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Generalized GCD Rings II

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Abstract. Greatest common divisors and least common multiples of quotients of elements of integral domains have been investigated by Lüneburg and further by Jäger. In this paper we extend these results to invertible fractional ideals. We also lift our earlier study of the greatest common divisor and least common multiple of finitely generated faithful multiplication ideals to finitely generated projective ideals.

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0. Introduction

Let R be a ring and K the total quotient ring of R. An integral (or fractional) ideal A of R is invertible if $AA^{-1} = R$, where $A^{-1} = \{x \in K : xA \subseteq R\}$.

Let I and J be ideals of R. Then $[I : J] = \{x \in R : xJ \subseteq I\}$ is an ideal of R. The annihilator of I, denoted by ann$I$, is $[0 : I]$. An ideal J of R is called a multiplication ideal if for every ideal $I \subseteq J$, there exists an ideal C of R such that $I = JC$, see [6], [16] and [23]. Let J be a multiplication ideal of R and $I \subseteq J$. Then $I = JC \subseteq [I : J]J \subseteq I$, so that $I = [I : J]J$. We also note that if J is a multiplication ideal of R, then $I \cap J = [I : J]J$ for every ideal J of R, see [30, Lemma 3.1]. A finitely generated (f.g.) ideal I of R is projective if and only if I is multiplication and ann$I = eR$ for some idempotent e, [31, Theorem 2.1] and [35, Theorem 11]. If I is a f.g. multiplication (equivalently f.g. locally principal) ideal of R such that ann$I$ is a pure ideal, then I is a flat ideal, [31, Theorem 2.2]. Every projective ideal is flat while a f.g. flat ideal I with f.g. annihilator is a projective ideal, [31, Corollary 4.3]. A ring R is called a p.p. ring if every principal ideal is projective, [14]. It is shown, [14, Proposition 1] that a ring R is p.p. if and only if R_M is an integral domain for every maximal ideal M of R and K, the quotient ring of R, is a von Neumann regular ring. If R is a p.p. ring, then a f.g. ideal I is projective if and only if it is flat, [31]. More results explaining the relationships between projective, multiplication and flat f.g. ideals can be found in [31] and [36]. On the other hand, invertible ideals are projective (and hence multiplication and flat) while f.g. projective (flat) ideals are either locally zero or locally invertible.

Let R be a ring. Let $F(R)$ be the group of invertible fractional ideals of R and $I(R)$ the semigroup of invertible integral ideals of R. In Part 1 we investigate the greatest common divisor and least common multiple of the elements of $F(R)$ and $I(R)$ generalizing the results of Jäger [21] and Lüneburg [27]. We show that if $A, B \in F(R)$, then $\mathrm{GCD}(A, B)$ exists if and only if $\mathrm{LCM}(A, B)$ exists and in this case $AB = \mathrm{GCD}(A, B)\mathrm{LCM}(A, B)$, [Theorem 1.3]. We also prove that $\mathrm{GCD}(A, B)$ exists if and only if $\mathrm{GCD}(CA, CB)$ exists and that $\mathrm{GCD}(CA, CB) = C\mathrm{GCD}(A, B)$, where $A, B, C \in F(R)$ [Corollary 1.4]. D. D. Anderson and D. F. Anderson [8] introduced the generalized GCD (GGCD) domains as those in which the intersection of any two invertible integral (fractional) ideals is an invertible ideal. Theorem 1.8 gives 40 equivalent conditions for an integral domain to be a GGCD-domain. Let R be a Bezout domain and K its quotient field. Lüneburg [27] studied the GCD and LCM of any two non-zero elements of K. Let $a, b \in K - \{0\}$. Then $a = \frac{u}{v}, b = \frac{x}{y}$, where $u, v, x, y \in R - \{0\}$ and $\gcd(u, v) = 1 = \gcd(x, y)$. Lüneburg proved that $\mathrm{GCD}(a, b) = \frac{\gcd(u,x)}{\mathrm{lcm}(v,y)}$ and $\mathrm{LCM}(a, b) = \frac{\mathrm{lcm}(u,x)}{\gcd(v,y)}$. Jäger [21] extended these results to GCD domains. We generalize Lüneburg's results to GGCD-domains. We show that if $A, B \in F(R)$, then A and B can be written as $A = IJ^{-1}, \quad B = KL^{-1}$ where $I, J, K, L \in I(R)$ and $\gcd(I, J) = R = \gcd(K, L)$, and $\mathrm{GCD}(A, B) = \gcd(I, K)\mathrm{lcm}(J, L)^{-1}$, and $\mathrm{LCM}(A, B) = \mathrm{lcm}(I, K)\gcd(J, L)^{-1}$, [Theorem 1.10 and Corollary 1.11]. At the end of Part 1 we study the greatest common divisor and least common multiple of infinite subsets of $F(R)$ and $I(R)$ [Theorem 1.12].

In [3] we investigated the greatest common divisor and least common multiple of f.g. faithful multiplication ideals. We also introduced a class of rings which we called generalized GCD (GGCD) rings in which the intersection of any two f.g. faithful multiplication ideals is a f.g. faithful multiplication ideal (equivalently the gcd of any two f.g. faithful multiplication ideals exists). The purpose of our work in Part 2 is to extend these results to f.g. projective ideals. Let R be a ring and $S(R)$ the semigroup of f.g. projective ideals of R. We show that if $A, B \in S(R)$ such that $\gcd(A, B)$ exists, then $\gcd(A, B) \in S(R)$. A similar result holds for $\mathrm{lcm}(A, B)$, [Theorem 2.1]. We also prove that if $A, B \in S(R)$, then $\mathrm{lcm}(A, B)$ exists if and only if $[A : B] \in S(R)$, [Theorem 2.2]. Theorem 2.4 establishes that for $A, B, C \in S(R), \quad \mathrm{lcm}(CA, CB)$ exists if and only if $\mathrm{lcm}(A + \mathrm{ann}C, B + \mathrm{ann}C)$ exists, and $\mathrm{lcm}(CA, CB) = C\mathrm{lcm}(A+\mathrm{ann}C, B+\mathrm{ann}C)$. Moreover, if $\gcd(CA, CB)$ exists, then $\gcd(A+\mathrm{ann}C, B+\mathrm{ann}C)$ exists and in this case $\gcd(CA, CB) = C\gcd(A+\mathrm{ann}C, B+\mathrm{ann}C)$. A relationship between $\mathrm{lcm}(A, B)$ and $\gcd(A, B)$ where $A, B \in S(R)$ is given in Corollary 2.5. We prove that if $\gcd(A, B)$ exists for all $A, B \in S(R)$ then $\mathrm{lcm}(A, B)$ exists for all $A, B \in S(R)$, and $AB = \gcd(A, B)\mathrm{lcm}(A, B)$. We then call a ring R a G*GCD-ring if $\gcd(A, B)$ exists for all $A, B \in S(R)$, generalizing GGCD-ring. We see that all the results of

[3, Section 3] concerning GGCD rings can easily be extended to G*GCD rings.

In Part 3 we introduce a new class of rings, generalizing the almost Prüfer domains defined by Anderson and Zafrullah [9]. We call a ring (possibly with zero divisors) an almost semihereditary ring (AS-ring) if R is a p.p. ring and for all $a, b \in R$, there exists a positive integer $n = n(a, b)$ such that $a^nR+b^nR \in S(R)$. Theorem 3.1 and Proposition 3.2 give several characterizations and properties of AS-rings.

All rings in this paper are commutative with 1. For the basic concepts, we refer the reader to [15], [16], [22], [23], and [34].

1. GCD and LCM of invertible ideals

Let R be a ring and $F(R)$ the group of invertible fractional ideals of R and $I(R)$ the semigroup of invertible integral ideals of R. If $I, J \in I(R)$, then I divides J $(I|J)$ if $J = IK$ for some ideal K of R. The common divisor of I and J which is divisible by every common divisor of I and J (if such exists) is denoted by $\gcd(I, J)$, and $\operatorname{lcm}(I, J)$ is defined analogously. The existence and arithmetic properties of these in the case of finitely generated faithful multiplication ideals are discussed in [3]. If $A, B \in F(R)$, then A divides B if there exists an integral ideal I of R such that $B = IA$. By analogy with the definitions of gcd and lcm, we define $\mathrm{GCD}(A, B)$ as a fractional ideal which is a common divisor of A and B divisible by every common divisor of A and B (if such exists). Similarly, we define $\mathrm{LCM}(A, B)$ as a fractional ideal which is a common multiple of A and B which divides every common multiple of A and B (if such exists).

If $B \in F(R)$ and A is any fractional ideal, then $A \subseteq B$ (and hence $A = AB^{-1}B$ where AB^{-1} is an integral ideal) if and only if $B|A$. Also, if $B \in F(R)$ and G is any fractional ideal such that $G|B$, then $G \in F(R)$. In particular, for all $A, B \in F(R)$, if $\mathrm{GCD}(A, B)$ exists, then it is in $F(R)$. Moreover, if $A, B \in F(R)$ have least common multiple, say $K = \mathrm{LCM}(A, B)$, then there exists a non-zero divisor $x \in R$ such that xA and xB are in $I(R)$ and x^2AB is a common multiple of A and B. Therefore $K|x^2AB$. As $x^2AB \in F(R), K \in F(R)$. Let $I, J \in I(R)$ and $A, B \in F(R)$. If $\gcd(I, J)$ (resp. $\operatorname{lcm}(I, J), \mathrm{GCD}(A, B), \mathrm{LCM}(A, B)$) does exist, then it is unique.

Let X be a fractional ideal of R. Then $X_v = (X^{-1})^{-1}$ is a fractional ideal of R. Suppose that $A, B \in F(R)$ such that $(A+B)_v \in F(R)$. Then $A = A_v \subseteq (A+B)_v$, and $B \subseteq (A+B)_v$, and hence $(A + B)_v$ is a common divisor of A and B. If G is any fractional ideal with $G|A$ and $G|B$, then $G \in F(R)$ and $A + B \subseteq G$. Hence $(A + B)_v \subseteq G_v = G$. Therefore, $G|(A + B)_v$, and $(A + B)_v = \mathrm{GCD}(A, B)$. Conversely, suppose that $G = \mathrm{GCD}(A, B)$ exists. Then $A + B \subseteq G$, and hence $G^{-1} \subseteq (A + B)^{-1} = A^{-1} \cap B^{-1}$. On the other hand, for all $x \in A^{-1} \cap B^{-1}, xR \subseteq A^{-1}$, and $xR \subseteq B^{-1}$. Hence $A \subseteq x^{-1}R$ and $B \subseteq x^{-1}R$. It follows that $x^{-1}R$ is a common divisor of A and B, and hence $x^{-1}R|G$. This implies that $G^{-1}|xR$, and hence $x \in G^{-1}$. Therefore, $A^{-1}\cap B^{-1} \subseteq G^{-1}$, and this gives that $G^{-1} = (A+B)^{-1}$, and hence $G = (A + B)_v$. So for all $A, B \in F(R), \mathrm{GCD}(A, B)$ exists if and only if $(A + B)_v \in F(R)$, and in this case $\mathrm{GCD}(A, B) = (A + B)_v$.

