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# Matrices over Centrally $\mathbb{Z}_2$-graded Rings

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**Abstract.** We introduce a new computational technique for $n \times n$ matrices, over a $\mathbb{Z}_2$-graded ring $R = R_0 \oplus R_1$ with $R_0 \subseteq Z(R)$, leading us to a new concept of determinant, which can be used to derive an invariant Cayley-Hamilton identity. An explicit construction of the inverse matrix $A^{-1}$ for any invertible $n \times n$ matrix $A$ over a Grassmann algebra $E$ is also obtained.

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## 1. Introduction

The main aim of the present paper is to introduce a new computational technique for matrices over certain $\mathbb{Z}_2$-graded rings. We shall consider $n \times n$ matrices over a $\mathbb{Z}_2$-graded ring $R = R_0 \oplus R_1$ with the property $R_0 \subseteq Z(R)$, where $Z(R)$ denotes the centre of $R$. For these matrices our method provides a possibility to use the classical determinant theory of matrices over commutative rings. The most important example for $\mathbb{Z}_2$-graded rings with the above mentioned property is the exterior (Grassmann) algebra

$$E = F\langle v_1, v_2, \ldots, v_i, \ldots \mid v_i v_j = -v_j v_i \text{ for all integers } 1 \leq i \leq j\rangle$$

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and the polynomial algebra $E[t]$, where $F$ is a field, $t$ is a commuting indeterminant. The $\mathbb{Z}_2$-gradings are $E = E_0 \oplus E_1$ and $E[t] = E_0[t] \oplus E_1[t]$ with $E_0$ being the subspace (subalgebra) generated by the monomials of even length and $E_1$ being the subspace generated by the monomials of odd length. We note that the $F$-algebra $M_n(E)$ of $n \times n$ matrices over the (infinite dimensional) exterior algebra $E$ with $\operatorname{char}(F) = 0$ is one of the most important objects of study in the theory of PI-algebras (see Kemer's structure theory of T-ideals in [2] and [3]).

First of all, our technique leads to a new concept of invariant determinant, which can be used to derive an invariant Cayley-Hamilton identity in $M_n(R)$. An immediate application of our results will provide a new explicit construction of the inverse matrix $A^{-1}$ for any invertible $n \times n$ matrix $A \in M_n(E)$.

Since the existence of a $\mathbb{Z}_2$-grading $R = R_0 \oplus R_1$ with the property $R_0 \subseteq Z(R)$ implies that $R$ is Lie nilpotent of index 2, it would be desirable to find the precise relationship between the concepts presented in the sequel and the Lie nilpotent determinant theory in [4]. The constructions in [4] are based on the use of the so called preadjoint, which is a natural but complicated generalization of the ordinary adjoint matrix. In defining our determinant, here we use only classical determinants and adjoints. Our results on $n \times n$ matrices over $R = R_0 \oplus R_1$ with $R_0 \subseteq Z(R)$ are similar to the results of [4] specialized to $n \times n$ matrices over Lie nilpotent rings of index 2. We believe that our present approach is easier to understand and gives more chance to find an explicit form of the Cayley-Hamilton equation (Newton formulae) for $n \geq 3$. Using sophisticated calculations, starting from the characteristic polynomial defined in [4], M. Domokos obtained Newton formulae for $2 \times 2$ matrices over the Grassmann algebra (see [1]).

## 2. $\mathbb{Z}_2$-gradings and skew polynomial rings

A $\mathbb{Z}_2$-grading of an (associative) ring $R$ is a pair $(R_0, R_1)$, where $R_0$ and $R_1$ are additive subgroups of $R$ such that $R = R_0 \oplus R_1$ and $R_i R_j \subseteq R_{i+j}$ for all $i, j \in \{0, 1\}$ and $i + j$ is taken modulo 2. The relation $R_0 R_0 \subseteq R_0$ ensures that $R_0$ is a subring of $R$. Now any element $r \in R$ can be uniquely written as $r = r_0 + r_1$, where $r_0 \in R_0$ and $r_1 \in R_1$. It is easy to see that the existence of $1 \in R$ implies that $1 \in R_0$. The function $\sigma : R \longrightarrow R$ defined by $\sigma(r_0 + r_1) = r_0 - r_1$ is a ring homomorphism (actually, it is an automorphism of $R$). A more general situation is, when $R$ is considered as a $C$-algebra for some commutative ring $C \subseteq Z(R)$ and $R = \bigoplus_{u \in S} R_u$ is graded by a subsemigroup $S \subseteq U(C)$ of the multiplicative group of units in $C$ (each $R_u \subseteq R$ is a $C$-submodule) and $\sigma(\sum_{u \in S} r_u) = \sum_{u \in S} u r_u$.

