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Napoleon's Theorem and Generalizations Through Linear Maps

Hellmuth Stachel

Institute of Geometry, Vienna University of Technology
Wiedner Hauptstr. 8-10/113, A-1040 Wien, Austria
e-mail: stachel@geometrie.tuwien.ac.at

Abstract. Recently J. Fukuta and Z. Čerin showed how regular hexagons can be associated to any triangle, thus extending Napoleon's theorem. The aim of this paper is to prove that these results are closely related to linear maps. This reflects better the affine character of some constructions and gives also rise to a few new theorems.

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1. Introduction

J. Fukuta showed in [4, 5] that to each triangle in the Euclidean plane E regular hexagons can be associated. In a slightly generalized form due to Z. Čerin [1] one of Fukuta's constructions applied to a given triangle $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$ reads as follows (Fig. 1):

- *Operation 1:* Divide all sides of $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$ in two given ratios, i.e., for given $\lambda, \overline{\lambda} \in \mathbb{R}$ define two point triples $\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3$ and $\overline{\mathbf{b}}_1\overline{\mathbf{b}}_2\overline{\mathbf{b}}_3$ as affine combinations $\mathbf{b}_i := \lambda\mathbf{a}_i + (1-\lambda)\mathbf{a}_{i+1}$, $\overline{\mathbf{b}}_i := (1-\overline{\lambda})\mathbf{a}_i + \overline{\lambda}\mathbf{a}_{i+1}$, $i = 1, 2, 3$, indices modulo 3.[1]
- *Operation 2:* Define six points $\mathbf{c}_1, \overline{\mathbf{c}}_1, \mathbf{c}_2, \overline{\mathbf{c}}_2, \mathbf{c}_3, \overline{\mathbf{c}}_3$ by building equally oriented equilateral triangles on the sides $\mathbf{b}_1\overline{\mathbf{b}}_1, \overline{\mathbf{b}}_1\mathbf{b}_2, \dots, \overline{\mathbf{b}}_3\mathbf{b}_1$ of the hexagon $\mathbf{H_b} := \mathbf{b}_1\overline{\mathbf{b}}_1\mathbf{b}_2\overline{\mathbf{b}}_2\mathbf{b}_3\overline{\mathbf{b}}_3$.
- *Operation 3:* Let $\mathbf{d}_1, \overline{\mathbf{d}}_1, \mathbf{d}_2, \dots, \overline{\mathbf{d}}_3$ be the centroids of the consecutive triples $\overline{\mathbf{c}}_3\mathbf{c}_1\overline{\mathbf{c}}_1$, $\mathbf{c}_1\overline{\mathbf{c}}_1\mathbf{c}_2$, $\dots$, $\mathbf{c}_3\overline{\mathbf{c}}_3\mathbf{c}_1$ in the hexagon $\mathbf{H_c} := \mathbf{c}_1\overline{\mathbf{c}}_1\mathbf{c}_2\overline{\mathbf{c}}_2\mathbf{c}_3\overline{\mathbf{c}}_3$.

Then the hexagon $\mathbf{H_d} := \mathbf{d}_1\overline{\mathbf{d}}_1\mathbf{d}_2\overline{\mathbf{d}}_2\mathbf{d}_3\overline{\mathbf{d}}_3$ *is regular.*

[1] In [4, 5] only the cases $\lambda = \overline{\lambda}$ have been treated.

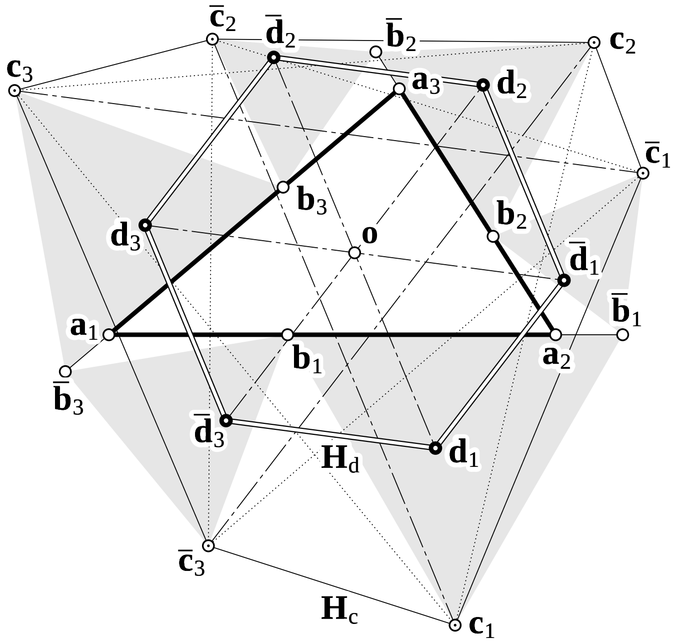

Figure 1. Fukuta's theorem

For $(\lambda, \overline{\lambda}) = (1, 1)$ (see Fig. 2) this statement is the "hexagonal" extension ([6], Theorem 4, or [9], Theorem IV) of Napoleon's theorem (cf. [3, p. 23] or [8]). Various generalizations of Fukuta's construction as presented in [1, 2] will be addressed in the sequel. The proofs given in [4, 5, 1, 2] are verifications using complex numbers. We show that these results are closely related to statements on linear maps (Lemma 2). This approach is not only more appropriate to the affine character of some constructions but it also leads to simplified proofs and a few new results (Corollaries 3, 5 and Theorem 6).

