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On the Intersection of Invariant Rings

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Abstract. Based on Weitzenböck's theorem and Nagata's counterexample for Hilbert's fourteenth problem we construct two finitely generated invariant rings $R, S \subset \mathbf{K}[x_1, x_2, \dots, x_n]$ s.t. the intersection $R \cap S$ is not finitely generated as a $\mathbf{K}$-algebra.

1. Introduction

Recently the author has provided an algorithm for computing the intersection of invariant rings of finite groups and for computing $\mathbf{K}$-vectorspace bases of the intersection of arbitrary graded finitely generated algebras up to a given degree, cf. [2]. One might ask if it is possible to extend the algorithm to compute the intersection of arbitrary finitely generated invariant rings. We give a negative answer by showing the existence of finitely generated invariant rings $R, S \subset \mathbf{K}[x_1, x_2, \dots, x_n]$ s.t. their intersection $R \cap S$ cannot be finitely generated. The example builds upon Weitzenböck's theorem and a counterexample of Nagata for Hilbert's fourteenth problem, which can be formulated as follows: *Let $\mathbf{K}$ be a field and $G \subseteq GL_n(\mathbf{K})$ be an algebraic subgroup. Is the invariant ring $\mathbf{K}[x_1, x_2, \dots, x_n]^G$ finitely generated as a $\mathbf{K}$-algebra?*

In 1958 Nagata gave a negative answer by using commutative groups (cf. [6]) and in 1965 he provided invariant rings of non-commutative groups which are not finitely generated (cf. [7]). Later, these examples were greatly simplified and extended by R. Steinberg (cf. [9]). Meanwhile, based on the work of Roberts (cf. [8]), several counterexamples of invariant rings of algebraic $\mathbf{G}_a$-actions have been found. We refer, e.g., to [3] and the references therein, and for a non-finitely generated invariant ring of a linear action of $\mathbf{G}_a^{12}$ on $\mathbf{K}^{19}$ we refer to [1].

2. Nagata's counterexample

We present Nagata's counterexample of 1965, given in [7]. Let G be an algebraic group and $\rho : G \to GL_n(\mathbf{K})$ be a linear representation. A polynomial $f \in \mathbf{K}[x_1, x_2, \ldots, x_n]$ is *invariant* w.r.t. G if $f(\rho(\sigma) \cdot \mathbf{x}) = f(\mathbf{x})$ for all $\sigma \in G$. The ring $\mathbf{K}[x_1, x_2, \ldots, x_n]^G$ consisting of all invariant polynomials w.r.t. G is called the *invariant ring* of G (ρ will be omitted). The invariant ring is *finitely generated* if there exist invariants $h_1, h_2, \ldots, h_m$ s.t. the map $\mathbf{K}[y_1, y_2, \ldots, y_m] \to \mathbf{K}[x_1, x_2, \ldots, x_n]^G$, sending y_i to h_i, is surjective.

For $r \geq s^2$, where $s \geq 4$, let a_{ij}, $i = 1, 2, 3$ and $1 \leq j \leq r$, be algebraic independent elements over the the field $\mathbf{k}$ of characteristic 0 ($\mathbf{k}$ is the prime field Π of the algebraic curve defined in Ch. III of (loc. cit.)). Let $\mathbf{k} \subset \mathbf{K}$ be a field extension containing the a_{ij}'s and set $n = 2r$. Consider the subgroup

$$G = \left\{ \begin{pmatrix} B_1 & 0 & \ldots & 0 \\ 0 & B_2 & \ldots & 0 \\ 0 & \ldots & \ddots & \ldots \\ 0 & \ldots & 0 & B_r \end{pmatrix} : B_i = \begin{pmatrix} c_i & c_i b_i \\ 0 & c_i \end{pmatrix} \right\} \subset GL_n(\mathbf{K})$$

where $\sum_{j=1}^r a_{1j} b_j = \sum_{j=1}^r a_{2j} b_j = \sum_{j=1}^r a_{3j} b_j = 0$ and $\prod_{i=1}^r c_i = 1$.