For a $\mathbb{Z}_2$-graded ring $R = R_0 \oplus R_1$ let us consider the skew polynomial ring $R[x, \sigma]$ in the skew indeterminate $x$. The elements of $R[x, \sigma]$ are left polynomials of the form $f(x) = a_0 + a_1 x + \cdots + a_k x^k$ with $a_0, a_1, \ldots, a_k \in R$. Besides the obvious addition, we have the following multiplication rule in $R[x, \sigma]$:

$$xr = \sigma(r)x \text{ for all } r \in R, \text{ i.e. that } x(r_0 + r_1) = (r_0 - r_1)x \text{ for all } r_0 \in R_0,\ r_1 \in R_1$$

and

$$(a_0 + a_1 x + \cdots + a_k x^k)(b_0 + b_1 x + \cdots + b_l x^l) = c_0 + c_1 x + \cdots + c_{k+l} x^{k+l},$$

where

$$c_m = \sum_{i+j=m, i\geq 0, j\geq 0} a_i \sigma^i(b_j).$$

Since $\sigma$ is an involution, $x^2$ is a central element of $R[x,\sigma]$: we have $\sigma(\sigma(r)) = r$ and $x^2 r = x\sigma(r)x = \sigma(\sigma(r))x^2 = rx^2$ for all $r \in R$, moreover $x^2$ commutes with the powers of $x$. Thus the ideal $(x^2) \lhd R[x,\sigma]$ generated by $x^2$ can be written as $(x^2) = R[x,\sigma]x^2 = x^2 R[x,\sigma]$. Consider the factor ring $R[x,\sigma]/(x^2)$, then for any element $f(x) \in R[x,\sigma]$ there exists exactly one left polynomial of the form $r + sx \in R[x,\sigma]$ in the residue class $f(x) + (x^2)$. Hence the elements of $R[x,\sigma]/(x^2)$ can be represented by linear left polynomials with coefficients in $R$ and the multiplication in $R[x,\sigma]/(x^2)$ is the following:

$$(r + sx)(p + qx) = rp + (rq + s\sigma(p))x,$$

where $r, s, p, q \in R$. The above observation ensures that $R[x,\sigma]/(x^2) = R \oplus Rx$ is a $\mathbb{Z}_2$-grading with $(Rx)(Rx) = \{0\}$. It follows that the $n \times n$ matrices $P, Q \in M_n(R[x,\sigma]/(x^2))$ can be uniquely written as $P = P' + P''x$ and $Q = Q' + Q''x$ for some $P', P'', Q', Q'' \in M_n(R)$ and that

$$(*) \qquad PQ = P'Q' + (P'Q'' + P''\sigma(Q'))x,$$

where $\sigma(Q') = [\sigma(q'_{ij})]$ is the natural action of $\sigma$ on $Q' = [q'_{ij}]$ and the products $P'Q', P'Q''$, $P''\sigma(Q')$ are taken in $M_n(R)$. It can be easily seen, that

$$\overline{R} = \{r_0 + s_1 x \mid r_0 \in R_0 \text{ and } s_1 \in R_1\} \subseteq \{r + sx \mid r, s \in R\} = R[x,\sigma]/(x^2)$$

is a subring of $R[x,\sigma]/(x^2)$. Indeed, $(r_0 + s_1 x)(p_0 + q_1 x) = r_0 p_0 + (r_0 q_1 + s_1 \sigma(p_0))x$, where $r_0 p_0 \in R_0$ and $r_0 q_1 + s_1 \sigma(p_0) = r_0 q_1 + s_1 p_0 \in R_1$ for all $r_0, p_0 \in R_0$ and $s_1, q_1 \in R_1$. In consequence, $\overline{R} = R_0 \oplus R_1 x$ is a $\mathbb{Z}_2$-grading. We note that $\overline{R}$ can be defined directly on the product $R_0 \times R_1$ with componentwise addition and taking the multiplication $(r_0, s_1)(p_0, q_1) = (r_0 p_0, r_0 q_1 + s_1 p_0)$. If $R_0 \subseteq Z(R)$ with $Z(R)$ being the centre of $R$, then $\overline{R}$ is commutative:

$$\begin{aligned}(r_0 + s_1 x)(p_0 + q_1 x) &= r_0 p_0 + (r_0 q_1 + s_1 p_0)x = \\ &= p_0 r_0 + (p_0 s_1 + q_1 r_0)x = (p_0 + q_1 x)(r_0 + s_1 x).\end{aligned}$$