2. Linear maps and isocentroidal triangles

We introduce a second plane E' with an equilateral "standard triangle" $\mathbf{s}'_1\mathbf{s}'_2\mathbf{s}'_3$. Then there is an *affine transformation* $\alpha\colon\ E' \to E,\ \mathbf{s}'_i \mapsto \mathbf{a}_i$. Under α the centroid $\mathbf{o}'$ of $\mathbf{s}'_1\mathbf{s}'_2\mathbf{s}'_3$ is mapped onto the centroid $\mathbf{o}$ of $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$. From now on we see the planes E and E' as two-dimensional vector spaces over $\mathbb{R}$ with zero vectors $\mathbf{o}$ and $\mathbf{o}'$, respectively. Then the affine transformation α can be represented by a *linear map* a (Fig. 3) with

$$\mathbf{a}_i = a(\mathbf{s}'_i) \quad \text{for} \quad i = 1, 2, 3. \tag{1}$$

Before we prove that the three Fukuta-operations listed above produce again linear maps, a brief view on notations and basic results from Linear Algebra: For any two vector spaces U, V let $L(U, V)$ denote the set of linear maps $U \to V$. This is again a vector space due to the definition

$$(\lambda g + \mu h)(\mathbf{u}) = \lambda g(\mathbf{u}) + \mu h(\mathbf{u}) \quad \text{for} \quad g, h \in L(U, V),\ \mathbf{u} \in U,\ \lambda, \mu \in \mathbb{R}.$$

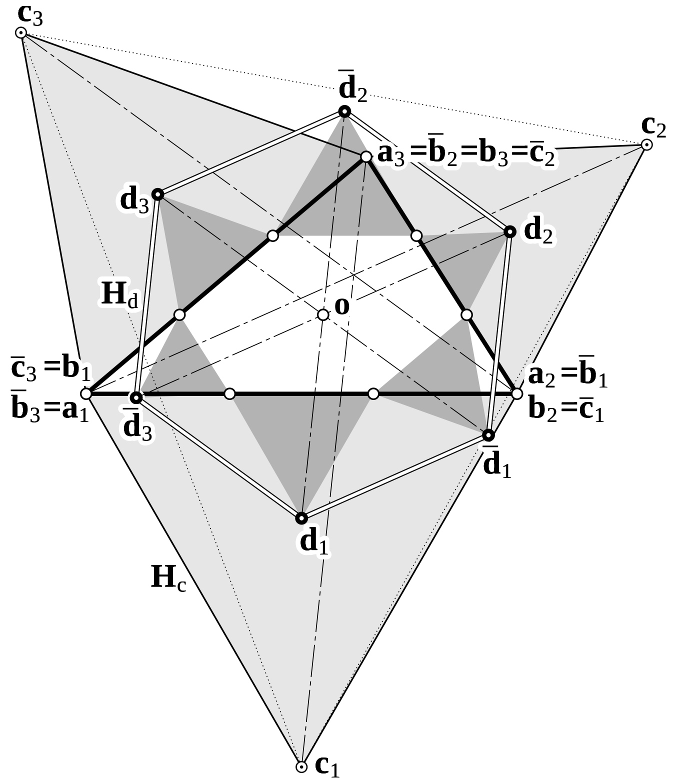

Figure 2. Napoleon's theorem

Let W be an additional vector space. Then for $k, l \in L(V, W)$ we can form the composites $k \circ g$ etc. obeying

$$k \circ (g + h) = k \circ g + k \circ h, \quad (k + l) \circ g = k \circ g + l \circ g.$$

A map $\mathcal{B}\colon U \times V \to W,\ (\mathbf{u}, \mathbf{v}) \mapsto \mathcal{B}(\mathbf{u}, \mathbf{v})$ is called bilinear if it is linear in each factor.

Lemma 1. *Let T_0 denote the set of ordered point triples in E with the centroid $\mathbf{o}$. Then there is a bijection*

$$\tau\colon\ L(E', E) \to T_0\,, \quad g \ \mapsto\ \mathbf{g}_1\mathbf{g}_2\mathbf{g}_3 := g(\mathbf{s}'_1)g(\mathbf{s}'_2)g(\mathbf{s}'_3).$$

Proof. $\mathbf{s}'_1 + \mathbf{s}'_2 + \mathbf{s}'_3 = \mathbf{o}'$ implies $g(\mathbf{s}'_1) + g(\mathbf{s}'_2) + g(\mathbf{s}'_3) = g(\mathbf{o}') = \mathbf{o}$. Conversely, g is uniquely defined by the images of $\mathbf{s}'_1$ and $\mathbf{s}'_2$. □

In this sense we can replace statements about point triples from T_0 by statements on linear maps of $L(E', E)$. The triple $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$ corresponds to $a \in L(E', E)$ according to (1). The cyclic permutation $\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1$ corresponds to $a \circ r'_3$ when $r'_3 \in L(E', E')$ denotes the rotation of E' about $\mathbf{o}'$ through 120° (see Fig. 3). $a \circ {r'_3}^2$ corresponds to $\mathbf{a}_3\mathbf{a}_1\mathbf{a}_2$. Since for all $\mathbf{x}' \in E'$ point $\mathbf{o}'$ is the centroid of the triple $\mathbf{x}'\ r'_3(\mathbf{x}')\ {r'_3}^2(\mathbf{x}')$, we get

$${r'_3}^2 + r'_3 + 1' = 0', \tag{2}$$

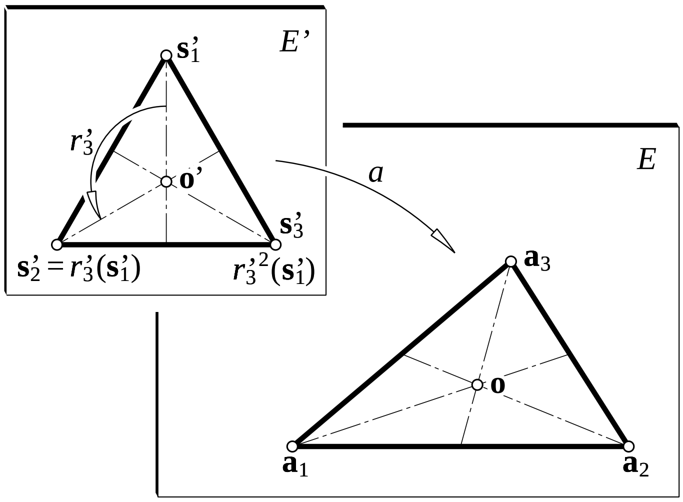

Figure 3. We identify the given triangles with linear maps

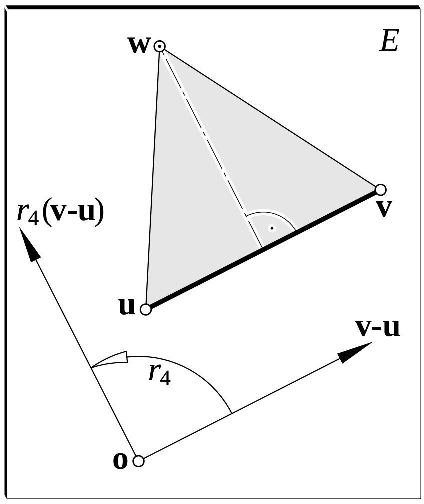

Figure 4. Building equilateral triangles

where $1'$ denotes the identity and $0'$ the zero-map of $L(E', E')$.[2] On the other hand we have ${r'_3}^3 = 1'$.