If $A, B \in F(R)$, then it is easily verified that $\mathrm{LCM}(A, B)$ exists if and only if $A \cap B \in F(R)$, and in this case $\mathrm{LCM}(A, B) = A \cap B$.

In this section we extend results on greatest common divisors and least common multiples of quotients of elements of integral domains given in [21] and [27] to fractional invertible ideals. The first lemma is mentioned in [3] and the second follows immediately by [3, Theorem 2.2].

Lemma 1.1. *Let R be a ring.*

1. *Suppose $A, B, C, D \in F(R)$ with $A \subseteq C$ and $B \subseteq D$. If $\mathrm{GCD}(A, B)$ and $\mathrm{GCD}(C, D)$ exist, then $\mathrm{GCD}(A, B) \subseteq \mathrm{GCD}(C, D)$.*
2. *If $A_1, \dots, A_n \in F(R)$ and $\mathrm{GCD}(A_1, \dots, A_n)$ and $\mathrm{GCD}(A_1, \dots, A_{n-1})$ exist, then*

$$\mathrm{GCD}(A_1, \dots, A_n) = \mathrm{GCD}(\mathrm{GCD}(A_1, \dots, A_{n-1}), A_n).$$

Lemma 1.2. *Let R be a ring and $A, B, C \in F(R)$, and $I, J, K \in I(R)$. Then*

1. *$\mathrm{LCM}(A, B)$ exists if and only if $\mathrm{LCM}(CA, CB)$ exists, and in this case*

$$\mathrm{LCM}(CA, CB) = C\mathrm{LCM}(A, B).$$

2. *If $\mathrm{GCD}(CA, CB)$ exists, then so too does $\mathrm{GCD}(A, B)$, and in this case*

$$\mathrm{GCD}(CA, CB) = C\mathrm{GCD}(A, B).$$

3. *$\mathrm{lcm}(I, J)$ exists if and only if $\mathrm{lcm}(KI, KJ)$ exists, and in this case*

$$\mathrm{lcm}(KI, KJ) = K\mathrm{lcm}(I, J).$$

4. *If $\gcd(KI, KJ)$ exists, then so too does $\gcd(I, J)$, and in this case*

$$\gcd(KI, KJ) = K \gcd(I, J).$$

Compare the next result with [3, Theorem 2.1] and [21, Theorem 3].

Theorem 1.3. *Let R be a ring and $A, B \in F(R)$. Then $\mathrm{GCD}(A, B)$ exists if and only if $\mathrm{LCM}(A, B)$ exists, and in this case, $AB = \mathrm{GCD}(A, B)\mathrm{LCM}(A, B)$.*

Proof. Suppose that $\mathrm{GCD}(A, B)$ exists. As noted earlier, $\mathrm{GCD}(A, B)^{-1} = A^{-1} \cap B^{-1}$, and hence $A^{-1} \cap B^{-1} \in F(R)$. It follows that $\mathrm{LCM}(A^{-1}, B^{-1})$ exists, and by Lemma 1.2(1), $AB\mathrm{LCM}(A^{-1}, B^{-1}) = \mathrm{LCM}(A, B)$ exists.

Conversely, assume that $\mathrm{LCM}(A, B)$ exists. Then again by Lemma 1.2(1), $\mathrm{LCM}(A^{-1}, B^{-1})$ exists, and hence $A^{-1} \cap B^{-1} \in F(R)$. But $A^{-1} \cap B^{-1} = (A + B)^{-1} \in F(R)$. Thus $(A + B)_v \in F(R)$, and hence $\mathrm{GCD}(A, B) = (A + B)_v$. Next, since $\mathrm{GCD}(A, B)^{-1} = \mathrm{LCM}(A^{-1}, B^{-1})$, we infer that

$$AB\mathrm{GCD}(A, B)^{-1} = AB\mathrm{LCM}(A^{-1}, B^{-1}) = \mathrm{LCM}(A, B),$$

and hence $AB = \mathrm{GCD}(A, B)\mathrm{LCM}(A, B)$.

Corollary 1.4. *Let R be a ring and $A, B, C \in F(R)$. If $\mathrm{GCD}(A, B)$ exists, then so too does $\mathrm{GCD}(CA, CB)$, and in this case $\mathrm{GCD}(CA, CB) = C\mathrm{GCD}(A, B)$.*

Proof. The existence of $\mathrm{GCD}(CA, CB)$ follows from Theorem 1.3 and Lemma 1.2(1). Also it is clear that

$$[C\mathrm{GCD}(A,B)]^{-1} = C^{-1}\mathrm{GCD}(A,B)^{-1} = C^{-1}\mathrm{LCM}(A^{-1},B^{-1})$$

$$= \mathrm{LCM}((CA)^{-1},(CB)^{-1}) = \mathrm{GCD}(CA,CB)^{-1},$$

and therefore, $C\mathrm{GCD}(A,B) = \mathrm{GCD}(CA,CB)$.

Corollary 1.5. *Let R be a ring and $I, J \in I(R)$. Then:*

1. *If* $\mathrm{GCD}(I,J)$ *exists, then so too does* $\gcd(I,J)$, *and in this case* $\mathrm{GCD}(I,J) = \gcd(I,J)$.
2. $\mathrm{LCM}(I,J)$ *exists if and only if* $\mathrm{lcm}(I,J)$ *exists, and in this case* $\mathrm{LCM}(I,J) = \mathrm{lcm}(I,J)$.

Proof. 1. Let $G = \mathrm{GCD}(I,J)$. Then $G \in F(R)$. Also there exists a non-zero divisor x such that $xG \in I(R)$. Now xG is a common divisor of xI and xJ. Let G' be an integral ideal which is a common divisor of xI and xJ. Then $x^{-1}G'|I$, and $x^{-1}G'|J$, and therefore $x^{-1}G'|G$. Hence, $G'|xG$, and $xG = \gcd(xI, xJ)$. By Lemma 1.2(4), $xG = x\gcd(I,J)$, and hence $G = \gcd(I,J)$.

2. Let $K = \mathrm{LCM}(I,J)$. Then clearly $K \in I(R)$, and hence $K = \mathrm{lcm}(I,J)$. The converse is obvious.

We make two remarks on Corollary 1.5. The first is [3, Theorem 2.1]. If $I, J \in I(R)$ such that $\mathrm{lcm}(I,J)$ exists, then by Corollary 1.5(2), $\mathrm{LCM}(I,J)$ exists and $\mathrm{LCM}(I,J) = \mathrm{lcm}(I,J)$. From Theorem 1.3, we infer that $\mathrm{GCD}(I,J)$ exists and $IJ = \mathrm{GCD}(I,J)\mathrm{LCM}(I,J)$, and by Corollary 1.5(1), we obtain that $\gcd(I,J)$ exists and $IJ = \gcd(I,J)\mathrm{lcm}(I,J)$.

The second remark is that the converse of Corollary 1.5(1) is not true. For example, let $R = k[x^2, x^3], k$ a field. then $\gcd(x^2R, x^3R) = R$, but $\mathrm{GCD}(x^2R, x^3R)$ does not exist. We can however, state the following.

Proposition 1.6. *Let R be a ring. Then:*

1. $\gcd(I,J)$ *exists for all* $I, J \in I(R)$ *if and only if* $\mathrm{GCD}(A,B)$ *exists for all* $A, B \in F(R)$.
2. $\mathrm{lcm}(I,J)$ *exists for all* $I, J \in I(R)$ *if and only if* $\mathrm{LCM}(A,B)$ *exists for all* $A, B \in F(R)$.

Proof. Let $A, B \in F(R)$. There exists a non-zero divisor $x \in R$ such that $xA, xB \in I(R)$. Suppose that $G = \gcd(xA, xB)$. Then $G \in I(R)$, and $G|xA, G|xB$. This implies that $x^{-1}G|A$, and $x^{-1}G|B$. Assume that G' is a fractional ideal of R such that $G'|A, G'|B$. Then $G' \in F(R)$ and $xG'|xA, xG'|xB$. It follows that $xG'|G$, and hence $G'|x^{-1}G$. This shows that $x^{-1}G = \mathrm{GCD}(A,B)$. The converse follows by Corollary 1.5(1). Part (2) is similar.

In the next theorem, we state some Ohm-type properties for GCD and LCM of invertible fractional ideals.

Theorem 1.7. *Let R be a ring and $A, B \in F(R)$ such that* $\mathrm{GCD}(A,B)$ *exists. Then:*

1. $\mathrm{LCM}(A,B)^k = \mathrm{LCM}(A^k, B^k)$ *for all* $k \in \mathbb{N}$.
2. $\mathrm{GCD}(A,B)^k = \mathrm{GCD}(A^k, B^k)$ *for all* $k \in \mathbb{N}$.
3. $[A : B]^k = [A^k : B^k]$ *for all* $k \in \mathbb{N}$.

Proof. (1) As $\mathrm{GCD}(A,B)$ exists, so too does $\mathrm{LCM}(A,B)$. There exists a non-zero divisor $x \in R$ such that $xA, xB \in I(R)$. By Lemma 1.2, Corollary 1.5(2) and [3, Theorem 2.6(i)], we have that

$$x^k\mathrm{LCM}(A,B)^k = (x\mathrm{LCM}(A,B))^k = \mathrm{LCM}(xA,xB)^k = \mathrm{lcm}(xA,xB)^k$$

$$= \mathrm{lcm}(x^kA^k, x^kB^k) = \mathrm{LCM}(x^kA^k, x^kB^k) = x^k\mathrm{LCM}(A^k,B^k).$$

Hence, $\mathrm{LCM}(A,B)^k = \mathrm{LCM}(A^k,B^k)$.

(2) By Theorem 1.3 and part (1), we get that $\mathrm{GCD}(A,B)^{-1} = \mathrm{LCM}(A^{-1},B^{-1})$, and hence

$$\mathrm{GCD}(A,B)^k = (\mathrm{GCD}(A,B)^{-1})^{-k} = \mathrm{LCM}(A^{-1},B^{-1})^{-k}$$

$$= (\mathrm{LCM}(A^{-1},B^{-1})^k)^{-1} = \mathrm{LCM}(A^{-k},B^{-k})^{-1} = \mathrm{GCD}(A^k,B^k).$$

(3) This follows since $\mathrm{LCM}(A,B) = [A:B]B$, and $\mathrm{LCM}(A^k,B^k) = [A^k:B^k]B^k$.