The condition $R_0 \subseteq Z(R)$ also implies the Lie nilpotence (of index 2) of $R$. For the elements $r, s \in R$ we have $r = r_0 + r_1$, $s = s_0 + s_1$ for some $r_0, s_0 \in R_0$ and $r_1, s_1 \in R_1$. Now $r_0, s_0 \in Z(R)$ implies that $[r_0, s] = [r_1, s_0] = 0$, whence we get $[r, s] = [r_0 + r_1, s] = [r_0, s] + [r_1, s] = [r_1, s_0 + s_1] = [r_1, s_0] + [r_1, s_1] = [r_1, s_1] = r_1 s_1 - s_1 r_1 \in R_0$. Thus $[r, s] \in Z(R)$, so we obtain that $[[r, s], w] = 0$ for all $r, s, w \in R$.

## 3. Computing with $n \times n$ matrices over a centrally $\mathbb{Z}_2$-graded ring

A $\mathbb{Z}_2$-grading $(R_0, R_1)$ of the ring $R$ is called central, if $R_0 \subseteq Z(R)$. Let $A = [a_{ij}] \in M_n(R)$ be an $n \times n$ matrix over a ring with a central $\mathbb{Z}_2$-grading, then $a_{ij} = a_{ij}^{(0)} + a_{ij}^{(1)}$ for some unique $a_{ij}^{(0)} \in R_0$ and $a_{ij}^{(1)} \in R_1$ for all integers $1 \leq i \leq n$ and $1 \leq j \leq n$, i.e. $A = A_0 + A_1$ with $A_0 = [a_{ij}^{(0)}] \in M_n(R_0)$ and $A_1 = [a_{ij}^{(1)}] \in M_n(R_1)$. The companion matrix of $A$ in $M_n(\overline{R})$ is defined as

$$A_0 + A_1 x = [a_{ij}^{(0)} + a_{ij}^{(1)} x].$$

Since $\overline{R}$ is commutative, the determinant and the adjoint of $A_0 + A_1 x$ are defined and can be written as

$$\det(A_0 + A_1 x) = d_0 + d_1 x \in \overline{R}$$

and

$$\mathrm{adj}(A_0 + A_1 x) = [b_{ij}^{(0)} + b_{ij}^{(1)} x] = B_0 + B_1 x \in M_n(\overline{R}),$$

where $d_0 \in R_0$ and $d_1 \in R_1$ are elements, $B_0 = [b_{ij}^{(0)}] \in M_n(R_0)$ and $B_1 = [b_{ij}^{(1)}] \in M_n(R_1)$ are $n \times n$ matrices, each of these objects is uniquely determined by $A$. Clearly, $d_0 = \det(A_0)$, $B_0 = \mathrm{adj}(A_0)$ and the elements $d_1, b_{ij}^{(1)} \in R_1$ are also polynomial expressions of the $a_{ij}^{(0)}$'s and the $a_{ij}^{(1)}$'s (it is not hard to give them explicitly).

**3.1. Theorem.** *The elements of the product matrices*

$$A(B_0 + B_1) = (A_0 + A_1)(B_0 + B_1) \text{ and } (B_0 + B_1)A = (B_0 + B_1)(A_0 + A_1)$$

*are contained in the subring $R_0[d_1]$ of $R$ generated by $d_1$ and the elements of $R_0$, namely*:

$$A(B_0 + B_1), (B_0 + B_1)A \in M_n(R_0[d_1]).$$

*Proof.* Since $d_0 + d_1 x$ is the determinant and $B_0 + B_1 x$ is the adjoint of $A_0 + A_1 x$, we have

$$(A_0 + A_1 x)(B_0 + B_1 x) = (d_0 + d_1 x)I,$$

in $M_n(\overline{R})$, where $I$ is the identity matrix. In view of

$$M_n(\overline{R}) = M_n(R_0 \oplus R_1 x) \subseteq M_n(R \oplus Rx) = M_n(R[x, \sigma]/(x^2))$$