Ad Operation 1: We define a class of bilinear maps by affine combinations with fixed coefficients

$$\mathcal{A}_\xi\colon\ L(E',E)\times L(E',E) \to L(E',E), \quad (g,h) \mapsto \mathcal{A}_\xi(g,h) := \xi g + (1-\xi)h. \tag{3}$$

Instead of applying Operation 1 to the given triangle $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$ we determine the linear maps

$$b := \mathcal{A}_\lambda(a, a\circ r'_3) \quad\text{and}\quad \overline{b} := \mathcal{A}_{\overline{\lambda}}(a\circ r'_3, a). \tag{4}$$

Due to Lemma 1 the images $\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3$ and $\overline{\mathbf{b}}_1\overline{\mathbf{b}}_2\overline{\mathbf{b}}_3$ of $\mathbf{s}'_1\mathbf{s}'_2\mathbf{s}'_3$ under b and $\overline{b}$, resp., are *isocentroidal*, i.e., they share the centroid $\mathbf{o}$ (compare [1], Theorems 2,3,4).

Ad Operation 2: Let $\mathbf{w}$ complete the given side $\mathbf{uv}$ to a positively oriented equilateral triangle (see Fig. 4). When $r_4 \in L(E,E)$ denotes the rotation of E about $\mathbf{o}$ through 90°, then we can set

$$\mathbf{w} = \tfrac{1}{2}\,(\mathbf{u}+\mathbf{v}) + \tfrac{\sqrt{3}}{2}\,r_4(\mathbf{v}-\mathbf{u}).$$

For $\mathbf{u} = g(\mathbf{x}')$ and $\mathbf{v} = h(\mathbf{x}')$ there is again a linear map $k \in L(E',E)$ with $\mathbf{w} = k(\mathbf{x}')$ for all $\mathbf{x}' \in E'$. We now generalize Operation 2 and replace the equilateral triangles by mutually similar ones:

- *Operation 2':* Define six points $\mathbf{c}_1, \overline{\mathbf{c}}_1, \mathbf{c}_2, \overline{\mathbf{c}}_2, \mathbf{c}_3, \overline{\mathbf{c}}_3$ by building equally oriented triangles of a given shape on the sides $\mathbf{b}_1\overline{\mathbf{b}}_1, \overline{\mathbf{b}}_1\mathbf{b}_2, \ldots, \overline{\mathbf{b}}_3\mathbf{b}_1$ of the hexagon $\mathbf{H_b}$.

We meet this operation by the definition

$$\mathcal{T}_{\xi\overline{\xi}}\colon\ L(E',E)\times L(E',E) \to L(E',E), \quad (g,h) \mapsto \mathcal{T}_{\xi\overline{\xi}}(g,h) := \mathcal{A}_\xi(g,h) + \overline{\xi}r_4\circ(h-g) \tag{5}$$

[2]Equation (2) expresses exactly the statement of the Cayley-Hamilton theorem for $r'_3 \in L(E',E')$.

with constant $\xi, \overline{\xi} \in \mathbb{R}$. Changing the sign of $\overline{\xi}$ means reflecting all affixed triangles in their baselines. The original Operation 2 is based on the specification $\xi = 1/2$ and $\overline{\xi} = \pm\sqrt{3}/2$.

The triangles $\mathbf{c}_1\mathbf{c}_2\mathbf{c}_3$ and $\overline{\mathbf{c}}_1\overline{\mathbf{c}}_2\overline{\mathbf{c}}_3$ resulting from Operation 2' correspond to

$$c := \mathcal{T}_{\xi\overline{\xi}}(b, \overline{b}) \quad \text{and} \quad \overline{c} := \mathcal{T}_{\xi\overline{\xi}}(\overline{b}, b \circ r_3'). \tag{6}$$

Ad Operation 3: Instead of determining the centroids for the triples of consecutive points we define the *"mean maps"* as the affine combinations

$$d := \tfrac{1}{3}\,(\overline{c} \circ {r_3'}^2 + c + \overline{c}) \quad \text{and} \quad \overline{d} := \tfrac{1}{3}\,(c + \overline{c} + c \circ r_3'). \tag{7}$$

From (7) and (2) we obtain

$$d + \overline{d} \circ r_3' = \tfrac{1}{3}\,(c + \overline{c}) \circ (\mathit{1}' + r_3' + {r_3'}^2) = \mathit{0},$$

where $\mathit{0}$ denotes the zero-map in $L(E', E)$. Substituting ${r_3'}^2 = -\mathit{1}' - r_3'$ in (7) yields

Theorem 1. *For any $c, \overline{c} \in L(E', E)$ the mean maps $d, \overline{d}$ defined in (7) obey*

$$d = \tfrac{1}{3}\,(c - \overline{c} \circ r_3') \quad \text{and} \quad \overline{d} = -d \circ {r_3'}^2 = d + d \circ r_3'.$$

Corollary 1. *For any two isocentroidal point triples $\mathbf{c}_1\mathbf{c}_2\mathbf{c}_3$ and $\overline{\mathbf{c}}_1\overline{\mathbf{c}}_2\overline{\mathbf{c}}_3$ the centroids of consecutive triples $\overline{\mathbf{c}}_3\mathbf{c}_1\overline{\mathbf{c}}_1, \ldots, \mathbf{c}_3\overline{\mathbf{c}}_3\mathbf{c}_1$ in the hexagon $\mathbf{H_c} = \mathbf{c}_1\overline{\mathbf{c}}_1\mathbf{c}_2\overline{\mathbf{c}}_2\mathbf{c}_3\overline{\mathbf{c}}_3$ constitute a hexagon $\mathbf{H_d}$ symmetric with respect to $\mathbf{o}$.*

$\mathbf{H_d}$ is an affine transform of a regular hexagon. The main diagonals $\mathbf{d}_i\overline{\mathbf{d}}_{i+1}$ of $\mathbf{H_d}$ and $\mathbf{c}_i\overline{\mathbf{c}}_{i+1}$ of $\mathbf{H_c}$ are parallel. Their lengths make the ratio $2:3$.