D. D. and D. F. Anderson [8] introduced the generalized GCD domains (GGCD-domains) as those for which the intersection of any two invertible integral ideals of is invertible. Equivalently, the intersection of any two invertible fractional ideals is invertible.

By combining Theorem 1.3 and Proposition 1.6, we can state the next result summarizing several equivalent criteria of [3, Theorem 3.1], [8, Theorem 1], [21, Theorem 5], and [24, Theorem 1], and including some extensions which follow by induction.

Theorem 1.8. *Let R be an integral domain and K its quotient field. Then the following are equivalent.*

1. *R is a GGCD-domain.*
2. *For all $a,b \in R-\{0\}$,* $\quad aR \cap bR \in I(R)$.
3. *For all $a,b \in K-\{0\}$,* $\quad aR \cap bR \in F(R)$.
4. *For all $a,b \in R-\{0\}$,* $\quad \mathrm{lcm}(aR,bR)$ *exists.*
5. *For all $a,b \in R-\{0\}$,* $\quad \mathrm{gcd}(aR,bR)$ *exists.*
6. *For all $a,b \in K-\{0\}$,* $\quad \mathrm{LCM}(aR,bR)$ *exists.*
7. *For all $a,b \in K-\{0\}$,* $\quad \mathrm{GCD}(aR,bR)$ *exists.*
8. *For all $a,b \in R-\{0\}$,* $\quad (aR+bR)_v \in I(R)$.
9. *For all $a,b \in K-\{0\}$,* $\quad (aR+bR)_v \in F(R)$.
10. *For all $a,b \in R-\{0\}$,* $\quad [aR:bR] \in I(R)$.
11. *For all $a,b \in K-\{0\}$,* $\quad [aR:bR] \in I(R)$.
12. *For all $a_1,\dots,a_n \in R-\{0\}$,* $\quad \bigcap_{i=1}^{n} a_iR \in I(R)$.
13. *For all $a_1,\dots,a_n \in K-\{0\}$,* $\quad \bigcap_{i=1}^{n} a_iR \in F(R)$.
14. *For all $a_1,\dots,a_n \in R-\{0\}$,* $\quad \mathrm{lcm}(a_1R,\dots,a_nR)$ *exists.*
15. *For all $a_1,\dots,a_n \in R-\{0\}$,* $\quad \mathrm{gcd}(a_1R,\dots,a_nR)$ *exists.*
16. *For all $a_1,\dots,a_n \in K-\{0\}$,* $\quad \mathrm{LCM}(a_1R,\dots,a_nR)$ *exists.*

17. *For all* $a_1, \ldots, a_n \in K - \{0\}$, $\quad \mathrm{GCD}(a_1R, \ldots, a_nR)$ *exists.*
18. *For all* $a_1, \ldots, a_n \in R - \{0\}$, $\quad (\sum_{i=1}^{n} a_iR)_v \in I(R)$.
19. *For all* $a_1, \ldots, a_n \in K - \{0\}$, $\quad (\sum_{i=1}^{n} a_iR)_v \in F(R)$.
20. *For all* $A, B \in F(R)$, $\quad \mathrm{LCM}(A, B)$ *exists.*
21. *For all* $A, B \in F(R)$, $\quad \mathrm{GCD}(A, B)$ *exists.*
22. *For all* $A, B \in F(R)$, $\quad (A + B)_v \in F(R)$.
23. *For all* $I, J \in I(R)$, $\quad \mathrm{lcm}(I, J)$ *exists.*
24. *For all* $I, J \in I(R)$, $\quad \gcd(I, J)$ *exists.*
25. *For all* $I, J \in I(R)$, $\quad (I + J)_v \in I(R)$.
26. *For all* $A, B \in F(R)$, $\quad [A : B] \in I(R)$.
27. *For all* $I, J \in I(R)$, $\quad [I : J] \in I(R)$.
28. *For all* $A_1, \ldots, A_n \in F(R)$, $\quad \bigcap_{i=1}^{n} A_i \in F(R)$.
29. *For all* $A_1, \ldots, A_n \in F(R)$, $\quad \mathrm{LCM}(A_1, \ldots, A_n)$ *exists.*
30. *For all* $A_1, \ldots, A_n \in F(R)$, $\quad \mathrm{GCD}(A_1, \ldots, A_n)$ *exists.*
31. *For all* $A_1, \ldots, A_n \in F(R)$, $\quad (\sum_{i=1}^{n} A_i)_v \in F(R)$.
32. *For all* $I_1, \ldots, I_n \in I(R)$, $\quad \bigcap_{i=1}^{n} I_i \in I(R)$.
33. *For all* $I_1, \ldots, I_n \in I(R)$, $\quad \mathrm{lcm}(I_1, \ldots, I_n)$ *exists.*
34. *For all* $I_1, \ldots, I_n \in I(R)$, $\quad \gcd(I_1, \ldots, I_n)$ *exists.*
35. *For all* $I_1, \ldots, I_n \in I(R)$, $\quad (\sum_{i=1}^{n} I_i)_v \in I(R)$.
36. *For all* $A \in F(R)$, $\quad R \cap A \in I(R)$.
37. *For all* $A \in F(R)$, $\quad \mathrm{LCM}(R, A)$ *exists.*
38. *For all* $A \in F(R)$, $\quad \mathrm{GCD}(R, A)$ *exists.*
39. *For all* $A \in F(R)$, $\quad (R + A)_v \in F(R)$.
40. *For all* $A \in F(R)$, $\quad [R : A] \in I(R)$.

The next result is a version of the Chinese Remainder Theorem for invertible fractional ideals. Compare with [3, Corollary 3.3].

Corollary 1.9. *Let R be a GGCD-domain. Then for all $A, B, C \in F(R)$,*

1. $\mathrm{LCM}(\mathrm{GCD}(A, B), C) = \mathrm{GCD}(\mathrm{LCM}(A, C), \mathrm{LCM}(B, C))$.
2. $\mathrm{GCD}(A, \mathrm{LCM}(B, C)) = \mathrm{LCM}(\mathrm{GCD}(A, B), \mathrm{GCD}(A, C))$.

Proof. (1) Let $G = \mathrm{GCD}(A, B)$. Then by Corollary 1.4, $\mathrm{GCD}(AG^{-1}, BG^{-1}) = R$, and hence by Lemma 1.1,

$$\begin{aligned}\mathrm{LCM}(G, C) &= \mathrm{LCM}(G, C)\mathrm{GCD}(AG^{-1}, BG^{-1})\\ &= \mathrm{GCD}(AG^{-1}\mathrm{LCM}(G, C), BG^{-1}\mathrm{LCM}(G, C))\\ &= \mathrm{GCD}(\mathrm{LCM}(AG^{-1}G, AG^{-1}C), \mathrm{LCM}(BG^{-1}G, BG^{-1}C))\\ &\subseteq \mathrm{GCD}(\mathrm{LCM}(A, AA^{-1}C), \mathrm{LCM}(B, BB^{-1}C))\\ &= \mathrm{GCD}(\mathrm{LCM}(A, C), \mathrm{LCM}(B, C)).\end{aligned}$$

The other inclusion is clearly true, and (1) follows.

(2) Using the fact that if R is a GGCD-domain then for all $X, Y \in F(R)$,

$$\mathrm{GCD}(X, Y)^{-1} = \mathrm{LCM}(X^{-1}, Y^{-1}), \text{and} \quad \mathrm{LCM}(X, Y)^{-1} = \mathrm{GCD}(X^{-1}, Y^{-1}),$$

and part (1), we have that

$$\begin{aligned}\mathrm{LCM}(\mathrm{GCD}(A, B), \mathrm{GCD}(A, C)) &= (\mathrm{LCM}(\mathrm{GCD}(A, B), \mathrm{GCD}(A, C))^{-1})^{-1}\\ &= (\mathrm{GCD}(\mathrm{GCD}(A, B)^{-1}, \mathrm{GCD}(A, C)^{-1})^{-1}\\ &= \mathrm{GCD}(\mathrm{LCM}(A^{-1}, B^{-1}), \mathrm{LCM}(A^{-1}, C^{-1}))^{-1}\\ &= \mathrm{LCM}(A^{-1}, \mathrm{GCD}(B^{-1}, C^{-1}))^{-1}\\ &= \mathrm{LCM}(A^{-1}, \mathrm{LCM}(B, C)^{-1})^{-1}\\ &= \mathrm{GCD}(A, \mathrm{LCM}(B, C)),\end{aligned}$$

as required.

Let R be a GGCD-domain. Let $A \in F(R)$. Then $A = IJ^{-1}$ for some $I, J \in I(R)$ with $\gcd(I, J) = R$. For example, there is a non-zero divisor $x \in R$ such that $xA \in I(R)$. Letting $D = \gcd(xR, xA)$, we may take $I = xAD^{-1}$ and $J = xD^{-1}$.

In the next two results, we use this observation to calculate the GCD and LCM of invertible fractional ideals in terms of gcd and lcm of invertible integral ideals, generalizing Lüneburg's results, [27, Theorems 1 and 5]. See also [21, Theorem 8].

Theorem 1.10. *Let R be a GGCD-domain and $A, B \in F(R)$ such that $A = IJ^{-1}$ and $B = KL^{-1}$ where $I, J, K, L \in I(R)$ and $\gcd(I, J) = R = \gcd(K, L)$. Then*

$$\mathrm{GCD}(A, B) = \gcd(I, K)\mathrm{lcm}(J, L)^{-1}.$$

Proof. It is enough to show that $JL\mathrm{GCD}(A, B) = \gcd(I, K)\gcd(J, L)$. It follows from Corollaries 1.4 and 1.5 that

$$JL\mathrm{GCD}(A, B) = JL\mathrm{GCD}(IJ^{-1}, KL^{-1}) = \mathrm{GCD}(IL, JK) = \gcd(IL, JK),$$

and by Lemma 1.1,

$$\gcd(IL, JK) \subseteq \gcd(\gcd(I, K)L, \gcd(I, K)J) = \gcd(I, K)\gcd(J, L).$$

On the other hand, let $G = \gcd(IL, JK)$. As $\gcd(I, J) = R = \gcd(K, L)$, we infer from [3, Proposition 2.3] that

$$\gcd(I, K) = \gcd(I, KJ), \quad \gcd(J, L) = \gcd(J, IL),$$

$$\gcd(K, I) = \gcd(K, IL), \quad \gcd(L, J) = \gcd(L, KJ).$$

Using these four equalities, [3, Proposition 2.3], and Lemma 1.1 we get that

$$\begin{aligned}
&\gcd(I, K) \gcd(J, L) = \gcd(I, KJ) \gcd(J, IL)\\
&\quad\subseteq \gcd(\gcd(I, K), KJ) \gcd(\gcd(J, L), IL)\\
&\quad= \gcd(\gcd((K, IL), KJ) \gcd(\gcd(L, KJ), IL)\\
&\quad= \gcd(K, \gcd(IL, KJ)) \gcd(\gcd(L, \gcd(IL, KJ))\\
&\quad= \gcd(K, G) \gcd(L, G) = \gcd(K \gcd(L, G), G \gcd(L, G))\\
&\quad= \gcd(\gcd(KL, KG), \gcd(GL, G^2)) = \gcd(\gcd(KL, KG), GL, G^2)\\
&\quad= \gcd(\gcd(GL, GK), KL, G^2) = \gcd(G \gcd(L, K), KL, G^2)\\
&\quad= \gcd(G, KL, G^2) = \gcd(KL, G) = \gcd(KL \gcd(I, J), G)\\
&\quad= \gcd(\gcd(IKL, JKL), G) \subseteq \gcd(\gcd(IL, JK), G) = G.
\end{aligned}$$

This finishes the proof of the theorem.

