q \\ -q & p \end{pmatrix}.$$

Thus, $\det R = p^2 + q^2$, and two cases are possible: $\operatorname{rank} R = 2$ and $\operatorname{rank} R = 0$. In the first case, a submanifold $X \subset \mathbb{R}P^5$ of rank 2 without singularities corresponds to the line $\gamma \subset \mathbb{C}P^2$, and in the second case, an $\mathbb{R}P^3$ corresponds to the line $\gamma \subset \mathbb{C}P^2$.

For a line γ in the plane $\mathbb{C}^0 P^2$, in formula (30) we have

$$\Omega_0^1 = \begin{pmatrix} \omega_0^2 & \omega_0^3 \\ 0 & \omega_0^2 \end{pmatrix}, \quad \Omega_1^2 = \begin{pmatrix} \omega_2^4 & \omega_2^5 \\ 0 & \omega_2^4 \end{pmatrix}, \quad R = \begin{pmatrix} p & q \\ 0 & p \end{pmatrix},$$

and $\det R = p^2$. If $p \neq 0$, then $\operatorname{rank} R = 2$, and the curvature R is not a zero divisor. If $p = 0, q \neq 0$, then $\operatorname{rank} R = 1$, and the curvature R is a nonvanishing zero divisor. If $p = q = 0$, then $R = 0$. The rank a submanifold X corresponding in $\mathbb{R}P^5$ to a line $\gamma \subset \mathbb{C}^0 P^2$ is equal to the rank of R. If $R = 0$, then the submanifold X is a flat subspace $\mathbb{R}P^3 \subset \mathbb{R}P^5$.

Thus, we have proved the following result.

Theorem 4. *The rank of the ruled submanifold X corresponding in $\mathbb{R}P^5$ to a smooth line $\gamma \subset \mathbb{A}P^2$, where $\mathbb{A} = \mathbb{C}$, $\mathbb{C}^1$, $\mathbb{C}^0$, is equal to the rank of the curvature of this line. For $A = \mathbb{C}$, this rank can be 2 or 0, and for $\mathbb{A} = \mathbb{C}^1$, $\mathbb{C}^0$, the rank can be 2, or 1, or 0.*

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