Proof. The points $\mathbf{c}_i = c(\mathbf{s}_i')$ and $\overline{\mathbf{c}}_{i+1} = \overline{c}(\mathbf{s}_{i+1}') = \overline{c} \circ r_3'(\mathbf{s}_i')$ are opposite in $\mathbf{H_c}$. In $\mathbf{H_d}$ point $\mathbf{d}_i = d(\mathbf{s}_i') = d \circ {r_3'}^2(\mathbf{s}_{i+1}')$ is opposite to $\overline{\mathbf{d}}_{i+1} = \overline{d}(\mathbf{s}_{i+1}') = \overline{d} \circ r_3'(\mathbf{s}_i') = -d(\mathbf{s}_i')$. Theorem 1 yields

$$\mathbf{c}_i - \overline{\mathbf{c}}_{i+1} = (c - \overline{c} \circ r_3')(\mathbf{s}_i') = 3d(\mathbf{s}_i') = 3\mathbf{d}_i = \tfrac{3}{2}\,(d - \overline{d} \circ r_3')(\mathbf{s}_i') = \tfrac{3}{2}\,(\mathbf{d}_i - \overline{\mathbf{d}}_{i+1}).$$

$\overline{d} = d + d \circ r_3'$ implies $\overline{\mathbf{d}}_i = \mathbf{d}_i + \mathbf{d}_{i+1}$, i.e., the quadrangle $\mathbf{o}\,\mathbf{d}_i\,\overline{\mathbf{d}}_i\,\mathbf{d}_{i+1}$ is a parallelogram. Hence $\mathbf{H_d}$ is the *affine transform* of a regular hexagon (see Fig. 5).

The first statement in Corollary 1 can also be concluded from the fact that due to Lemma 1 point $\mathbf{o}$ is the centroid of the point set $\{\mathbf{c}_1, \overline{\mathbf{c}}_1, \ldots, \overline{\mathbf{c}}_3\}$. Therefore $\mathbf{o}$ is the midpoint between the centroids of any complementary triples selected from this set. □

Remark 1. The Operations 1, 2' or 3 can also be applied to maps $g \in L(\mathbb{R}^n, E)$ without destroying their linearity. Even affine maps remain affine. In Descriptive Geometry this has already been used in [7] for generating new parallel views (axonometries) from two given views of any 3D object (see also [10]).

3. Similarities

In the sense of Lemma 1 equilateral triangles $\mathbf{g}_1\mathbf{g}_2\mathbf{g}_3$ in E correspond to similarities $g \in L(E', E)$ since the preimage $\mathbf{s}'_1\mathbf{s}'_2\mathbf{s}'_3$ is supposed equilateral.

A linear map $g \in L(E', E)$ is a *similarity* if and only if it preserves orthogonality. This means that any vector $\mathbf{x}' \in E'$ and its image under the rotation r'_4 of E' about $\mathbf{o}'$ through $90°$ are mapped on two vectors corresponding under $\pm r_4$, i.e.,[3]

$$g \circ r'_4 = \varepsilon\, r_4 \circ g \quad \text{for} \quad \varepsilon \in \{1, -1\}. \tag{8}$$

Similarities with the same ε *constitute a subspace* $\mathcal{S}_\varepsilon(E', E) \subset L(E', E)$ since (8) and $h \circ r'_4 = \varepsilon\, r_4 \circ h$ imply for all $\lambda, \mu \in \mathbb{R}$

$$(\lambda g + \mu h) \circ r'_4 = \varepsilon\, r_4 \circ (\lambda g + \mu h).$$

Fig. 3 reveals the orthogonality between $\mathbf{x}'$ and $r'_3(\mathbf{x}') - {r'_3}^2(\mathbf{x}') = (r'_3 - {r'_3}^2)(\mathbf{x}')$. More precisely and due to (2) we obtain

$$r'_4 = \tfrac{1}{\sqrt{3}}\,(r'_3 - {r'_3}^2) = \tfrac{1}{\sqrt{3}}\,(\mathit{1}' + 2r'_3). \tag{9}$$

The following lemma will be useful in the sequel:

Lemma 2. *For given* $g \in L(E', E)$ *the linear map*

$$h := \alpha\, g + \beta\, g \circ r'_3 + r_4 \circ (\gamma\, g + \delta\, g \circ r'_3)$$

is a similarity if the coefficients $\alpha, \ldots, \delta \in \mathbb{R}$ *obey*

$$\gamma = \frac{\varepsilon}{\sqrt{3}}\,(2\beta - \alpha) \quad \text{and} \quad \delta = \frac{\varepsilon}{\sqrt{3}}\,(\beta - 2\alpha).$$

For linearly independent $\{g,\ g \circ r'_3,\ r_4 \circ g,\ r_4 \circ g \circ r'_3\}$ *this sufficient condition is also necessary.*

Proof. By straightforward computation we obtain with (2)

$$\begin{aligned} h \circ r'_4 &= \tfrac{1}{\sqrt{3}}\, h \circ (\mathit{1}' + 2r'_3) = \\ &= \tfrac{1}{\sqrt{3}}\,[(\alpha - 2\beta) g + (2\alpha - \beta) g \circ r'_3] + \tfrac{1}{\sqrt{3}}\, r_4 \circ [(\gamma - 2\delta) g + (2\gamma - \delta) g \circ r'_3]\,. \end{aligned}$$

On the other hand due to ${r_4}^2 = -\mathit{1}$ we get

$$r_4 \circ h := -\gamma\, g - \delta\, g \circ r'_3 + r_4 \circ (\alpha\, g + \beta\, g \circ r'_3).$$

We conclude by comparing coefficients that condition (8) is fulfilled when $\alpha, \ldots, \delta \in \mathbb{R}$ obey the equations given in Lemma 2. $\square$

[3] Also in [9] the operators r'_3 and r'_4 are used for proving Napoleon's theorem, however in a different way.