Corollary 1.11. *Let R be a GGCD-domain and A, B as in Theorem 1.10. Then*

$$\mathrm{LCM}(A, B) = \mathrm{lcm}(I, K) \gcd(J, L)^{-1}.$$

Proof. From Theorem 1.3 we have that $AB = \mathrm{GCD}(A, B)\mathrm{LCM}(A, B)$, and from Theorem 1.10 we obtain that

$$\mathrm{LCM}(A, B) = AB\mathrm{GCD}(A, B)^{-1} = IJ^{-1}KL^{-1}(\gcd(I, K)\mathrm{lcm}(J, L)^{-1})^{-1}$$

$$= IK \gcd(I, K)^{-1}J^{-1}L^{-1}\mathrm{lcm}(J, L) = \mathrm{lcm}(I, K) \gcd(J, L)^{-1},$$

and the result is proved.

Let R be a ring and S a non-empty subset of $I(R)$. We define $G = \gcd(S)$ as an integral ideal which is a common divisor of all elements of S and which is divisible by all common divisors of all elements of S. In the analogous way we define $\mathrm{lcm}(S)$, and if $S \subseteq F(R)$, we also define $\mathrm{GCD}(S), \mathrm{LCM}(S)$ analogously.

Any finite set S of invertible integral ideals has an invertible common divisor and common multiple (for example R, $\prod_{I \in S} I$ respectively). Any finite set S of n invertible fractional ideals also has an invertible common divisor and common multiple. For example, there exists a non-zero divisor x such that for all $A \in S$, $\quad xA \in I(R)$, so $x^{-1}R$ and $x^n \prod_{A \in S} A$ are invertible common divisor and common multiple of S respectively.

However, if S is an infinite set of invertible ideals, then it is not necessarily true that S has an invertible common divisor or a common multiple. Therefore, in the next result we assume the existence of invertible common divisor and common multiple. It is not difficult to see that if $S \subseteq F(R)$, then GCD(S) exists if and only if $(\sum_{A\in S} A)_v \in F(R)$, and in this case, GCD$(S) = (\sum_{A\in S} A)_v$. Also, LCM$(S)$ exists if and only if $\bigcap_{A\in S} A \in F(R)$, and in this case LCM$(S) = \bigcap_{A\in S} A$. GCD$(S)$ exists if and only if LCM(S) exists, and in this case, GCD$(S)^{-1} =$ LCM(S^{-1}), and LCM$(S) =$ GCD(S^{-1}), where $S^{-1} = \{A^{-1} : A \subseteq S\}$.

The final result of this section should be compared with [21, Theorem 9].

Theorem 1.12. *Let R be a ring. The following are equivalent.*

1. *For all non-empty $S \subseteq F(R)$ with common divisor,* GCD(S) *exists.*
2. *For all non-empty $S \subseteq F(R)$ with common multiple in $F(R)$,* LCM(S) *exists and is in $F(R)$.*
3. *For all non-empty $S \subseteq I(R)$ with common multiple in $I(R)$,* lcm(S) *exists and is in $I(R)$.*
4. *For all non-empty $S \subseteq I(R)$ with common divisor,* gcd(S) *exists.*

Proof. (1) $\Rightarrow$ (2). Let $S \subseteq F(R)$ such that S has a common multiple in $F(R)$. Then S^{-1} has a common divisor in $F(R)$, and hence GCD(S^{-1}) exists in $F(R)$. But GCD$(S^{-1}) = (\sum_{A\in S} A^{-1})_v \in F(R)$. It follows that $\bigcap_{A\in S} A = (\sum_{A\in S} A^{-1})^{-1} \in F(R)$, and hence LCM$(S)$ exists.

(2) $\Rightarrow$ (1), Let $S \subseteq F(R)$ such that S has a common divisor. Then S^{-1} has a common multiple in $F(R)$, and therefore LCM(S^{-1}) exists. It follows that $\bigcap_{A\in S} A^{-1} \in F(R)$, and hence $(\sum_{A\in S} A)^{-1} \in F(R)$. This implies that $(\sum_{A\in S} A)_v \in F(R)$, and hence GCD$(S)$ exists.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (4). Let $S \subseteq I(R)$, and let H be the set of all common divisors of S. Then $H \subseteq I(R)$ and H is non-empty as $R \in H$. Also, H has a common multiple (in fact every $J \in S$ is a common multiple of H). Hence, H has a least common multiple, $K \in I(R)$. Clearly, K is a common divisor of S. Let K' be any common divisor of S. Then $K' \in I(R)$ and $K' \in H$, so that $K'|K$. Hence $K = \gcd(S)$.

(4) $\Rightarrow$ (1). Let $S \subseteq F(R)$ and let X be a common divisor of S. Then $X \in F(R)$. For each $A \in S$, there exists $I_A \in I(R)$ such that $A = I_A X$. Let $M = \{I_A : A \in S\}$. Let $G = \gcd(M)$. Then $G|X^{-1}A$ and hence $XG|A$, for all $A \in S$. Assume now that G' is another common divisor of S. Then $G' \in F(R)$, and $X^{-1}G'|X^{-1}A$, so that $X^{-1}G'|I_A$ for all $A \in S$. There exists a non-zero divisor $y \in R$ such that $yX^{-1}G' \in I(R)$. Also $yX^{-1}G'|yI_A$ for all $A \in S$. By the assumption, $\gcd\{yI_A : A \in S\}$ exists, and also

$$\gcd\{yI_A : A \in S\} = y\gcd\{I_A : A \in S\} = yG.$$

It follows that $yX^{-1}G'|yG$, and hence $X^{-1}G'|G$. This implies that $G'|XG$, and this shows that $XG =$ GCD(S).

It is easy to see that an integral domain is a Prüfer GCD-domain if and only if it is a Bezout domain, and that a Prüfer domain need not be a GCD-domain. Clearly any GCD-domain is a GGCD-domain, and any Prüfer domain is a GGCD-domain.

We define a *pseudo-generalized GCD domain* (PGGCD-domain) to be an integral domain in which every non-empty set of invertible ideals which has a common divisor has a greatest common divisor. Theorem 1.12 gives several equivalent conditions. Dedekind domains are PGGCD-domains. Every PGGCD-domain is a GGCD-domain, but the converse is not true. Let E be the ring of entire functions, and let P be a maximal free ideal of E. Set $K = E/P$, where K is a proper extension of the field $\mathbb{C}$ of complex numbers. Let $t \in K$ be transcendental over $\mathbb{C}$ and let V_0 be a non-trivial valuation domain on $\mathbb{C}(t)$. Then V_0 can be extended to a non-trivial valuation domain V on K. Define $\phi : E \to K = E/P$ as a canonical homomorphism and $R = \phi^{-1}(V)$, see [15, Example 8.4.1]. Then R is a Prüfer domain and hence a GGCD-domain. P is a noninvertible divisorial ideal of R. Hence P and P^{-1} are noninvertible (and hence not f.g.) integral (fractional) ideals of R. If X is a set of generators of P, and $S = \{p^{-1}R : p \in X\}$, then R is an invertible common multiple of S, but $\bigcap_{p\in X} p^{-1}R = P^{-1}$ is not in $F(R)$, and hence $\mathrm{LCM}(S)$ does not exist. This shows that R is not a PGGCD-domain.

A PGGCD-domain need not be a GCD domain. For example let R be the ring of integers of the quadratic field $\mathbb{Q}(\sqrt{d})$, where d is a square-free non-zero integer. Then R is a Dedekind domain ([16], [34]) and hence is a PGGCD-domain. R is a Bezout domain if and only if $d \in \{-1, -2, -3, -7, -11, -19, -43, -67, -163\}$, see [34]. Therefore if we take $d < 0$ outside the previous set, then R is a Prüfer domain but not a Bezout domain, and hence not a GCD-domain.

2. gcd and lcm of projective ideals

A projective module is characterized, see [23], as a direct summand of a free module. If R is an integral domain and A a fractional ideal of R, then A is invertible if and only if A is a projective $R-$module, see [16]. It is also well known that projective ideals are multiplication, see [36]. The converse is studied for the finitely generated case in [31], [35], and [36]. It is proved that a f.g. ideal I of R is a projective ideal if and only if I is multiplication and $\mathrm{ann}I = eR$ for some idempotent e, see [31, Theorem 2.1] and [35, Theorem 11]. Let R be a ring and M a maximal ideal of R. If I is a f.g. projective ideal of R, then I_M is principal, [31], and $\mathrm{ann}(I_M) = eR_M$ for some idempotent e. As R_M is local, either e or $1-e$ is a unit in R_M. If e is a unit, then $I_M = 0_M$. Otherwise $1-e$ is a unit and hence $e = 0$. In this case, I_M is invertible. For details about projective ideals, see also [13], [14], [20], and [33].

In [3] we investigated the gcd and lcm of f.g. faithful multiplication ideals of a ring R. In this note we generalize these results to f.g. projective ideals. Let R be a ring and $S(R)$ the semigroup of f.g. projective ideals of R.

This first result should be compared with [3, Lemmas 1.4 and 1.5].

Theorem 2.1. *Let R be a ring and $A, B \in S(R)$. Then*

1. *If* $\gcd(A, B)$ *exists, then it is in* $S(R)$.
2. *If* $\mathrm{lcm}(A, B)$ *exists, then it is in* $S(R)$.