and $\sigma(B_0) = B_0$, the application of $(*)$ gives that

$$A_0 B_0 + (A_0 B_1 + A_1 B_0)x = d_0 I + (d_1 I)x,$$

where $A_0 B_0$ and $A_0 B_1 + A_1 B_0$ are taken in $M_n(R)$. Using the unique $r_0 + s_1 x$ form (with $r_0 \in R_0$ and $s_1 \in R_1$) of the elements in $\overline{R}$ and matching the coefficients of $x$ in the left and the right side of the above equation, we obtain the following identity in $M_n(R)$:

$$A_0 B_1 + A_1 B_0 = d_1 I.$$

Thus

$$A(B_0 + B_1) = (A_0 + A_1)(B_0 + B_1) = (A_0B_0 + A_1B_1) + (A_0B_1 + A_1B_0) = \\ (A_0B_0 + A_1B_1) + d_1$$

and $A_0B_0 + A_1B_1 \in M_n(R_0)$ imply that $A(B_0 + B_1) \in M_n(R_0[d_1])$. The similar statement on the product $(B_0 + B_1)A$ can be proved analogously. $\square$

The condition $R_0 \subseteq Z(R)$ implies that the subring $R_0[d_1] \subseteq R$ is commutative (the elements of $R_0[d_1]$ are polynomials of $d_1$ with coefficients in $R_0$). As a consequence of Theorem 3.1 the determinant and the adjoint of the matrices $A(B_0 + B_1), (B_0 + B_1)A \in M_n(R_0[d_1])$ are defined: $\det(A(B_0 + B_1))$ is called the right determinant (with respect to the given central $\mathbb{Z}_2$-grading $R = R_0 \oplus R_1$) and $(B_0 + B_1)\mathrm{adj}(A(B_0 + B_1))$ is called the right adjoint (with respect to the given central $\mathbb{Z}_2$-grading $R = R_0 \oplus R_1$) of the matrix $A \in M_n(R)$. We use the following notations:

$$\mathrm{rdet}(A) = \det(A(B_0 + B_1)) \quad \text{and} \quad \mathrm{radj}(A) = (B_0 + B_1)\mathrm{adj}(A(B_0 + B_1)).$$

Since $A(B_0 + B_1)\mathrm{adj}(A(B_0 + B_1)) = \det(A(B_0 + B_1))I$ in $M_n(R_0[d_1])$, we immediately obtain (in $M_n(R)$) that:

$$A\ \mathrm{radj}(A) = \mathrm{rdet}(A)I.$$

**3.2. Proposition.** (i) *If* $T \in \mathrm{GL}_n(R_0)$ *is an invertible matrix and* $A \in M_n(R)$, *then* $\mathrm{rdet}(TAT^{-1}) = \mathrm{rdet}(A)$ *and* $\mathrm{radj}(TAT^{-1}) = T(\mathrm{radj}(A))T^{-1}$.
(ii) *If* $A \in M_n(R_0)$, *then* $\mathrm{rdet}(A) = (\det(A))^n$ *and* $\mathrm{radj}(A) = (\det(A))^{n-1}\mathrm{adj}(A)$.

*Proof.* (i) In view of $TA_0T^{-1} \in M_n(R_0)$ and $TA_1T^{-1} \in M_n(R_1)$, the companion matrix of $TAT^{-1} = TA_0T^{-1} + TA_1T^{-1}$ is $TA_0T^{-1} + TA_1T^{-1}x$. Using $\mathrm{adj}(A_0 + A_1x) = B_0 + B_1x$, we obtain that

$$\mathrm{adj}(TA_0T^{-1} + TA_1T^{-1}x) = \mathrm{adj}(T(A_0 + A_1x)T^{-1}) = T(\mathrm{adj}(A_0 + A_1x))T^{-1} = \\ = T(B_0 + B_1x)T^{-1} = (TB_0T^{-1}) + (TB_1T^{-1})x.$$

It follows that

$$\mathrm{rdet}(TAT^{-1}) = \det(TAT^{-1}(TB_0T^{-1} + TB_1T^{-1})) = \\ = \det(TA(B_0 + B_1)T^{-1}) = \det(A(B_0 + B_1)) = \mathrm{rdet}(A)$$

and

$$\mathrm{radj}(TAT^{-1}) = (TB_0T^{-1} + TB_1T^{-1})\mathrm{adj}(TAT^{-1}(TB_0T^{-1} + TB_1T^{-1})) = \\ = T(B_0 + B_1)T^{-1}\mathrm{adj}(TA(B_0 + B_1)T^{-1}) = \\ = T(B_0 + B_1)T^{-1}T\ \ (\mathrm{adj}(A(B_0 + B_1)))\,T^{-1} = T(\mathrm{radj}(A))T^{-1}.$$