Theorem 2. *If we set* $\xi = 1/2$ *and* $\overline{\xi} = -\varepsilon\sqrt{3}/2$ *in* (6), *then for any* $b, \overline{b} \in L(E', E)$ *the linear map* d *defined in* (7) *as well as* $\overline{d}$ *and* $(c - \overline{c}\circ r_3')$ *are similarities.*

Proof. It is sufficient to prove that $3d = c - \overline{c}\circ r_3'$ is a similarity since $\overline{d} = -d\circ {r_3'}^2$ differs from d by a rotation in E' and the reflection of E in $\mathbf{o}$. From (7) and (6) we obtain

$$c = \xi b + (1-\xi)\overline{b} + \overline{\xi} r_4\circ(\overline{b} - b) \quad \text{and} \quad \overline{c} = \xi\overline{b} + (1-\xi) b\circ r_3' + \overline{\xi} r_4\circ(b\circ r_3' - \overline{b}),$$

hence due to Theorem 1 and (2)

$$3d = \left[b + (1-\xi) b\circ r_3' + \overline{\xi} r_4\circ b\circ r_3'\right] + \left[(1-\xi)\overline{b} - \xi\overline{b}\circ r_3' + \overline{\xi} r_4\circ(\overline{b} + \overline{b}\circ r_3')\right]. \qquad (10)$$

Lemma 2 applied to both terms gives the sufficient conditions

$$0 = 1 - 2\xi, \quad \overline{\xi} = -\frac{\varepsilon}{\sqrt{3}}(1+\xi), \quad \overline{\xi} = \frac{\varepsilon}{\sqrt{3}}(\xi - 2),$$

which are only true for Fukuta's choice.[4] □

Corollary 2. (Čerin [1]) *Let two isocentroidal point triples* $\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3$ *and* $\overline{\mathbf{b}}_1\overline{\mathbf{b}}_2\overline{\mathbf{b}}_3$ *be given. When the Operations 2 and 3 are applied to the hexagon* $\mathbf{H_b}$, *then the resulting hexagon* $\mathbf{H_d}$ *is regular with center* $\mathbf{o}$.

Proof. According to the Theorems 1 and 2 the hexagon $\mathbf{H_d}$ consists of two centrally symmetric equilateral triangles. □

The similarities d and $\overline{d}$ addressed in Theorem 2 are

$$\begin{aligned} d &:= \tfrac{1}{6}\left[2b + b\circ r_3' - \varepsilon\sqrt{3}\, r_4\circ b\circ r_3'\right] + \tfrac{1}{6}\left[\overline{b} - \overline{b}\circ r_3' - \varepsilon\sqrt{3}\, r_4\circ(\overline{b} + \overline{b}\circ r_3')\right] \\ \overline{d} &:= \tfrac{1}{6}\left[b + 2b\circ r_3' + \varepsilon\sqrt{3}\, r_4\circ b\right] + \tfrac{1}{6}\left[2\overline{b} + \overline{b}\circ r_3' - \varepsilon\sqrt{3}\, r_4\circ\overline{b}\circ r_3'\right]. \end{aligned} \qquad (11)$$

Remark 2. Lemma 2 shows that in the generic case, i.e., for two indeterminate isocentroidal triples $\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3, \overline{\mathbf{b}}_1\overline{\mathbf{b}}_2\overline{\mathbf{b}}_3$, the Operations 2' and 3 will not produce a regular hexagon unless equilateral triangles $\mathbf{b}_1\overline{\mathbf{b}}_1\mathbf{c}_1, \overline{\mathbf{b}}_1\mathbf{b}_2\overline{\mathbf{c}}_1, \dots$ are erected on the sides of $\mathbf{H_b}$. So, only equilateral triangles have this general "regularizing" effect. In [2], Theorem 9, an algebraic condition Θ is given for special cases where already isosceles affixed triangles lead to a regular hexagon $\mathbf{H_d}$.

4. Further results and special cases

The following modification of Operation 2' has been introduced in [4] and discussed in [1, 2][5]:

- *Operation 4:* Define six points $\mathbf{e}_1, \overline{\mathbf{e}}_1, \mathbf{e}_2, \overline{\mathbf{e}}_2, \mathbf{e}_3, \overline{\mathbf{e}}_3$ by building equally oriented triangles of a given shape on the small diagonals $\mathbf{b}_1\mathbf{b}_2, \overline{\mathbf{b}}_1\overline{\mathbf{b}}_2, \dots, \overline{\mathbf{b}}_3\overline{\mathbf{b}}_1$ of the hexagon $\mathbf{H_b}$.

[4] For $\varepsilon = +1$, i.e., $\overline{\xi} < 0$, the affixed regular triangles are in the right halfplane of the oriented sides of the hexagon $\mathbf{H_b} = \mathbf{b}_1\overline{\mathbf{b}}_1\mathbf{b}_2\overline{\mathbf{b}}_2\mathbf{b}_3\overline{\mathbf{b}}_3$ (see Fig. 1).

[5] However only isosceles affixed triangles were used in [1, 2].

We now apply the Operations 4 and 3 to $\mathbf{H_b}$ and obtain from

$$e := \mathcal{T}_{\eta\overline{\eta}}(b, b\circ r_3), \quad \overline{e} := \mathcal{T}_{\eta\overline{\eta}}(\overline{b}, \overline{b}\circ r_3) \tag{12}$$

$$\begin{aligned} 3f := \ & \overline{e}\circ r_3'^{\,2} + e + \overline{e} = e - \overline{e}\circ r_3' = [\eta b + (1-\eta) b\circ r_3' + \overline{\eta} r_4\circ(-b + b\circ r_3')] + \\ & + \left[(1-\eta)\overline{b} + (1-2\eta)\overline{b}\circ r_3' + \overline{\eta} r_4\circ(\overline{b} + 2\overline{b}\circ r_3')\right]. \end{aligned} \tag{13}$$

The relative position of the hexagons $\mathbf{H_d}$ and $\mathbf{H_f}$ (see Fig. 5) is subject of