Proof. For (1), Suppose that $G = \gcd(A, B)$. Let $\mathrm{ann}A = e_1R$ and $\mathrm{ann}B = e_2R$ for some idempotents e_1 and e_2. Then

$$\mathrm{ann}(A + B) = \mathrm{ann}A \cap \mathrm{ann}B = e_1R \cap e_2R = e_1e_2R = eR,$$

and e is idempotent. It follows that $eA = 0 = eB$, and hence $(1-e)R$ is a common divisor of A and B. Hence $(1-e)R|G$, and therefore $eG = 0$. This shows that $\text{ann}(A+B) = eR \subseteq \text{ann}G$. The other inclusion is obviously true, and hence $\text{ann}G = \text{ann}(A+B)$. As $G|A$, there exists an ideal I of R such that $A = IG$. It follows that

$$A + \text{ann}A = IG + \text{ann}A = (I + \text{ann}A)(G + \text{ann}A),$$

and therefore $G + \text{ann}A|A + \text{ann}A$. Since $A + \text{ann}A = [A^2 : A]$, [11, p.430] and [2, Lemma 1.2], we have from [35, Corollary 1 of Theorem 10] that $A + \text{ann}A$ is a f.g. multiplication ideal. Also, it is easy to see that

$$\text{ann}(A + \text{ann}A) = \text{ann}A \cap \text{ann}(\text{ann}A) = 0,$$

i.e. $A + \text{ann}A$ is a f.g. faithful multiplication ideal of R. It follows from [3, Lemma 1.4] that $G + \text{ann}A$ is a f.g. faithful multiplication ideal. Similarly, $G + \text{ann}B$ is a f.g. faithful multiplication ideal. Also

$$(G + \text{ann}A) + (G + \text{ann}B) = (G + \text{ann}A) + \text{ann}B = [(G + \text{ann}A)B : B]$$

is a f.g. faithful multiplication ideal of R [35, Corollary 1 of Theorem 10]. Next, since $\text{ann}A + \text{ann}B = e_1R + e_2R = (e_1 + e_2 - e_1e_2)R$, which is f.g. multiplication, we infer from [4, Theorem 2.1] that

$$(G + \text{ann}A) \cap (G + \text{ann}B) = G + (\text{ann}A \cap \text{ann}B) = G + \text{ann}(A + B) = G + \text{ann}G.$$

It follows by [35, Lemma 7], [4, Theorem 2.3] that $G+\text{ann}G$ is multiplication. But $G\cap\text{ann}G = 0$, for if $x \in G \cap \text{ann}G$, then $x \in G$ and $x = re$, $r \in R$, $eG = 0$. This implies that $x = re = re^2 \in eG = 0$. It follows from [35, Theorem 8], [4, Theorems 3.6 and 4.2] that G is a multiplication ideal of R. Finally, since $A + B \subseteq G$ and $\text{ann}(A + B) = \text{ann}G$, it follows from [25, Corollary 1 of Lemma 1.5] that G is f.g. and by [31, Theorem 2.1], $G \in S(R)$, and part (1) of the theorem is concluded.

For part (2), let $K = \text{lcm}(A, B)$. We first show that $\text{ann}K = \text{ann}(AB)$. $AB \subseteq K$ since $K|AB$, so $\text{ann}K \subseteq \text{ann}(AB)$. Let $\text{ann}(AB) = eR$ for some idempotent e. As $A|K$, we have $eK \subseteq K \subseteq A$. Also, $eKA \subseteq eBA = 0$. It follows that $eK \subseteq A \cap \text{ann}A = 0$, and hence $\text{ann}(AB) = eR \subseteq \text{ann}K$. This shows that $\text{ann}K = \text{ann}(AB)$. Next, since $K|AB$, we have that $K + \text{ann}K|AB + \text{ann}K$. But

$$AB + \text{ann}K = AB + \text{ann}(AB) = [A^2B^2 : AB]$$

which is a f.g. multiplication ideal, see [11, p. 430] and [35, Corollary 1 of Theorem 10]. Moreover, it is clearly faithful. Therefore by [3, Lemma 1.4], we have that $K + \text{ann}K$ is a f.g. faithful multiplication ideal of R. Finally, since $K \cap \text{ann}K = 0$, we infer from [35, Lemma 7] that K is multiplication (see also [4, Theorems 3.6 and 4.2] and [12, Theorem 2.2]). Next, as $AB \subseteq K$ and $\text{ann}(AB) = \text{ann}K$, we get from [25, Corollary 1 of Lemma 1.5] that K is f.g., and by [31, Theorem 2.1], $K \in S(R)$. This finishes the proof of the theorem.

Recall that a ring R is called an arithmetical ring if every f.g. ideal of R is multiplication. R is a semihereditary ring if every f.g. ideal of R is projective. R is an f.f. ring if every f.g.

ideal of R is flat. A f.g. ideal I is flat if I is multiplication and ann$I$ is a pure ideal, [31]. It is proved [30, Theorem 2.5] that if R is an arithmetical ring and A, B are f.g. ideals of R such that ann$B$ is f.g., then $[A : B]$ is f.g. (and hence multiplication). From this result, it follows immediately that if R is a semihereditary ring and A, B are f.g. ideals of R, then $[A : B]$ is f.g. (and hence projective). These results have been generalized to modules by P. F. Smith [35, Theorem 10 and its two corollaries]. On the other hand, the Ohm property, $(A \cap B)^k = A^k \cap B^k$, for ideals A, B of a ring R is proved for ideals of Prüfer domains [17] and semihereditary rings [33].

It is further known that if A and B are f.g. multiplication (projective, flat) ideals of R such that $A + B$ is multiplication (projective, flat) then this Ohm property holds, see [29], [32] and [28] respectively. This result has been generalized for multiplication ideals (not necessarily f.g.) in [1]. We proved in [3] that if A, B are f.g. faithful multiplication ideals of a ring R such that $A \cap B$ is f.g. faithful multiplication (which is equivalent to the existence of $\mathrm{lcm}(A, B)$), then this Ohm property is satisfied.

In the next result we generalize the above results and more to f.g. projective ideals. It enables simpler proofs of most of the results in [24].

Let R be a ring and $A, B \in S(R)$. Then $\mathrm{lcm}(A, B)$ exists (and hence by Theorem 2.1, $\mathrm{lcm}(A, B) \in S(R)$) if and only if $A \cap B \in S(R)$; and in this case, $\mathrm{lcm}(A, B) = A \cap B$.

Theorem 2.2. *Let R be a ring and $A, B \in S(R)$ such that $\mathrm{lcm}(A, B)$ exists. Then the following are true.*

1. $[A : B] \in S(R)$.
2. $(A \cap B)^k = A^k \cap B^k$ *for all* $k \in \mathbb{N}$.
3. $\mathrm{lcm}(A, B)^k = \mathrm{lcm}(A^k, B^k)$ *for all* $k \in \mathbb{N}$.
4. $[A : B]^k = [A^k : B^k]$ *for all* $k \in \mathbb{N}$.
5. $C(A \cap B) = CA \cap CB$ *for every* $C \in S(R)$.
6. $C\mathrm{lcm}(A, B) = \mathrm{lcm}(CA, CB)$ *for every* $C \in S(R)$.

Proof. (1) By [35, Corollary 2 of Theorem 10], $[A : B]$ is a multiplication ideal. We now show that $\mathrm{ann}[A : B] = \mathrm{ann}(A + \mathrm{ann}B)$. Obviously, $\mathrm{ann}[A : B] \subseteq \mathrm{ann}(A + \mathrm{ann}B)$. On the other hand let $x \in \mathrm{ann}(A + \mathrm{ann}B)$. Then $xA = 0$, and $x \in \mathrm{ann}(\mathrm{ann}B)$. For each $h \in [A : B]$, $hx \in \mathrm{ann}B \cap \mathrm{ann}(\mathrm{ann}B) = 0$. Hence $x \in \mathrm{ann}[A : B]$, and therefore $\mathrm{ann}(A + \mathrm{ann}B) \subseteq \mathrm{ann}[A : B]$. It follows from [25, Corollary 1 of Lemma 1.5] that $[A : B]$ is f.g. and hence by [31, Theorem 2.1], $[A : B] \in S(R)$.

(2) It is enough to prove the result locally. Thus we may assume the R is a local ring. If $A = 0$ or $B = 0$, the result is trivial. Let A and B be invertible. Then by [3, Theorem 2.6],

$$(A \cap B)^k = \mathrm{lcm}(A, B)^k = \mathrm{lcm}(A^k, B^k) = A^k \cap B^k.$$

(3) By (2), $\mathrm{lcm}(A, B)^k = (A \cap B)^k = A^k \cap B^k$. Hence $A^k \cap B^k \in S(R)$, and therefore $\mathrm{lcm}(A^k, B^k)$ exists and $\mathrm{lcm}(A, B)^k = \mathrm{lcm}(A^k, B^k)$.

(4) Again, it suffices to prove the result locally. Thus we may assume that R is a local ring. If $B = 0$, then both sides of the relation equal R. Suppose that B is invertible (and hence B^k is invertible). As $A \cap B = [A : B]B$ and $A^k \cap B^k = [A^k : B^k]B^k$, we infer that $[A : B]^k B^k = [A^k : B^k]B^k$, and therefore $[A : B]^k = [A^k : B^k]$.

(5) Again, we prove the result locally. If $C = 0$, the result is trivial. Assume that C is invertible. It follows from [3, Theorem 2.2(i)] that

$$C(A \cap B) = C\mathrm{lcm}(A, B) = \mathrm{lcm}(CA, CB) = CA \cap CB.$$

(6) $C\mathrm{lcm}(A, B) = C(A \cap B) = CA \cap CB$, and therefore $CA \cap CB \in S(R)$. Hence, $\mathrm{lcm}(CA, CB)$ exists and $C\mathrm{lcm}(A, B) = \mathrm{lcm}(CA, CB)$.

Theorem 2.3. *Let R be a ring and $A, B \in S(R)$. Then*

1. lcm*(A,B) exists if and only if* $\mathrm{lcm}(A+\mathrm{ann}(AB), B+\mathrm{ann}(AB))$ *exists, and in this case,*
$$\mathrm{lcm}(A + \mathrm{ann}(AB), B + \mathrm{ann}(AB)) = \mathrm{lcm}(A, B) + \mathrm{ann}(AB).$$
2. *If $G = \mathrm{gcd}(A, B)$ exists, then so too does* $\mathrm{gcd}(A+\mathrm{ann}(AB), B+\mathrm{ann}(AB))$, *and in this case,*
$$\mathrm{gcd}(A + \mathrm{ann}(AB), B + \mathrm{ann}(AB)) = G + \mathrm{ann}(AB).$$
3. *If $G = \mathrm{gcd}(A, B)$ exists, then so too does* $\mathrm{gcd}(A + \mathrm{ann}G, B + \mathrm{ann}G)$, *and in this case,*
$$\mathrm{gcd}(A + \mathrm{ann}G, B + \mathrm{ann}G) = G + \mathrm{ann}G.$$

Proof. First of all we observe that $A + \mathrm{ann}(AB) = [A^2B : AB]$, and $B + \mathrm{ann}(AB) = [AB^2 : AB]$, and these are f.g. faithful multiplication ideals (and hence f.g. projective).