(ii) Since $A \in M_n(R_0)$ implies that $A_0 = A$ and $A_1 = 0$, from $\mathrm{adj}(A_0 + A_1x) = B_0 + B_1x$ we get that $B_0 = \mathrm{adj}(A)$ and $B_1 = 0$. Thus

$$\mathrm{rdet}(A) = \det(A(B_0 + B_1)) = \det(A_0B_0) = \det(\det(A)I) = (\det(A))^n$$

and

$$\begin{aligned}\mathrm{radj}(A) &= (B_0 + B_1)\mathrm{adj}(A(B_0 + B_1)) = B_0\mathrm{adj}(A_0B_0) = \\ &= \mathrm{adj}(A)\mathrm{adj}(\det(A)I) = \mathrm{adj}(A)(\det(A))^{n-1}I = (\det(A))^{n-1}\mathrm{adj}(A).\end{aligned}$$ □

If $(R_0, R_1)$ is a $\mathbb{Z}_2$-grading of the ring $R$, then $(R_0[t], R_1[t])$ is a natural $\mathbb{Z}_2$-grading of the polynomial ring $R[t]$ of the commuting indeterminant $t$. It is straightforward to see that $(R[t])[x, \sigma_t]/(x^2) \cong (R[x, \sigma]/(x^2))[t]$ and $\overline{R[t]} = (R_0[t]) \oplus (R_1[t])x \cong (R_0 \oplus R_1x)[t] = \overline{R}[t]$ are ring isomorphisms, where $\sigma_t : R[t] \longrightarrow R[t]$ is the natural extension of $\sigma$. For a central $\mathbb{Z}_2$-grading $(R_0, R_1)$, the induced $\mathbb{Z}_2$-grading $(R_0[t], R_1[t])$ of $R[t]$ is also central: $R_0[t] \subseteq Z(R[t])$.

We define the right characteristic polynomial (with respect to the given central $\mathbb{Z}_2$-grading $R = R_0 \oplus R_1$) of a matrix $A \in M_n(R)$ as the right determinant (with respect to the induced central $\mathbb{Z}_2$-grading $R[t] = R_0[t] \oplus R_1[t]$) of the matrix $tI - A \in M_n(R[t])$, where $I$ is the identity matrix in $M_n(R)$:

$$\chi_A(t) = \mathrm{rdet}(tI - A) = \lambda_0 + \lambda_1t + \cdots + \lambda_kt^k \in R[t],\ \lambda_0, \lambda_1, \ldots, \lambda_k \in R \text{ and } \lambda_k \neq 0.$$

Since $\mathrm{GL}_n(R_0) \subseteq \mathrm{GL}_n(R_0[t])$, an immediate consequence of Proposition 3.2 is that $\chi_{TAT^{-1}}(t) = \chi_A(t)$ for any invertible matrix $T \in \mathrm{GL}_n(R_0)$.

**3.3. Proposition.** *If $\chi_A(t) = \lambda_0 + \lambda_1t + \cdots + \lambda_kt^k$ is the right characteristic polynomial of the $n \times n$ matrix $A \in M_n(R)$, then $k = n^2$ and $\lambda_{n^2} = 1,\ \lambda_0 = \mathrm{rdet}(-A)$.*

*Proof.* If $A = A_0 + A_1$ with $A_0 \in M_n(R_0)$ and $A_1 \in M_n(R_1)$, then $tI - A = (tI - A_0) + (-A_1)$ with $tI - A_0 \in M_n(R_0[t])$ and $-A_1 \in M_n(R_1[t])$. The companion matrix of $tI - A$ in $M_n(\overline{R[t]}) \cong M_n(\overline{R}[t])$ is $(tI - A_0) + (-A_1)x = tI - (A_0 + A_1x)$ (here $\overline{R[t]} \cong \overline{R}[t]$ is a commutative ring). It is well known that each of the elements in the diagonal of $\mathrm{adj}(tI - (A_0 + A_1x))$ is a polynomial in $\overline{R}[t]$ with leading term $t^{n-1}$. The non-diagonal entries in $\mathrm{adj}(tI - (A_0 + A_1x))$ are polynomials in $\overline{R}[t]$ of degree less than $n - 1$. In consequence, the matrices $B_0(t) \in M_n(R_0[t])$ and $B_1(t) \in M_n(R_1[t])$ in