Theorem 3. *For all $b, \overline{b} \in L(E', E)$ the maps d based on constants $\xi, \overline{\xi}$ and f with constants $\eta = \frac{1}{3}(1+\xi)$ and $\overline{\eta} = \frac{1}{3}\overline{\xi}$ obey*

$$f = \mathcal{A}_{2/3}(d, d\circ r_3') = \mathcal{A}_{1/3}(\overline{d}\circ r_3'^{\,2}, \overline{d}\,), \quad \overline{f} = f + f\circ r_3' = \mathcal{A}_{1/3}(d, d\circ r_3').$$

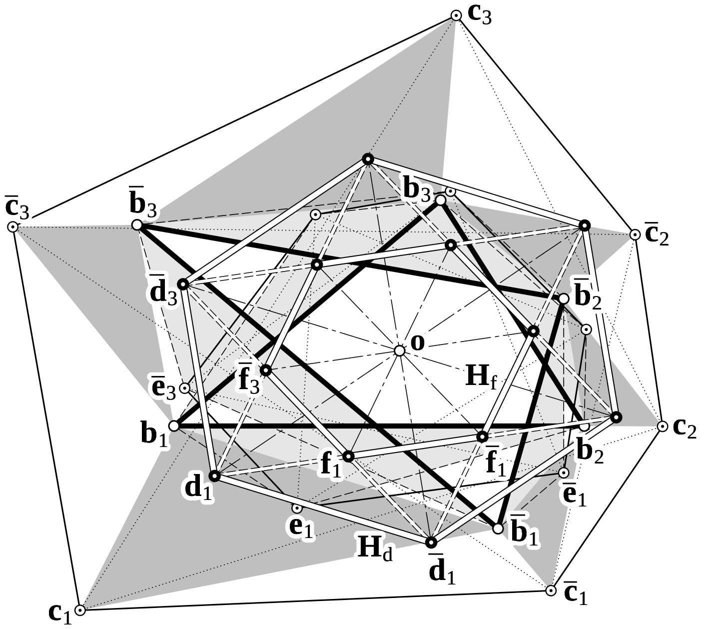

Figure 5. The hexagons $\mathbf{H_d}$ and $\mathbf{H_f}$ under the conditions of Theorem 3

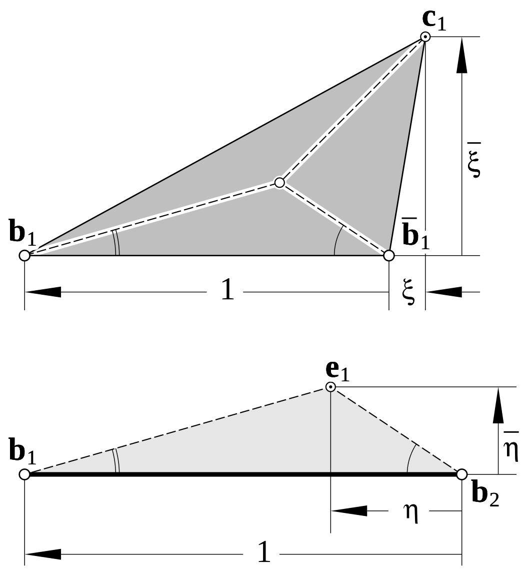

Figure 6. Relation between the affixed triangles for $\mathbf{H_d}$ and $\mathbf{H_f}$

Proof. Equation (10) implies

$$\begin{aligned} & 3(2d + d\circ r_3') = \\ & = (1+\xi)b + (2-\xi)b\circ r_3' + \overline{\xi}\circ r_4\circ(-b + b\circ r_3') + (2-\xi)\overline{b} + (1-2\xi)\overline{b}\circ r_3' + \overline{\xi}(\overline{b} + 2\overline{b}\circ r_3'). \end{aligned}$$

The comparison of coefficients between this linear map and $3f$ in (13) reveals that

$$1 + \xi = 3\eta \quad \text{and} \quad \overline{\xi} = 3\overline{\eta} \tag{14}$$

are sufficient for $3f = 2d + d\circ r_3'$. Theorem 1 implies the other equations since

$$\begin{aligned} \tfrac{1}{3}\,\overline{d}\circ r_3'^{\,2} + \tfrac{2}{3}\,\overline{d} &= \tfrac{1}{3}(d\circ r_3'^{\,2} + d) + \tfrac{2}{3}(d + d\circ r_3') = \tfrac{2}{3}\,d + \tfrac{1}{3}\,d\circ r_3' = f, \\ f + f\circ r_3' &= \left(\tfrac{2}{3}\,d + \tfrac{1}{3}\,d\circ r_3'\right)\circ(-r_3'^{\,2}) = \tfrac{2}{3}\,d + \tfrac{2}{3}\,d\circ r_3' - \tfrac{1}{3}\,d = \tfrac{1}{3}\,d + \tfrac{2}{3}\,d\circ r_3'. \end{aligned}$$

□

Corollary 3. *For any two isocentroidal triples* $\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3$ *and* $\overline{\mathbf{b}}_1\overline{\mathbf{b}}_2\overline{\mathbf{b}}_3$ *the vertices of* $\mathbf{H_f}$ *are the trisection points of the small diagonals of* $\mathbf{H_d}$ *(see Fig. 5) provided the parameters* $\xi, \overline{\xi}$ *and* $\eta, \overline{\eta}$ *of the affixed triangles obey* (14).

The geometric meaning of (14) is expressed in Fig. 6: The triangles built in Operation 4 have to be directly similar to one third of the triangles erected in Operation 2', i.e., to the subtriangle with vertices $\mathbf{b}_1$, $\overline{\mathbf{b}}_1$ and the centroid of $\mathbf{b}_1\overline{\mathbf{b}}_1\mathbf{c}_1$.

Theorem 4. *If we set* $\eta = 1/2$ *and* $\overline{\eta} = -\varepsilon/2\sqrt{3}$ *in* (12)*, then for all* $b, \overline{b} \in L(E', E)$ *the linear map* f *defined in* (13) *as well as* $\overline{f}$ *and* $(e - \overline{e}\circ r_3')$ *are similarities.*

Proof. Theorem 2 gives sufficient conditions for the regularity of $\mathbf{H_d}$. Due to Theorem 3 this implies the regularity of $\mathbf{H_f}$, provided the parameters $\eta, \overline{\eta}$ in Operation 4 obey (14), i.e.,

$$\eta = \frac{1}{2}, \quad \overline{\eta} = -\frac{\varepsilon}{2\sqrt{3}}\,.^{4}$$

In this case the triangles $\mathbf{b}_i\mathbf{b}_{i+1}\mathbf{e}_i, \dots$ are isosceles with base angles 30° (see [1], Fig. 3). □

Corollary 4. (Čerin [1]) *Let two isocentroidal point triples* $\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3$ *and* $\overline{\mathbf{b}}_1\overline{\mathbf{b}}_2\overline{\mathbf{b}}_3$ *be given. When the Operations 4 and 3 are applied to the hexagon* $\mathbf{H_b}$ *using isosceles affixed triangles with* 30° *base angles, then the resulting hexagon* $\mathbf{H_f}$ *is regular with center* $\mathbf{o}$.