(1) Suppose that $K = \mathrm{lcm}(A, B)$ exists. Then $\mathrm{ann}(AB) = \mathrm{ann}K$ by Theorem 2.1. Also, $K + \mathrm{ann}K = [K^2 : K]$ is a f.g. faithful multiplication ideal and a common multiple of $A + \mathrm{ann}K$ and $B + \mathrm{ann}K$. Assume that K' is another common multiple of $A + \mathrm{ann}K$ and $B + \mathrm{ann}K$. Then $K'K$ is a common multiple of AK and BK, and by Theorem 2.2(2),

$$K'K \subseteq AK \cap BK \subseteq A^2 \cap B^2 = (A \cap B)^2 = K^2.$$

It follows that $K' \subseteq K + \mathrm{ann}K$, and hence $(K + \mathrm{ann}K)|K'$, and this shows that

$$K + \mathrm{ann}K = \mathrm{lcm}(A + \mathrm{ann}K, B + \mathrm{ann}K).$$

Suppose now that $\mathrm{lcm}(A + \mathrm{ann}(AB), B + \mathrm{ann}(AB))$ exists. Then by Theorem 2.2(6), $\mathrm{lcm}(A^2B, AB^2)$ exists, and again by Theorem 2.2(1), $[A^2B : AB^2] \in S(R)$. We now show that $[A^2B : AB^2] = [A : B] + \mathrm{ann}(AB)$. Obviously $[A^2B : AB^2] \supseteq [A : B] + \mathrm{ann}(AB)$. On the other hand, let $y \in [A^2B : AB^2]$. Then $yB(AB) \subseteq A(AB)$, and hence $yB \subseteq A + \mathrm{ann}(AB)$. It follows that $y \in [A + \mathrm{ann}(AB) : B]$. But $A + \mathrm{ann}(AB)$ is a f.g multiplication ideal. Thus by [4, Corollary 1.2],

$$\begin{aligned} y \in [A : B] + [\mathrm{ann}(AB) : B] &= [A : B] + [[0 : AB] : B] \\ &\subseteq [A : B] + [0 : AB^2] \subseteq [A : B] + [0 : A^2B^2] = [A : B] + \mathrm{ann}(A^2B^2). \end{aligned}$$

As $AB \in S(R)$, we have by [32, Corollary 2.4] that $\mathrm{ann}(AB) = \mathrm{ann}(A^2B^2)$, and therefore $y \in [A : B] + \mathrm{ann}(AB)$, and hence, $[A^2B : AB^2] \subseteq [A : B] + \mathrm{ann}(AB)$. Next, we prove that $[A : B] \cap \mathrm{ann}(AB) = \mathrm{ann}B$. Obviously, $[A : B] \cap \mathrm{ann}(AB) \supseteq \mathrm{ann}B$. On the other hand, let $x \in [A : B] \cap \operatorname{ann}(AB)$. Then $xB \subseteq A$, and $x \in \operatorname{ann}(AB) = eR$ for some idempotent e. Hence $x = re, r \in R$, and $eAB = 0$. It follows that $(reB)(eB) \subseteq eAB = 0$, and hence $x \in \operatorname{ann}(B^2)$. But B is projective and $\operatorname{ann}B = \operatorname{ann}(B^2)$, [32, Corollary 2.4]. It follows that $[A : B] \cap \operatorname{ann}(AB) \subseteq \operatorname{ann}B$. Since each of $[A : B] + \operatorname{ann}(AB)$ and $[A : B] \cap \operatorname{ann}(AB)$ is a projective ideal (and hence multiplication), it follows from [35, Theorem 8] that $[A : B]$ is multiplication. See also [4, Theorems 3.6 and 4.2]. As we mentioned in the proof of Theorem 2.2(1), $\operatorname{ann}[A : B] = \operatorname{ann}(A + \operatorname{ann}B)$, and hence by [25, Corollary of Lemma 1.5], $[A : B]$ is f.g., and hence $[A : B] \in S(R)$. Finally, as $A \cap B = [A : B]B$, we see that $A \cap B \in S(R)$, so that $\operatorname{lcm}(A, B)$ exists, and the first part of the theorem is proved.

(2) Let $G = \gcd(A, B)$. Then $G + \operatorname{ann}(AB)$ is a f.g. faithful multiplication ideal and a common divisor of $A + \operatorname{ann}(AB)$ and $B + \operatorname{ann}(AB)$. Let G' be another common divisor of $A + \operatorname{ann}(AB)$ and $B + \operatorname{ann}(AB)$. Then G' is a f.g. faithful multiplication ideal [3, Lemma 1.4]. As $A \subseteq G'$ and $B \subseteq G'$, we have that G' is a common divisor of A and B, and hence $G'|G$. It follows that $G \subseteq G'$. But $\operatorname{ann}(AB) \subseteq G'$. Thus $G + \operatorname{ann}(AB) \subseteq G'$, and hence $G'|G + \operatorname{ann}(AB)$. This shows that

$$G + \operatorname{ann}(AB) = \gcd(A + \operatorname{ann}(AB), B + \operatorname{ann}(AB)).$$

(3) From the proof of Theorem 2.1(1), we know that $G + \operatorname{ann}G$ is a f.g. faithful multiplication ideal of R (and hence is projective). From [35, Corollary 1 of Theorem 10] and [31, Corollary 3.4], we have that the following ideals are f.g. projective:

$$A + \operatorname{ann}A = [A^2 : A],$$

$$A + \operatorname{ann}B = [AB : B],$$

$$(A + \operatorname{ann}A) + (A + \operatorname{ann}B) = (A + \operatorname{ann}A) + \operatorname{ann}B = [(A + \operatorname{ann}B)B : B].$$

We infer from [35, Lemma 7] and [30, Corollary 3.4] that

$$A + \operatorname{ann}G = A + \operatorname{ann}(A + B) = A + (\operatorname{ann}A \cap \operatorname{ann}B) = (A + \operatorname{ann}A) \cap (A + \operatorname{ann}B)$$

is a f.g. multiplication ideal of R, and hence by [31, Theorem 2.1], $A + \operatorname{ann}G \in S(R)$. Similarly, $B + \operatorname{ann}G \in S(R)$. Clearly, $G + \operatorname{ann}G$ is a common divisor of $A + \operatorname{ann}G$ and $B + \operatorname{ann}G$. Suppose that G' is another common divisor of $A + \operatorname{ann}G$ and $B + \operatorname{ann}G$. Then from the proof of Theorem 2.1(1), we have that $G' + \operatorname{ann}((A + B) + \operatorname{ann}G) = G'$ is a multiplication ideal of R. Moreover, G' is a common divisor of A and B, and hence $G'|G$. This implies that $G \subseteq G'$. But $\operatorname{ann}G \subseteq G'$. Hence $G + \operatorname{ann}G \subseteq G'$, and $G'|G + \operatorname{ann}G$. This proves that

$$G + \operatorname{ann}G = \gcd(A + \operatorname{ann}G, B + \operatorname{ann}G),$$

as required.

Let R be a ring and $A, B \in S(R)$ such that $K = \operatorname{lcm}(A, B)$ exists. Then by the above theorem, $\operatorname{lcm}(A + \operatorname{ann}K, B + \operatorname{ann}K)$ exists and equals $K + \operatorname{ann}K$. By [3, Theorem 2.1],

$\gcd(A+\text{ann}K, B+\text{ann}K)$ exists. As $K|AB$, there exists an ideal G of R such that $AB = KG$. Then also by [3, Theorem 2.1],

$$\begin{aligned}\gcd(A+\text{ann}K, B+\text{ann}K) &= [(A+\text{ann}K)(B+\text{ann}K) : K+\text{ann}K] \\ &= [AB+\text{ann}K : K+\text{ann}K] = [GK+\text{ann}K : K+\text{ann}K] \\ &= [(G+\text{ann}K)(K+\text{ann}K) : K+\text{ann}K] = G+\text{ann}K.\end{aligned}$$

Also, by [1, Proposition 2.1] and [3, Theorem 2.6(ii)] we have for every positive integer n,

$$G^n + \text{ann}K = \gcd(A^n + \text{ann}K, B^n + \text{ann}K).$$

We conjecture that the ideal G in the above remark is $\gcd(A, B)$. If this is true, then as one would expect, $AB = \gcd(A, B)\text{lcm}(A, B)$, and for every positive integer k, $\gcd(A, B)^k = \gcd(A^k, B^k)$.

As a consequence of Theorem 2.3 we give the next result which generalizes [3, Theorem 2.5].

Corollary 2.4. *Let R be a ring. If $\gcd(A, B)$ exists for all $A, B \in S(R)$, then $\text{lcm}(A, B)$ exists for all $A, B \in S(R)$.*

Proof. $\gcd(A, B)$ exists for all $A, B \in S(R)$, hence for all f.g. faithful multiplication ideals of R. Hence we get from [3, Theorem 2.5] that $\text{lcm}(A + \text{ann}(AB), B + \text{ann}(AB))$ exists, and hence by Theorem 2.3(1), $\text{lcm}(A, B)$ exists.

The next theorem should be compared with [3, Theorem 2.2].

Theorem 2.5. *Let R be a ring and $A, B, C \in S(R)$. Then*

1. $\text{lcm}(CA, CB)$ *exists if and only if* $\text{lcm}(A + \text{ann}C, B + \text{ann}C)$ *exists, and in this case,*

$$\text{lcm}(CA, CB) = C\text{lcm}(A + \text{ann}C, B + \text{ann}C).$$

2. *If* $\gcd(CA, CB)$ *exists, then so too does* $\gcd(A + \text{ann}C, B + \text{ann}C)$*, and in this case,*

$$\gcd(CA, CB) = C \gcd(A + \text{ann}C, B + \text{ann}C).$$

Proof. (1) Let $K = \text{lcm}(CA, CB)$. Then $K \subseteq C$ and $[K : C] \in S(R)$, (see [35, Corollary 1 of Theorem 10], [25, Corollary 1 of Lemma 1.5], and [31, Theorem 2.1]). Also, $A + \text{ann}C$ and $B + \text{ann}C \in S(R)$, and $[K : C]$ is a common multiple of them. Suppose that K' is another common multiple of $A + \text{ann}C$ and $B + \text{ann}C$. Then CK' is a common multiple of CA and CB, and therefore $K|CK'$. It follows that $K' \subseteq [K : C]$ and $[K : C]|K'$. This implies that

$$[K : C] = \text{lcm}(A + \text{ann}C, B + \text{ann}C),$$

and

$$K = [K : C]C = C\text{lcm}(A + \text{ann}C, B + \text{ann}C).$$

The converse follows by Theorem 2.2(4).