Theorem 1. (Nagata 1965) *The invariant ring* $\mathbf{K}[x_1, x_2, \ldots, x_n]^G$ *is not finitely generated.*

Proof. We refer to Theorem 1, Chapter III in [7]. □

Remark 1. Actually Nagata proved that the invariant ring is an ideal transform $T(I, R)$ where $I \subset R$ is an ideal and R a Noetherian integral domain. Ideal transforms are inherently non-terminating and provide counterexamples to the (generalized) Zariski problem, but there are several conditions for $T(I, R)$ being finitely generated (cf., e.g., Chapter V of [7]). Serre proved that if R satisfies condition S_2 then for any ideal $I \subset R$ there exist $f, g \in R$ s.t. $T(I, R) = T(f, R) \cap T(g, R)$ where $T(f, R)$ and $T(g, R)$ are finitely generated (cf. Section 7.1 of [11]).

3. Construction of the invariant rings

Let a_{ij}, $i = 1, 2, 3$ and $1 \leq j \leq r$, be algebraic independent elements over $\mathbf{k}$ and $\mathbf{k} \subset \mathbf{K}$ be a field extension containing the the a_{ij}'s (as in the previous section), let $r \geq s^2, s \geq 4$, and $n = 2r$. In order to obtain the counterexample we define two groups T and H s.t.

$$\mathbf{K}[x_1, x_2, \ldots, x_n]^T \cap \mathbf{K}[x_1, x_2, \ldots, x_n]^H = \mathbf{K}[x_1, x_2, \ldots, x_n]^G$$

and show that $\mathbf{K}[x_1, x_2, \ldots, x_n]^H$ is not finitely generated and that the group H contains subgroups H', H'' s.t. the invariant rings $\mathbf{K}[x_1, x_2, \ldots, x_n]^{H'}$ and $\mathbf{K}[x_1, x_2, \ldots, x_n]^{H''}$ are finitely generated, but their intersection is not finitely generated. Consider the groups

$$T = \left\{ \begin{pmatrix} c_1 & 0 & \ldots & \ldots & 0 \\ 0 & c_1 & 0 & \ldots & 0 \\ \vdots & \ldots & \ddots & \ldots & \vdots \\ 0 & \ldots & 0 & c_r & 0 \\ 0 & \ldots & \ldots & 0 & c_r \end{pmatrix} : \prod_{i=1}^r c_i = 1 \right\} \subset GL_n(\mathbf{K})$$

and

$$H_k = \left\{ \begin{pmatrix} B_1 & 0 & \dots & 0 \\ 0 & B_2 & \dots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & \dots & 0 & B_k \end{pmatrix} : B_i = \begin{pmatrix} 1 & b_i \\ 0 & 1 \end{pmatrix} \right\} \subset GL_{2k}(\mathbf{K})$$

where $k = 4, \dots, r$, and $\sum_{j=1}^{k} a_{1j} b_j = \sum_{j=1}^{k} a_{2j} b_j = \sum_{j=1}^{r} a_{3j} b_j = 0$. Note that both groups are closed, but only T is reductive.

Proposition 1. *If T acts algebraically on an affine $\mathbf{K}$-algebra R then R^T is finitely generated. In particular, the invariant ring $\mathbf{K}[x_1, x_2, \dots, x_n]^T$ is finitely generated.*

Proof. The group T is a closed subgroup of the r-torus $(\mathbf{K}^*)^r$, hence T is reductive and the invariant ring is finitely generated, cf. e.g., Chapter II.3 of [5]. $\square$

In the sequel we define a linear action of $H_k/H_{k-1}, (k \geq 4)$, on $\mathbf{K}^n$ and we show, by using Weitzenböcks theorem (cf. [10]), that the invariant rings of H_4 and H_k/H_{k-1} are finitely generated. We obtain the desired counterexample from $\mathbf{K}[x_1, x_2, \dots, x_n]^{H_k} = \mathbf{K}[x_1, x_2, \dots, x_n]^{H_{k-1}} \cap \mathbf{K}[x_1, x_2, \dots, x_n]^{H_k/H_{k-1}}$.