The similarity addressed in Theorem 4 reads

$$\begin{aligned} f &:= \tfrac{1}{6}\left[b + b\circ r_3' + \tfrac{\varepsilon}{\sqrt{3}}\, r_4\circ(b - b\circ r_3')\right] + \tfrac{1}{6}\left[\overline{b} - \tfrac{\varepsilon}{\sqrt{3}}\, r_4\circ(\overline{b} + 2\overline{b}\circ r_3')\right] \\ \overline{f} &:= \tfrac{1}{6}\left[b\circ r_3' + \tfrac{\varepsilon}{\sqrt{3}}\, r_4\circ(2b + b\circ r_3')\right] + \tfrac{1}{6}\left[\overline{b} + \overline{b}\circ r_3' + \tfrac{\varepsilon}{\sqrt{3}}\, r_4\circ(\overline{b} - \overline{b}\circ r_3')\right]. \end{aligned} \tag{15}$$

In [1], Fig. 4, both regular hexagons $\mathbf{H_d}$ and $\mathbf{H_f}$ are displayed (note Corollary 3).

The Corollaries 2 and 4 reveal that Operation 1 does not influence the regularity of $\mathbf{H_d}$ and $\mathbf{H_f}$. This has already been pointed out in [1]. Nevertheless, we want to figure out the dependence of these hexagons from the triangle $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$:

Upon substitution of (4) in (11) we obtain

$$\begin{aligned} d &= \tfrac{1}{2}\left[\lambda a + \overline{\lambda} a\circ r_3'\right] - \tfrac{\varepsilon}{2\sqrt{3}}\, r_4\circ\left[(\lambda - 2\overline{\lambda})a + (2\lambda - \overline{\lambda})a\circ r_3'\right] \\ \overline{d} &= \tfrac{1}{2}\left[(\lambda - \overline{\lambda})a + \lambda a\circ r_3'\right] + \tfrac{\varepsilon}{2\sqrt{3}}\, r_4\circ\left[(\lambda + \overline{\lambda})a - (\lambda - 2\overline{\lambda})a\circ r_3'\right]. \end{aligned} \tag{16}$$

On the other hand the substitution $b = \mathcal{A}_\mu(a, a\circ r_3')$ and $\overline{b} = \mathcal{A}_{\overline{\mu}}(a\circ r_3', a)$ in (15) gives

$$\begin{aligned} f &= \tfrac{1}{6}\left[(2\mu - \overline{\mu})a + (\mu + \overline{\mu})a\circ r_3'\right] + \tfrac{\varepsilon}{2\sqrt{3}}\, r_4\circ\left[\overline{\mu}a - (\mu - \overline{\mu})a\circ r_3'\right] \\ \overline{f} &= \tfrac{1}{6}\left[(\mu - 2\overline{\mu})a + (2\mu - \overline{\mu})a\circ r_3'\right] + \tfrac{\varepsilon}{2\sqrt{3}}\, r_4\circ\left[\mu a + \overline{\mu}a\circ r_3'\right]. \end{aligned} \tag{17}$$

The following lemma clarifies the condition mentioned in Lemma 2:

Lemma 3. *For any $g \in L(E', E)$ the set $\{g,\, g\circ r_3',\, r_4\circ g,\, r_4\circ g\circ r_3'\}$ is linear dependent if and only if g is a similarity.*

Proof. There are orthonormal bases in E' and E such that the associated matrix $M(d)$ of d has diagonal form. Since $g + 2g\circ r_3' = \sqrt{3}\, g\circ r_4'$ due to (9), we can replace r_3' by r_4' in the given set before proving the linear dependence. We get

$$M(g) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},\ M(g\circ r_4') = \begin{pmatrix} 0 & -\alpha \\ \beta & 0 \end{pmatrix},\ M(r_4\circ g) = \begin{pmatrix} 0 & -\beta \\ \alpha & 0 \end{pmatrix},\ M(r_4\circ g\circ r_4') = \begin{pmatrix} -\beta & 0 \\ 0 & -\alpha \end{pmatrix}.$$

These matrices are linearly dependent if and only if $\alpha = \pm\beta$, hence $g\circ r_4' = \pm r_4\circ g$. $\square$

Theorem 5. *For each non-equilateral triangle $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$ there is a linear bijection*

$$\sigma\colon\ \mathbb{R}^2 \to \mathcal{S}_\varepsilon(E', E), \quad (\lambda, \overline{\lambda}) \mapsto d$$

of the parameters used in Operation 1 onto the – either direct or indirect – similarities resulting from Operations 1, 2 and 3.

Proof. With Lemma 3 the set $\{a,\, a\circ r_3',\, r_4\circ a,\, r_4\circ a\circ r_3'\}$ is a basis of the four-dimensional vector space $L(E', E)$. Due to Lemma 2 any similarity is uniquely defined by the coefficients of a and $a\circ r_3'$ when represented as a linear combination of this basis. Hence Theorem 5 results immediately from (16). $\square$