For (2), let $G = \gcd(CA, CB)$. As $C|CA, C|CB$, we have $C|G$ and hence $G \subseteq C$. But $G \in S(R)$ by Theorem 2.1. Thus $[G : C] \in S(R)$. Now $A + \text{ann}C, B + \text{ann}C \subseteq [G : C]$, and hence $[G : C]$ is a common divisor of $A + \text{ann}C$ and $B + \text{ann}C$. Suppose that $G'|A + \text{ann}C$, $G'|B + \text{ann}C$. Then $CG'|CA$ and $CG'|CB$. It follows that $CG'|G$, and hence there exists an ideal F of R such that $G = FCG'$. Next,

$$[G : C] = [FCG' : C] = FG' + \text{ann}C = (F + \text{ann}C)(G' + \text{ann}C).$$

But $\text{ann}C \subseteq G'$. Thus $[G : C] = (F + \text{ann}C)G'$, and hence $G'|[G : C]$. It follows that $[G : C] = \gcd(A + \text{ann}C, B + \text{ann}C)$, and $G = [G : C]C = C \gcd(A + \text{ann}C, B + \text{ann}C)$. This completes the proof of the theorem.

It is easy to see that Lemma 1.1 remains true for f.g. projective ideals. Moreover we mention two further corollaries of Theorem 2.5(2). They may be compared with [3, Proposition 2.3 and Lemma 2.4] respectively. The first is an extension of Euclid's lemma to f.g. projective ideals.

Corollary 2.6. *Let R be a ring and $A, B, C \in S(R)$ such that $\gcd(BA, BC)$ exists and $\gcd(A, C) = R$. Then $\gcd(A, BC) = \gcd(A, B)$.*

Proof. By Theorem 2.5(2), $\gcd(A + \text{ann}B, C + \text{ann}B)$ exists and

$$\gcd(BA, BC) = B \gcd(A + \text{ann}B, C + \text{ann}B).$$

From Lemma 1.1, we have that

$$R = \gcd(A, C) \subseteq \gcd(A + \text{ann}B, C + \text{ann}B) \subseteq R,$$

hence $\gcd(BA, BC) = B$, and

$$\gcd(A, B) = \gcd(A, \gcd(BA, BC)) = \gcd(\gcd(A, BA), BC) = \gcd(A, BC).$$

Corollary 2.7. *Let R be a ring and $A, B \in S(R)$ such that $G = \gcd(A, B)$ exists. Then $\gcd([A : G], [B : G]) = R$.*

Proof. As $A \subseteq G, B \subseteq G$ and G is projective (and hence multiplication), $A = [A : G]G$, $B = [B : G]G$. It follows from Theorem 2.5(2) that

$$G = \gcd([A : G]G, [B : G]G) = G \gcd([A : G] + \text{ann}G, [B : G] + \text{ann}G).$$

But $\text{ann}G \subseteq [A : G]$, and $\text{ann}G \subseteq [B : G]$. Thus $G = G \gcd([A : G], [B : G])$, and therefore $R = \gcd([A : G], [B : G]) + \text{ann}G$. Again, $\text{ann}G \subseteq \gcd([A : G], [B : G])$, and the result is established.

It may be worth noting that Corollary 2.4 can also be proved by using Corollary 2.7 and the same argument as that used in [3, Theorem 2.5], from which it also follows that $AB = \gcd(A, B)\text{lcm}(A, B)$.

In [3] we introduced a class of rings called generalized GCD rings. A ring R was called a GGCD ring if $\gcd(A, B)$ exists for all f.g. faithful multiplication ideals of R (equivalently, the intersection of every two f.g. faithful multiplication ideals of R is f.g. faithful multiplcation). S.Glaz [18],[19] defined a ring R to be a GGCD ring if the following two conditions hold:

(1) R is a p.p. ring.

(2) The intersection of any two f.g. flat ideals of R is a f.g. flat ideal of R.

As f.g. flat and f.g. projective ideals coincide in p.p. rings, one can replace Condition (2) by

(2′) The intersection of any two f.g. projective ideals of R is a f.g. projective ideal of R.

It is proved [18, Theorem 3.3] that if $aR \cap bR$ is a f.g. projective ideal for any two non zero divisors a, b of R, then $aR \cap bR$ is a f.g. projective ideal for any elements a, b of R. Thus a ring R is a GGCD ring as defined by Glaz if the following conditions are satisfied:

(1) R is a p.p. ring.

(2″) The intersection of any two invertible ideals of R is invertible.

As every f.g. faithful multiplication ideal of a ring R is projective, it follows that a GGCD ring as defined by Glaz is a GGCD ring by our definition. In fact, Condition (2) alone implies GGCD by our definition. The converse is not true. For example, arithmetical rings are GGCD rings by our definition but not necessarily by that of Glaz. $\mathbb{Z}_{12}$ is such an example, being an arithmetical ring but not a p.p. ring. Both definitions coincide, however, if R is an integral domain.

We now call a ring R a *G*GCD ring* if $\gcd(A, B)$ exists for all f.g. projective ideals of R. This implies that the intersection of every two f.g. projective ideals of R is f.g. projective. It is clear that this class of rings includes our GGCD rings, semihereditary rings, f.f. rings (and hence flat rings), von Neumann regular rings, arithmetical rings, Prüfer domains and GGCD-domains. Also it is obvious that the concepts G*GCD ring and GGCD-domain coincide when R is an integral domain.

Let R be a G*GCD ring and $A, B \in S(R)$. Then by Corollary 2.7,

$$\gcd([A : G], [B : G]) = R.$$

By Corollary 2.5 and Theorem 2.2, $K = \operatorname{lcm}(A, B)$ exists and $[A : B], [B : A] \in S(R)$. Therefore

$$\gcd([A : B], [B : A]) = \gcd([K : B], [K : A]) = R.$$

In fact, all the results of [3, Section 3] concerning GGCD rings can be extended to G*GCD rings. The proofs are routine modifications of those given.

3. Almost semihereditary rings

Anderson and Zafrullah [9] introduced several classes of integral domains.

AB. Almost Bezout domain: domain R in which for all $a, b \in R - \{0\}$ there exists $n = n(a, b)$ such that $a^n R + b^n R$ is principal.

AV. Almost valuation domain: domain R in which for all $a, b \in R - \{0\}$, there exists $n = n(a, b)$ such that $a^n R \subseteq b^n R$ or $b^n R \subseteq a^n R$.

AP. Almost Prüfer domain: domain R in which for all $a, b \in R - \{0\}$, there exists $n = n(a, b)$ such that $a^n R + b^n R$ is invertible.

AGCD. Almost greatest common divisor domain: domain R in which for all $a, b \in R - \{0\}$, there exists $n = n(a, b)$ such that $a^n R \cap b^n R$ is principal.

These classes of domains are studied further in [10] and [24]. In this note, we generalize AP-domains to rings with zero divisors. A ring R is called an *almost semihereditary ring (AS-ring)* if the following conditions are satisfied:

1. R is a p.p. ring, i.e. every principal ideal of R is projective.
2. For all $a, b \in R$, there exists a positive integer $n = n(a, b)$ such that $a^n R + b^n R$ is projective.

For basic properties of p.p. rings, see [13] and [14]. Clearly, AP-domains and semihereditary rings are AS-rings. The polynomial ring $R = K[x, y]$ over a field K is not an AS-ring, since $xR, yR \in S(R)$, but for all $n \in \mathbb{N}$, $(xR)^n + (yR)^n \notin S(R)$.

The next theorem shows several equivalent conditions for a ring to be an AS-ring. Compare with [9, Lemma 4.3 and Theorem 5.8].

Theorem 3.1. *Let R be a p.p. ring. Then the following are equivalent:*

1. *For all $a, b \in R$, there exists $n = n(a, b)$ such that $a^n R + b^n R \in S(R)$.*
2. *For all $a_1, \ldots, a_m \in R$, there exists $n = n(a_1, \ldots, a_m)$ such that $\sum\limits_{i=1}^{m} a_i^n R \in S(R)$.*
3. *R_P is an AV-domain for every prime ideal P of R.*
4. *R_M is an AV-domain for every maximal ideal M of R.*
5. *For all $a, b \in R$, there exists $n = n(a, b)$ such that*
$$[a^n R : b^n R] + [b^n R : a^n R] = R.$$
6. *For all $a, b \in R$, there exists $x, y, r, s \in \mathbb{R}$ and $n = n(a, b)$ such that*
$$\begin{pmatrix} x & r \\ s & 1-x \end{pmatrix} \begin{pmatrix} b^n \\ -a^n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
7. *For all $a_1, \ldots, a_m; b_1, \ldots, b_r \in R$, there exists $n = n(a_1, \ldots, b_r)$ such that* $[\sum\limits_{i=1}^{m} a_i^n R : \sum\limits_{i=1}^{r} b_i^n R] + [\sum\limits_{i=1}^{r} b_i^n R : \sum\limits_{i=1}^{m} a_i^n R] = R.$
8. *For all $A, B \in S(R)$ there exists $n = n(A, B)$ such that $A^n + B^n \in S(R)$.*

Proof. (1) $\Rightarrow$ (2) : Let $a_1, \ldots, a_m \in R$. For each i, j there exists $n_{ij} = n_{ij}(a_i, a_j)$ such that $a_i^{n_{ij}} R + a_j^{n_{ij}} R \in S(R)$. Put $n = \prod\limits_{i,j} n_{ij}$ and $\hat{n}_{ij} = \dfrac{n}{n_{ij}}$. Then from [1, Proposition 2.1] and [32, Theorem 2.1 and Corollary 4.3], we have that
$$(a_i^{n_{ij}} R + a_j^{n_{ij}} R)^{\hat{n}_{ij}} = a_i^n R + a_j^n R$$

is projective. It follows by [35, Theorem 8] that $\sum_{i=1}^{n} a_i^n R$ is multiplication. See also [4, Theorem 2.3]. As R is p.p., $\sum_{i=1}^{n} a_i^n R \in S(R)$.

(2) $\Rightarrow$ (1) : Clear.

(1) $\Rightarrow$ (3) : Let P be a prime ideal of R. Let I, J be principal ideals of R_P. Then there exist $a, b \in R$ such that $I = aR_P$ and $J = bR_P$. There exists $n = n(a, b)$ such that $a^n R + b^n R \in S(R)$. Hence $(a^n R + b^n R)_P$ is principal, and either $(a^n R + b^n R)_P = a^n R_P$ which implies that $b^n R_P \subseteq a^n R_P$, or $(a^n R + b^n R)_P = b^n R_P$ which implies that $a^n R_P \subseteq b^n R_P$. It follows that $I^n \subseteq J^n$ or $J^n \subseteq I^n$, and hence R_P is an AV-ring. But R_P is an integral domain, since R is p.p. [13, Proposition 1]. Thus R_P is an AV-domain.