Theorem 2. (Weitzenböck 1932) *Let $\mathbf{K}$ be a field of characteristic 0 and V be any finite-dimensional rational $\mathbf{G}_a$-module. Then the invariant ring $\mathbf{K}[V]^{\mathbf{G}_a}$ is finitely generated.*

Proof. We refer, e.g., to Theorem 10.1. in [4]. $\square$

In the sequel we denote the nullspace of the matrix $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ a_{31} & a_{32} & \dots & a_{3k} \end{pmatrix}$ by N_k and note that N_k has dimension $k-3$, provided that $k \geq 3$. The embedding of $N_k \hookrightarrow \mathbf{K}^r$ by setting the additional coordinates to 0 will be omitted. The groups H_k can be identified with the nullspace N_k of A_k via the morphism of additive groups

$$\psi_k : N_k \ni \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{pmatrix} \mapsto \begin{pmatrix} B_1 & 0 & \dots & 0 \\ 0 & B_2 & \dots & 0 \\ 0 & \dots & \ddots & \vdots \\ 0 & \dots & 0 & B_k \end{pmatrix}, B_i = \begin{pmatrix} 1 & b_i \\ 0 & 1 \end{pmatrix}.$$

We also omit the induced embedding of $H_k \hookrightarrow GL_n(\mathbf{K})$ for $1 \leq k \leq r$ by the embedding of N_k and note that $\psi_{k+1}(N_k) = \psi_k(N_k)$. In the sequel fix a basis $\beta_1, \beta_2, \dots \beta_{k-3}$ of N_k for $4 \leq k \leq r$ s.t. $\beta_1, \beta_2, \dots \beta_{k-4}$ is a basis of N_{k-1} and β_{k-3} extends the basis of N_{k-1} to a basis of N_k. Note that the groups $N_1 = N_2 = N_3 = \{0\}$ and that N_k/N_{k-1} is isomorphic to $\mathbf{G}_a$ for $4 \leq k \leq r$ via the mapping $\sum_{i=1}^{k-4} \lambda_i \beta_i + \lambda \beta_{k-3} \mapsto \lambda$. The map is well defined since $\beta_1, \beta_2, \dots, \beta_{k-4}, \beta_{k-3}$ form a basis, and bijective, hence an isomorphism of additive groups.

These isomorphisms are used to define a linear action for each H_k/H_{k-1} on $\mathbf{K}[x_1, x_2, \dots, x_n]$ in such a way that the corresponding invariant rings are finitely generated. Firstly, we define the representation ρ'_k of N_k by

$$\begin{aligned} \rho'_k : N_k &\rightarrow GL_n(\mathbf{K}), \\ \sum_{i=1}^{k-4} \lambda_i\beta_i + \lambda\beta_{k-3} &\mapsto \psi_k(\lambda\beta_{k-3}). \end{aligned}$$

Note that ρ'_k is well defined since $\beta_1, \beta_2, \ldots \beta_{k-3}$ form a basis, that ρ'_k has kernel N_{k-1} and yields a linear representation of N_k/N_{k-1} on $\mathbf{K}^n$. By applying Weitzenböck's theorem we obtain the following result.

Proposition 2. *For $4 \leq k \leq r$ the invariant ring $\mathbf{K}[x_1, x_2, \ldots, x_n]^{H_k/H_{k-1}}$ is finitely generated.*

Proof. As noted above, the group $\mathbf{G}_a$ is isomorphic to N_k/N_{k-1} and to H_k/H_{k-1} by sending $\lambda \in \mathbf{G}_a$ to $[\lambda\beta_{k-3}]$ or to $[\psi_k(\lambda\beta_{k-3})]$ respectively. Let ϕ_k be the inverse of the isomorphism ψ_k and define the linear representation ρ_k of H_k/H_{k-1} by $\rho_k([\sigma]) := \rho'_k(\phi_k(\sigma))$. The representation is well defined because $\ker \rho'_k \circ \phi_k = H_{k-1}$. Since H_k/H_{k-1} is isomorphic to $\mathbf{G}_a$ and acts linearly on $\mathbf{K}^n$ via ρ_k, the invariant ring $\mathbf{K}[x_1, x_2, \ldots, x_n]^{H_k/H_{k-1}}$ is finitely generated by Weitzenböck's theorem. $\square$