Corollary 5. *For each regular hexagon $\mathbf{H}$ centered at $\mathbf{o}$ there is pair of constants $\lambda, \overline{\lambda} \in \mathbb{R}$ such that $\mathbf{H} = \mathbf{H_d}$ results from a given non-equilateral triangle $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$ by applying the Operations 1, 2 and 3. In the same way there is a pair $\mu, \overline{\mu} \in \mathbb{R}$ of constants in Operation 1 such that $\mathbf{H} = \mathbf{H_f}$ results from $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$ by the Operations 1, 4 and 3 ($\eta, \overline{\eta}$ according to Theorem 3). Suppose, the affixed triangles have the same orientation in Operations 2 and 4. Then for any $(\nu, \overline{\nu}) \in \mathbb{R}^2$ the following parameters and only these give the same hexagon $\mathbf{H}$ – up to cyclic permutations:*

$$\begin{array}{llll}
\text{as } \mathbf{H_d}: & (\lambda, \overline{\lambda}) = & (\nu, \overline{\nu}), & (\nu - \overline{\nu}, \nu), & (-\overline{\nu}, \nu - \overline{\nu}), \\
 & & (-\nu, -\overline{\nu}), & (-\nu + \overline{\nu}, -\nu), & (\overline{\nu}, -\nu + \overline{\nu}), \\
\text{as } \mathbf{H_f}: & (\mu, \overline{\mu}) = & (\nu + \overline{\nu}, -\nu + 2\overline{\nu}), & (-\nu + 2\overline{\nu}, -2\nu + \overline{\nu}), & (-2\nu + \overline{\nu}, -\nu - \overline{\nu}), \\
 & & (-\nu - \overline{\nu}, \nu - 2\overline{\nu}), & (\nu - 2\overline{\nu}, 2\nu - \overline{\nu}), & (2\nu - \overline{\nu}, \nu + \overline{\nu}).
\end{array}$$

Proof. The linear maps d, $\overline{d}$, $d\circ r_3'$, $\overline{d}\circ r_3' = -d$, $d\circ {r_3'}^2 = -\overline{d}$ and $\overline{d}\circ {r_3'}^2 = -d\circ r_3'$ define the same regular hexagon. $\square$

Remark 3. Due to Corollary 5 there is no "distinguished" hexagon among all regular hexagons centered at $\mathbf{o}$.

Theorem 6. *For any non-equilateral triangle $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$ the cases presented in Theorems 2 and 4 are the only one where the Operations 1, 2' and 3 or 1, 4 and 3, resp., give regular hexagons in the real plane.*

Proof. Following the proof of Theorem 2 we express b and $\overline{b}$ in (10) in terms of a and get

$$3d = \left[(2\lambda-\overline{\lambda}) + \xi(2\overline{\lambda}-\lambda\right] a + \left[(\lambda+\overline{\lambda}) + \xi(\overline{\lambda}-2\lambda)\right] a\circ r_3' + \overline{\xi} r_4\circ\left[(\lambda-2\overline{\lambda})a + (2\lambda-\overline{\lambda})a\circ r_3'\right].$$

Due to Lemma 3 the set $\{a,\ a\circ r_3',\ r_4\circ a,\ r_4\circ a\circ r_3'\} \subset L(E',E)$ is linearly independent. Hence Lemma 2 implies the following necessary and sufficient conditions:

$$\begin{aligned} \lambda\xi + (\lambda - 2\overline{\lambda})\tfrac{\varepsilon}{\sqrt{3}}\overline{\xi} &= \overline{\lambda} \\ \overline{\lambda}\xi + (2\lambda - \overline{\lambda})\tfrac{\varepsilon}{\sqrt{3}}\overline{\xi} &= \overline{\lambda} - \lambda. \end{aligned}$$

Under $q(\lambda,\overline{\lambda}) := \lambda^2 - \lambda\overline{\lambda} + \overline{\lambda}^2 \neq 0$ this system of linear equations has the unique solution $\xi = 1/2$, $\overline{\xi} = -\varepsilon\sqrt{3}/2$.

The quadratic form q is positive definite. However, if $\lambda, \overline{\lambda} \in \mathbb{C}$ are admitted, then the two linear equations can also be linearly dependent. In this exceptional case there is a free choice for the third vertex of any affixed triangle on a line passing through the solution given in Theorem 2.

According to Theorem 3 the hexagon $\mathbf{H_f}$ (parameters $\eta, \overline{\eta}$) is regular if and only if $\mathbf{H_d}$ is regular for $\xi, \overline{\xi}$ obeying (14). □

Example 1. In order to get the hexagonal extension of Napoleon's theorem (see Fig. 2), we set in (16) $b = a$, $\overline{b} = a\circ r_3'$, i.e. $\lambda = \overline{\lambda} = 1$. This gives

$$d = \tfrac{1}{2}\left[a + a\circ r_3' + \tfrac{\varepsilon}{\sqrt{3}} r_4\circ(a - a\circ r_3')\right], \quad \overline{d} = d + d\circ r_3'.$$

Corollary 5 reveals that the same hexagon shows up as $\mathbf{H_d}$ for $(\lambda,\overline{\lambda}) = (0,1), (1,0), (-1,-1), (0,-1), (-1,0)$ and as $\mathbf{H_f}$ for $(\mu,\overline{\mu}) = (2,1), (1,-1), (1,2), (-2,-1), (-1,1), (-1,-2)$.

According to [6] the same hexagon arises as $\mathbf{H_c}$ when we build equilateral triangles on the middle third of the sides of $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$ (see Fig. 2), i.e.,

$$b := \mathcal{A}_{2/3}(a, a\circ r_3') = \tfrac{1}{3}(2a + a\circ r_3'), \quad \overline{b} := \mathcal{A}_{2/3}(a\circ r_3', a) = \tfrac{1}{3}(a + 2a\circ r_3'),$$

$$c = \mathcal{T}_{1/2\ -\varepsilon\sqrt{3}/2}(b,\overline{b}) = \tfrac{1}{2}\left[b + \overline{b} - \varepsilon\sqrt{3}\, r_4\circ(\overline{b} - b)\right] = \tfrac{1}{2}\left[a + a\circ r_3' + \tfrac{\varepsilon}{\sqrt{3}} r_4\circ(a - a\circ r_3')\right].$$

Because of $\overline{b}\circ r_3' = -b$ the hexagon $\mathbf{H_b}$ is symmetric with respect to $\mathbf{o}$. Hence $\mathbf{H_c}$ is centrally symmetric, too, and this implies $\overline{c} = -c\circ r_3'^2 = c + c\circ r_3'$.

It is easy to prove with Lemmas 2 and 3 that for a non-equilateral triangle $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$ the Operations 1 and 2 produce a regular hexagon $\mathbf{H_c}$ if and only if $\lambda = \overline{\lambda} = 2/3$.

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