$$\mathrm{adj}((tI - A_0) + (-A_1)x) = B_0(t) + B_1(t)x$$

have the following properties: each non-diagonal entry of $B_0(t)$ and each entry of $B_1(t)$ is of degree (in $t$) less than $n - 1$, moreover the leading term of each diagonal element in $B_0(t)$ is $t^{n-1}$. Thus each element in the diagonal of the product matrix $(tI - A)(B_0(t) + B_1(t))$ is a polynomial with leading term $t^n$. Since the non-diagonal entries in $(tI - A)(B_0(t) + B_1(t))$ are of degree less or equal than $n - 1$, we obtain that the leading term of the right characteristic polynomial $\det((tI - A)(B_0(t) + B_1(t))) = \mathrm{rdet}(tI - A) = \chi_A(t)$ is $(t^n)^n = t^{n^2}$, i.e. that $k = n^2$ and $\lambda_{n^2} = 1$.

To prove $\lambda_0 =$rdet$(-A)$, let adj$(-A_0 - A_1 x) = C_0 + C_1 x$ with $C_0 \in M_n(R_0)$ and $C_1 \in M_n(R_1)$. Now

$$\mathrm{adj}(tI - (A_0 + A_1 x)) = (C_0 + C_1 x) + C(t)t$$

for some $C(t) \in M_n(\overline{R}[t])$, whence we get that $B_0(t) + B_1(t) = (C_0 + C_1) + H(t)t$ for some $H(t) \in M_n(R[t])$. It follows, that

$$\begin{aligned}\chi_A(t) = \mathrm{rdet}(tI - A) = \det((tI - A)(B_0(t) + B_1(t))) = \\ = \det(H(t)t^2 - AH(t)t + C_0 t + C_1 t - A(C_0 + C_1)).\end{aligned}$$

Since $A(C_0+C_1)$ does not contain $t$, we deduce that the constant term in $\chi_A(t)$ is rdet$(-A) = \det(-A(C_0 + C_1))$. □

**3.4. Theorem.** *If $\chi_A(t) \in R[t]$ is the right characteristic polynomial of an $n \times n$ matrix $A \in M_n(R)$ over a centrally $\mathbb{Z}_2$-graded ring $R = R_0 \oplus R_1$ and $h(t) \in R[t]$ is arbitrary, then the left substitution of $A$ into the product polynomial $\chi_A(t)h(t) = \mu_0 + \mu_1 t + \cdots + \mu_m t^m$ is zero: $I\mu_0 + A\mu_1 + \cdots + A^m \mu_m = 0$.*

*Proof.* Using

$$(tI - A)(U_0 + U_1 t + \cdots + U_{m-1} t^{m-1}) = (\mu_0 + \mu_1 t + \cdots + \mu_m t^m)I$$

in $M_n(R[t]) \cong (M_n(R))[t]$ with $(\mathrm{radj}(tI - A))h(t) = U_0 + U_1 t + \cdots + U_{m-1} t^{m-1}$ and $U_i \in M_n(R)$ for the indices $0 \leq i \leq m-1$, we can proceed as in the proof of Theorem 4.2 in [4]. □

## 4. The inverse formula for $n \times n$ matrices over the Grassmann algebra

An element $g$ of $E = F\langle v_1, v_2, \ldots, v_i, \ldots \mid v_i v_j = -v_j v_i \text{ for all integers } 1 \leq i \leq j\rangle$ can be uniquely written in the form

$$g = c_g + \sum_{1 \leq i_1 < i_2 < \ldots < i_k} c_g(i_1, i_2, \ldots, i_k) v_{i_1} v_{i_2} \ldots v_{i_k} \ ,$$

where $c_g, c_g(i_1, i_2, \ldots, i_k) \in F$. Now $\gamma(g) = c_g$ defines an $F$-algebra homomorphism $\gamma : E \to F$ and $\gamma$ naturally extends to an $F$-algebra homomorphism $\overline{\gamma} : M_n(E) \to M_n(F)$ of the matrix algebras. If $N = A - \overline{\gamma}(A)$, then it is easy to see that $BN$ is a nilpotent matrix for all $B \in M_n(E)$. The existence of the inverse matrix $(\overline{\gamma}(A))^{-1}$ in $M_n(F)$ implies the existence of the inverse of $A = \overline{\gamma}(A)(I + (\overline{\gamma}(A))^{-1}N)$ in $M_n(E)$:

$$A^{-1} = (I + (-(\overline{\gamma}(A))^{-1}N) + (-(\overline{\gamma}(A))^{-1}N)^2 + \cdots + (-(\overline{\gamma}(A))^{-1}N)^{m-1})(\overline{\gamma}(A))^{-1},$$

where $m$ is the index of the nilpotence of $(\overline{\gamma}(A))^{-1}N$. Thus $\det(\overline{\gamma}(A)) \neq 0$ implies the existence of $A^{-1} \in M_n(E)$. On the other hand, $AB = I$ in $M_n(E)$ implies that $\overline{\gamma}(A)\overline{\gamma}(B) = \overline{\gamma}(AB) = \overline{\gamma}(I) = I$ in $M_n(F)$, whence we get that $\det(\overline{\gamma}(A)) \neq 0$. In consequence, the existence of $A^{-1}$ in $M_n(E)$ is equivalent to $\det(\overline{\gamma}(A)) \neq 0$.

**4.1. Theorem.** *For a matrix $A \in M_n(E)$ we have $A = A_0 + A_1$ for some unique $A_0 \in M_n(E_0)$ and $A_1 \in M_n(E_1)$. If $A$ is invertible, then*

$$A^{-1} = (\mathrm{adj}(A_0) + \alpha_1(A))\mathrm{adj}\,(A(\mathrm{adj}(A_0) + \alpha_1(A)))\,\{\det\,(A(\mathrm{adj}(A_0) + \alpha_1(A)))\}^{-1},$$

*where* $\mathrm{adj}(A_0 + A_1x) = B_0 + B_1x$ *in* $M_n(\overline{E})$ *with* $B_0 = \mathrm{adj}(A_0) \in M_n(E_0)$*,* $B_1 = \alpha_1(A) \in M_n(E_1)$ *and* $\det(A(\mathrm{adj}(A_0) + \alpha_1(A)))$ *is an invertible element of* $E$*.*

*Proof.* In view of $\overline{\gamma}(A_1) = \overline{\gamma}(B_1) = 0$, $\overline{\gamma}(A_0) = \overline{\gamma}(A)$ and $\det(\overline{\gamma}(A)) \neq 0$, we can write that

$$\begin{aligned}\gamma(\mathrm{rdet}(A)) &= \gamma(\det((A_0 + A_1)(B_0 + B_1))) = \det(\overline{\gamma}((A_0 + A_1)(B_0 + B_1))) = \\ &= \det(\overline{\gamma}(A_0 + A_1)\overline{\gamma}(B_0 + B_1)) = \det(\overline{\gamma}(A_0)\overline{\gamma}(B_0)) = \det(\overline{\gamma}(A_0B_0)) = \\ &= \gamma(\det(A_0B_0)) = \gamma(\det(\det(A_0)I)) = \gamma((\det(A_0))^n) = \\ &= (\gamma(\det(A_0)))^n = (\det(\overline{\gamma}(A_0)))^n = (\det(\overline{\gamma}(A)))^n \neq 0,\end{aligned}$$

whence we get that $\mathrm{rdet}(A)$ is an invertible element of $E$. From $A\ \mathrm{radj}(A) = \mathrm{rdet}(A)I$, the right multiplication by $(\mathrm{rdet}(A))^{-1}$ gives that $A^{-1} = \mathrm{radj}(A)(\mathrm{rdet}(A))^{-1}$, where $\mathrm{radj}(A) = (B_0 + B_1)\mathrm{adj}(A(B_0 + B_1))$ and $\mathrm{rdet}(A) = \det(A(B_0 + B_1))$. □

**4.2. Remark.** The idea of considering the companion matrix $A_0 + A_1x$ arose in the following way. If $A \in M_n(E)$ with $A = A_0 + A_1$ and $v_i$ is a generator of $E$ not occurring in the elements of $A$, then $A$ can be completely read off the matrix $A_0 + A_1v_i$ and $A_0 + A_1v_i \in M_n(E_0)$ lies in a commutative environment. Thus the use of $A_0 + A_1v_i$ instead of $A$ is a natural challenge.

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