Proposition 3. *For $4 \leq k \leq r$ the invariant ring $\mathbf{K}[x_1, x_2, \ldots, x_n]^{H_k}$ equals*

$$\mathbf{K}[x_1, x_2, \ldots, x_n]^{H_k} = \mathbf{K}[x_1, x_2, \ldots, x_n]^{H_{k-1}} \cap \mathbf{K}[x_1, x_2, \ldots, x_n]^{H_k/H_{k-1}}$$

Proof. If $f \in \mathbf{K}[x_1, x_2, \ldots, x_n]^{H_{k-1}} \cap \mathbf{K}[x_1, x_2, \ldots, x_n]^{H_k/H_{k-1}}$, then f is invariant w.r.t. $\psi_k(\sum_{i=1}^{k-4} \lambda_i\beta_i)$ and $\psi_k(\lambda\beta_{k-3})$ for $\lambda_1, \lambda_2, \ldots, \lambda_{k-4}, \lambda \in \mathbf{K}$. In particular, f is invariant w.r.t. $\psi_k(\sum_{i=1}^{k-4} \lambda_i\beta_i + \lambda\beta_{k-3})$. Since $\beta_1, \beta_2, \ldots, \beta_{k-3}$ form a basis of N_k and ψ_k is an isomorphism, the polynomial f is invariant w.r.t. the group H_k. The converse inclusion is obvious. $\square$

We obtain the counterexample by showing that some $\mathbf{K}[x_1, x_2, \ldots, x_n]^{H_k}$, $5 \leq k \leq r$, cannot be finitely generated.

Theorem 3. *There exists $5 \leq k \leq r$ s.t.*

$$\mathbf{K}[x_1, x_2, \ldots, x_n]^{H_{k-1}} \cap \mathbf{K}[x_1, x_2, \ldots, x_n]^{H_k/H_{k-1}}$$

is not finitely generated.

Proof. Firstly, assume that $\mathbf{K}[x_1, x_2, \ldots, x_n]^{H_r}$ is finitely generated. The group T acts on $\mathbf{K}[x_1, x_2, \ldots, x_n]^{H_r}$ since it is the normalizer of H_r. By Proposition 1, $(\mathbf{K}[x_1, x_2, \ldots, x_n]^{H_r})^T$ is finitely generated and, since

$$\begin{aligned} (\mathbf{K}[x_1, x_2, \ldots, x_n]^{H_r})^T &= \mathbf{K}[x_1, x_2, \ldots, x_n]^T \cap \mathbf{K}[x_1, x_2, \ldots, x_n]^{H_r} \\ &= \mathbf{K}[x_1, x_2, \ldots, x_n]^G \end{aligned}$$

is a contradiction to Nagata's theorem, the ring $\mathbf{K}[x_1, x_2, \ldots, x_n]^{H_r}$ cannot be finitely generated. Therefore let $k \leq r$ be the minimal index s.t. $\mathbf{K}[x_1, x_2, \ldots, x_n]^{H_k}$ is not finitely generated. By Proposition 2, $k > 4$ and $\mathbf{K}[x_1, x_2, \ldots, x_n]^{H_k/H_{k-1}}$ is finitely generated. By assumption, the ring $\mathbf{K}[x_1, x_2, \ldots, x_n]^{H_{k-1}}$ is finitely generated, but the intersection $\mathbf{K}[x_1, x_2, \ldots, x_n]^{H_k} = \mathbf{K}[x_1, x_2, \ldots, x_n]^{H_{k-1}} \cap \mathbf{K}[x_1, x_2, \ldots, x_n]^{H_k/H_{k-1}}$ is not finitely generated. $\square$

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