(3) $\Rightarrow$ (4) : Obvious.

(4) $\Rightarrow$ (1) : Let M be a maximal ideal of R. Let $a, b \in R$. There exists $n = n(a, b, M)$ such that $a^n R_M \subseteq b^n R_M$ or $b^n R_M \subseteq a^n R_M$. It follows that $a^n R_M + b^n R_M$ is principal. By [9, Lemma 4.7], there exists $N = N(a, b)$ such that $a^N R_M + b^N R_M$ is principal, and hence $a^N R + b^N R$ is multiplication. As R is a p.p. ring, $a^N R + b^N R \in S(R)$.

(1) $\Rightarrow$ (5) : Let $a, b \in R$. There exists $n = n(a, b)$ such that $a^n R + b^n R$ is projective. The result follows by [31, Corollary 4.2], see also [30, Lemma 3.3] and [4, Corollary 1.4].

(5) $\Rightarrow$ (4) : Let M be a maximal ideal of R. Let I, J be principal ideals of R_M. There exist $a, b \in R$ such that $I = aR_M$ and $J = bR_M$. There exists $n = n(a, b)$ such that $[a^n R : b^n R] + [b^n R : a^n R] = R$, and hence, $[a^n R_M : b^n R_M] + [b^n R_M : a^n R_M] = R_M$. It follows that either $[a^n R_M : b^n R_M] = R_M$ which implies that $b^n R_M \subseteq a^n R_M$, i.e. $J^n \subseteq I^n$, or $[b^n R_M : a^n R_M] = R_M$ which gives that $a^n R_M \subseteq b^n R_M$, i.e. $I^n \subseteq J^n$. Hence, R_M is an AV-domain.

(5) $\Rightarrow$ (6) : There exist $x, y \in R$ such that $x + y = 1$ with $x \in [a^n R : b^n R]$ and $y \in [b^n R : a^n R]$. Hence there exist $r, s \in R$ such that $xb^n = ra^n$ and $ya^n = sb^n$. Thus $\begin{pmatrix} x & r \\ s & 1-x \end{pmatrix} \begin{pmatrix} b^n \\ -a^n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(6) $\Rightarrow$ (5) : Clear.

(5) $\Rightarrow$ (7) : Let $a_1, \ldots, a_m; b_1, \ldots, b_r \in R$. For all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, r\}$ there exist $n_{ij} = n_{ij}(a_i, b_j)$ such that $[a_i^{n_{ij}} R : b_j^{n_{ij}} R] + [b_j^{n_{ij}} R : a_i^{n_{ij}} R] = R$. Let $n = \prod_{i,j} n_{ij}$ and $\hat{n}_{ij} = \dfrac{n}{n_{ij}}$. It follows from [1, Lemma 3.5] that $[a_i^n R : b_j^n R] + [b_j^n R : a_i^n R] = R$, and hence, $[\sum_{i=1}^{m} a_i^n R : b_j^n R] + [\sum_{j=1}^{r} b_j^n R : a_i^n R] = R$. For each $j = 1, \ldots, r$, and $l = 1, \ldots, m$, we have

$$[\sum_{i=1}^{m} a_i^n R : b_j^n R] + [\sum_{k=1}^{r} b_k^n R : a_l^n R] = R.$$

Hence, by [4, Corollary 1.2],

$$[\sum_{i=1}^{m} a_i^n R : b_j^n R] + [\sum_{k=1}^{r} b_k^n R : \sum_{i=1}^{m} a_i^n R] = R.$$

Similarly for each j, and the result follows.

(7) $\Rightarrow$ (8) : Let $A, B \in S(R)$, $A = \sum_{i=1}^{m} a_i R$, $B = \sum_{j=1}^{k} b_j R$. Then for all $n \in \mathbb{N}$, $A^n = \sum_{i=1}^{m} a_i^n R$, and $B^n = \sum_{j=1}^{k} b_j^n R$. By [30, Lemma 3.3], $A^n + B^n$ is multiplication and hence projective, since R is a p.p. ring.

(8) $\Rightarrow$ (7) : See [30, Lemma 3.3], [4, Corollary 1.4], and [35, Corollary 3 of Theorem 1].

The next result generalizes some results on AP-domains. Compare with [9, Lemma 4.5 and Theorem 4.10].

Proposition 3.2. *Let R be an AS-ring. Then the following are true:*

1. *For all $A, B \in S(R)$, there exists $n = n(A, B)$ such that $A^n \cap B^n \in S(R)$.*
2. *R_S is an AS-ring for every multiplicative set S.*
3. *R/P is an AP-domain for every prime ideal P of R.*
4. *Every overring of R is an AS-ring.*

Proof. (1) By Theorem 3.1, there exists $n = n(A, B)$ such that $A^n + B^n \in S(R)$. Hence by [30, Corollary 3.4], [4, Corollary 2.4], and [35, Proposition 12], $A^n \cap B^n$ is f.g. multiplication. As R is p.p., $A^n \cap B^n$ is projective.

(2) Clearly, if R is p.p., then so too is R_S. Let I, J be principal ideals of R_S. Then $I = aR_S$ and $J = bR_S$ for some $a, b \in R$. By the above theorem, there exists $n = n(a, b)$ such that $[a^n R : b^n R] + [b^n R : a^n R] = R$, and hence $[a^n R_S : b^n R_S] + [b^n R_S : a^n R_S] = R_S$, that is, $[I^n : J^n] + [J^n : I^n] = R_S$, and again by Theorem 3.1, R_S is an AS-ring.

(3) Let I, J be principal ideals of R/P. Then for some $a, b \in R$, $I = (a + P)R/P$ and $J = (b + P)R/P$. Since R is an AS-ring, there exists $n = n(a, b)$ such that

$$[a^n R : b^n R] + [b^n R : a^n R] = R,$$

and hence

$$[(a + P)^n R/P : (b + P)^n R/P] + [(b + P)^n R/P : (a + P)^n R/P] = R/P.$$

It follows that $[I^n : J^n] + [J^n : I^n] = R/P$. R/P is p.p. since it is an integral domain, and hence by Theorem 3.1, R/P is an AS-ring, hence an AP-domain.

(4) Let $R \subseteq T \subseteq K$, where T is an overring of R and K is the total quotient ring of T (and of R). Let $a, b \in T$. There exists a non-zero-divisor $r \in R$ such that $ra, rb \in R$. Then for some $n \in \mathbb{N}$, $(ra)^n R + (rb)^n R$ is projective. Hence $(ra)^n T + (rb)^n T$ is projective. As $r^n T(a^n T + b^n T)$ is projective, and $r^n T$ is invertible, we infer that $a^n T + b^n T$ is projective. Finally, since R is p.p., T is also p.p., and the result follows.

In our last two theorems, we characterize AB-domains and then generalize [9, Lemma 4.5] concerning AP domains.

Theorem 3.3. *Let R be an integral domain. Then R is an AP- and an AGCD-domain if and only if R is an AB-domain.*

Proof. Assume that R is an AP- and AGCD-domain. Let $a, b \in R - \{0\}$. Then there exist $m, n \in \mathbb{N}$ such that $a^m R + b^m R$ is invertible and $a^n R \cap b^n R$ is principal. It follows from [17] that $a^{mn} R + b^{mn} R = (a^m R + b^m R)^n$, which is invertible, and by Theorems 1.7 and 2.2 we have that $(a^n R \cap b^n R)^m = a^{mn} R \cap b^{mn} R$ is principal. By [26, Theorem 3]

$$(a^{mn} R + b^{mn} R)(a^{mn} R \cap b^{mn} R) = a^{mn} b^{mn} R,$$

and hence by [7, Theorem 1],

$$a^{mn} R + b^{mn} R = [a^{mn} b^{mn} R : (a^{mn} R \cap b^{mn} R)].$$

Next, as R is an AGCD-domain, we have from [10], [24, Theorem 3.2] and [37] that there exists $k \in \mathbb{N}$ such that $[a^{mnk} b^{mnk} R : (a^{mn} R \cap b^{mn} R)^k]$ is principal. Since

$$a^{mn} b^{mn} R + (a^{mn} R \cap b^{mn} R) = a^{mn} R \cap b^{mn} R$$

is invertible and hence flat, we infer again from [28, Corollary 3.2] that

$$[a^{mnk} b^{mnk} R : (a^{mn} R \cap b^{mn} R)^k] = [a^{mn} b^{mn} R : (a^{mn} R \cap b^{mn} R)]^k,$$

and this finally gives that

$$a^{mnk} R + b^{mnk} R = (a^{mn} R + b^{mn} R)^k = [a^{mn} b^{mn} R : (a^{mn} R \cap b^{mn} R)]^k$$

is principal, and hence R is an AB-domain.

Conversely, clearly an AB-domain is an AP-domain. Let $a, b \in R - \{0\}$. There exists $n \in \mathbb{N}$ such that $a^n R + b^n R$ is principal (hence invertible). It follows from [26, Theorem 3] that

$$(a^n R + b^n R)(a^n R \cap b^n R) = a^n b^n R,$$

and hence

$$a^n R \cap b^n R = a^n b^n R (a^n R + b^n R)^{-1}$$

is principal. Hence R is an AGCD-domain.

It is proved in [22, Theorem 101] that if R is a Prüfer domain with quotient field K, and L is an algebraic extension of K, then the integral closure T of R in L is also a Prüfer domain. The next theorem is a generalization of this fact to AP-domains. First we give a lemma.

Lemma 3.4. *Let R be an AV-domain. Then any overring T of R is an AV-domain.*

Proof. Let $x, y \in T$. Then there exists $0 \neq r \in R$ such that $rx, ry \in R$, and hence for some $n \in \mathbb{N}$, either $r^n x^n R \subseteq r^n y^n R$ or $r^n y^n R \subseteq r^n x^n R$. Hence $r^n x^n T \subseteq r^n y^n T$ or $r^n y^n T \subseteq r^n x^n T$, and this gives that $x^n T \subseteq y^n T$ or $y^n T \subseteq x^n T$. Hence T is an AV-domain.

Theorem 3.5. *Let R be an AP-domain with quotient field K, and L an algebraic extension of K. Then the integral closure T of R in L is an AP-domain.*

Proof. Let M be a maximal ideal of T. Then $P = M \cap R$ is a maximal ideal of R. Let $S = R \setminus P$. Then T_S is the integral closure of $R_S = R_P$ in L, and T_S is an overring of R_P. As R_P is an AV-domain ([9, Theorem 5.8] and Theorem 3.1) we infer from Lemma 3.4 that T_S is an AV-domain, and hence an AP-domain. Hence $T_M = (T_S)_{M_S}$ is an AV-domain. By [9, Theorem 5.8], T is an AP-domain.

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